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Perturbations of Fredholm operators*

by

ROBERT B. ISRAEL (Winnipeg)

Abstract. We consider the class of bounded linear operators between Banach spaces which are admissible perturbations for the Fredholm operators (including the unbounded ones). This class includes the strictly singular and strictly cosingular operators. Various conditions are given under which it coincides with one of these classes, or consists of all bounded linear operators between the two spaces.

Let X and Y be infinite-dimensional Banach spaces. Let $\Phi_c(X, Y)$ be the class of closed, densely-defined Fredholm operators from X into Y. The class $F_c(X, Y)$ of "admissible perturbations" for $\Phi_c(X, Y)$ consists of all bounded linear operators A such that $A+T$ is Fredholm for all $T \in \Phi_c(X, Y)$. In this paper we investigate the problem of characterizing the class $F_c(X, Y)$.

If we restrict our attention to bounded Fredholm operators T another class, which we call $F(X, Y)$, is obtained. Since the existence of a bounded Fredholm operator from X to Y implies that one of X and Y is isomorphic to a subspace of finite codimension in the other, little generality is lost in considering only $F(X)$, i.e. $F(X, X)$. $F(X)$ has been investigated in [7], [8], [9], [12] and [13], mainly using Banach algebra techniques. It is not difficult to show that if there exists a bounded Fredholm operator from X to Y, $F_c(X, Y) = F(X, Y)$. Gohberg, Markus and Fel'dman ([2], Section 3, Prop. 3) state in effect that $F_c(X, Y) = F(X, Y)$ for any X and Y. However, a counterexample to this assertion can easily be constructed in certain cases where no bounded Fredholm operator from X to Y exists, so that $F(X, Y)$ includes all bounded operators from X to Y.

The following notation will be used. All spaces X, Y, Z will be infinite-dimensional Banach spaces, and all operators will be linear. $L(X, Y)$ is the class of bounded operators from X into Y, $L_c(X, Y)$ the class of

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closed densely-defined operators from X into Y, $\Phi(X, Y)$ the class of Fredholm operators in $L(X, Y)$, and $\Phi_c(X, Y)$ the class of Fredholm operators in $L_c(X, Y)$. Thus $F(X, Y)$ consists of the admissible perturbations for $\Phi(X, Y)$, while $F_c(X, Y)$ consists of the admissible perturbations for $\Phi_c(X, Y)$. $K(X, Y)$ is the class of compact operators in $L(X, Y)$. The domain of an operator A will be denoted $\mathfrak{D}(A)$, the range $\mathfrak{R}(A)$ and the null space $\mathfrak{N}(A)$.

Two classes of operators which are always admissible perturbations for Fredholm operators are the strictly singular operators, denoted $S_0(X, Y)$ and the strictly cosingular operators $S_1(X, Y)$. An operator $A \in L(X, Y)$, is strictly singular if every subspace of X for which the restriction of A has a bounded inverse is finite-dimensional. $A \in L(X, Y)$ is strictly cosingular if there is no infinite-dimensional Banach space Z and $S \in L(Y, Z)$ such that SA is surjective. See [11] for basic properties of strictly cosingular operators and relations between them and strictly singular operators.

A result of Kato ([6], Thm. 8, proved more simply in [3], Thm. V.2.1) shows that $S_0(X, Y) \subseteq F_c(X, Y)$. Vladimirskiĭ [15] shows that members of $S_1(X, Y)$ are admissible perturbations for the bounded "semi-Fredholm" operators $\Phi_-(X, Y)$ (i.e. those with closed range of finite codimension in Y), and hence for the Fredholm operators. Essentially the same proof works in the unbounded case, i.e. $S_1(X, Y) \subseteq F_c(X, Y)$. We sketch the proof in Lemma 3. In general, equality need not hold in these inclusions, even if $X = Y$. Gohberg, Markus and Fel'dman [2], using Kadec's construction [5] of a subspace of $L_p(0, 1)$ isomorphic to l_q (for $1 < p < q < 2$) note that there is a bounded linear operator V on $X = L_p(0, 1) \oplus l_q$ which is not strictly singular but has a strictly singular adjoint. Thus by [11], Prop. 3 and the reflexivity of X, V is strictly cosingular but not strictly singular, while V^* is strictly singular but not strictly cosingular. Using V and V^* we can produce an operator in $L(X \oplus X^*)$ which is the sum of a strictly singular and a strictly cosingular operator (hence is in $F(X \oplus X^*)$), but is neither strictly singular nor strictly cosingular.

We first prove some preliminary lemmas, and then some criteria for $F_c(X, Y) = L(X, Y)