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Received May 25, 1973 (683)

STUDIA MATHEMATICA, T. LII. (1974)

On linear properties of separable conjugate spaces of C^*-algebras

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Abstract. It is proved that the conjugate space of a separable C^*-algebra in which any hermitian element has a countable spectrum is isometric to $(\sum_{n=1}^{\infty} N(H_n))_1$ where $N(H_n)$ are nuclear operators on separable Hilbert spaces H_n. This implies that a C^*-algebra with a separable conjugate space has a Schauder basis.

The present paper is a study of some linear properties of a class of C^*-algebras which can be considered as a generalization of spaces of continuous functions on countable compact spaces. We prove an isometric representation of a conjugate space of such an algebra. This result can be considered as a generalization to the C^*-algebra setting of a theorem of Rudin [5]. The method of the proof was influenced by [7]. From our representation theorem we deduce some corollaries on the linear structure of such algebras.

Our terminology on C^*-algebras agrees with that of [6] and our terminology on Banach spaces is that usually adopted in Banach space theory (cf. [3]).

DEFINITION. A C^*-algebra is called *countably scattered* if it is separable and each abelian $*$-subalgebra has a scattered spectrum.

LEMMA 1. *The class of countably scattered C^*-algebras is closed under taking $*$-subalgebras, $*$-homomorphic images and sums in the sense of* c_0.

Proof. Obvious from the definition and the following

SUBLEMMA. *A C^*-algebra X is countably scattered iff X is separable and every hermitian element in X has a countable spectrum.*

Recall that a W^*-algebra is a C^*-algebra X isometric to a conjugate space of some Banach space X_*. This space X_* is unique (cf. [6], 1.13.3).

LEMMA 2. *Let* (Ω, μ) *be a measure space which is a disjoint sum of sets of finite measure* (call such a space a *localizable measure space*) *and let W be a factor, i.e., a W^*-algebra such that an element commuting with any other is a multiple of identity. If every central projection in* $L_\infty(\Omega, W)$ *contains a minimal projection, then* (Ω, μ) *is purely atomic.*

Recall that a *central projection* is a projection which commutes with any element of an algebra. By $L_\infty(\Omega, W)$ we mean the space of all W-valued, essentially bounded, *weakly locally measurable functions. With pointwise operations this space is a W^*-algebra.

Proof. It is easily checked that every central projection in $L_\infty(\Omega, W)$ has the form $\chi_A I$ where χ_A is an indicator function of a set $A \subset \Omega$ and I means the identity in W. Moreover, such a projection is non-zero if $\mu(A) > 0$ and is minimal iff A is an atom of Ω. So the assumptions of the lemma imply that any set of positive measure contains an atom. So Ω is purely atomic.

Recall that by the Sherman theorem (cf. [6], 1.17.2) if X is a C^*-algebra, then X^{**} is a W^*-algebra. Recall also that $B(H)$ is a W^*-algebra and its predual is $N(H)$, the space of all nuclear operators on a space H.

LEMMA 3. *If X is a separable C^*-algebra, then each irreducible W^*-representation of X^{**} acts on a separable Hilbert space* (a *W^*-representation* is a representation continuous from the $\sigma(X^{**}, X^*)$ topology into $\sigma(B(H), N(H))$).

Proof. Let $\varphi\colon X^{**}\to B(H)$ be an irreducible W^*-representation of X^{**}. Then $\varphi(X)$ is norm-separable and dense in $\varphi(X^{**})$ in the strong operator topology. Since $\varphi(X)$ has a separable invariant subspace and φ is irreducible, we conclude that H is separable.

LEMMA 4. *If X is a separable Banach space, then X^* is not isomorphic to $(\sum_{\alpha\in A} Y_\alpha)_1$ where* $\operatorname{card} A > \aleph_0$ *and Y_α are separable conjugate spaces.*

Proof. If X^* is isomorphic to $(\sum_{\alpha\in A} Y_\alpha)_1$ with $\operatorname{card} A > \aleph_0$, then $X^* \supset l_1(A)$ and, by [2], $X^* \supset L_1(0,1)$. But this implies that $L_1(0,1) \subset (\sum_{n=1}^{\infty} Y_{\alpha_n})_1$ for some sequence $(\alpha_n) \subset A$. Since $(\sum_{n=1}^{\infty} Y_{\alpha_n})_1$ is a separable conjugate space, this contradicts the classical theorem of Gelfand [1] (cf. [2]).

LEMMA 5. *Let X be a countably scattered C^*-algebra. Then X contains a minimal abelian projection, i.e., a minimal projection p such that pXp is an abelian algebra.*

Proof. Let B be a maximal abelian $*$-subalgebra of X. Since the spectrum of B is scattered and metrizable, we infer that B is isometric with $C_0(\alpha)$ for some countable ordinal α. ($C_0(\alpha)$ means the space of all continuous functions f which are defined on the set of all ordinals less than or equal to α equipped with interval topology, and such that $f(\alpha) = 0$.) To see this, use the Gelfand representation theorem and the Mazurkiewicz–Sierpiński theorem [4]. Let p be an isolated point of α, $p \neq \alpha$, and let χ_p be the indicator function of p. $\chi_p \in B$ and it is a minimal projection in B. This means that for $y \in B$ there is a scalar $y(p)$ such that $\chi_p y = \chi_p y \chi_p = y(p)\chi_p$.

We shall show that χ_p is a minimal projection in X.

Let us consider $x \in X$ and $y \in B$. Then

$$(\chi_p x \chi_p) y = \chi_p x \chi_p \chi_p y \chi_p = y(p)\chi_p x \chi_p = y(p)\chi_p\chi_p x \chi_p = \chi_p y \chi_p \chi_p x \chi_p = y\chi_p x \chi_p.$$

Since B is maximal, we infer that $\chi_p x \chi_p \in B$. So if we put $x(p) = \chi_p x \chi_p(p)$, we obtain

$$\chi_p x \chi_p = \chi_p \chi_p x \chi_p \chi_p = x(p)\chi_p;$$

so χ_p is a minimal projection in X. Moreover, χ_p is an abelian projection.

THEOREM. *If X is a countably scattered C^*-algebra, then X^* is isometric to*

$$\Big(\sum_{n=1}^{S} N(H_n)\Big)_1$$

where H_n are separable Hilbert spaces and S is finite or ∞.

Proof. Let us consider X^{**} and let $z \in X^{**}$ be a central projection. Xz is a $*$-homomorphic image of X, and so, by Lemma 1 and Lemma 5, Xz contains an abelian projection e. Since Xz is w^*-dense in $X^{**}z$, e is an abelian and minimal projection in $X^{**}z$. So each central projection in X^{**} contains a minimal abelian projection. Hence by [6], 2.3.2 and 2.3.3,

$$X^{**} = \Big(\sum_{\beta\in K} Z_\beta \bar{\otimes} B(H_\beta)\Big)_\infty$$

where K is some set of cardinals, H_β is a Hilbert space of dimension β and Z_β is an abelian W^*-algebra. Since X^{**} has an irreducible W^*-representation onto each $B(H_\beta)$ (cf. [6], 1.22.11), by Lemma 3, K does not contain any uncountable cardinal, and so $N(H_\beta)$ are separable for all $\beta \in K$. So, by [6], 1.18.1 and 1.22.13,

$$X^{**} = \Big(\sum_{\beta\in K} L_\infty(\Omega_\beta, B(H_\beta))\Big)_\infty,$$

where $(\Omega_\beta, \mu_\beta)$ are localizable measure spaces. By Lemma 2 each $(\Omega_\beta, \mu_\beta)$ is purely atomic, and so $X^* = (\sum_{\alpha\in A} N(H_\alpha))_1$ (cf. [6], 1.13.3) where H_α are separable Hilbert spaces. Now Lemma 4 gives us the required representation ($N(H_\alpha)$ is a conjugate space).

Now let us state the following obvious

PROPOSITION 1. *The space $(\sum_{n=1}^{S} N(H_n))_1$ has a basis (x_n) such that there exists a sequence of indices (k_r) such that span $\{x_n\}_{n=1}^{k_r}$ is isometric to $\sum_{i=1}^{S} N(H_i)_1$ where S is finite and H_i are finite-dimensional Hilbert spaces.*

Now a result of Johnson, Rosenthal and Zippin [3], as stated in [10] Lemma 2, implies

COROLLARY 1. *A countably scattered C^*-algebra has a basis (x_n) such that there is a constant C and a sequence of indices (k_r) such that the Banach–Mazur distance from span $\{x_n\}_{n=1}^{k_r}$ to some finite-dimensional C^*-algebra is less than C.*

Remark 1. Since any infinite-dimensional C^*-algebra contains c_0, the results of [8] can be used to obtain some special types of bases.

Remark 2. It follows from the results of [9] and the easily provable fact that the Szlenk index of $C(\alpha)$ is greater than α that for a countably scattered C^*-algebra X there exists a countable ordinal β such that X does not contain subspaces isomorphic to $C(\alpha)$ for $\alpha \geqslant \beta$.

Remark 3. It is obvious that a C^*-algebra X is countably scattered iff X^* is separable, and so by [7] X is countably scattered iff it is a G.C.R. algebra with countable composition series.

Remark 4. The space X has the Schur property if the weak sequential convergence implies the norm convergence. It is well known that the space l_1, being the conjugate space of an abelian countably scattered C^*-algebra, has this property. We should like to discuss this property for conjugate spaces of other countably scattered C^*-algebras. If all H_n are finite-dimensional, then $(\sum_{n=1}^{\infty} N(H_n))_1$ has the Schur property which can easily be proved. If at least one of H_n is infinite-dimensional, then $(\sum_{n=1}^{\infty} N(H_n))_1$ does not have the Schur property since it contains l_2. However, Theorem IV.1 of [11] easily implies that in this case the following is true:

If (f_n) is a sequence of positive elements in $(\sum_{n=1}^{\infty} N(H_n))_1$ weakly convergent to f, then $\|f_n - f\| \to 0$.

This result can also be proved directly. Let us remark also that Theorem IV. 1 of [11], which is apparently formulated in a more general form, is in fact equivalent to the above statement.

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Received Juli 14, 1973 (701)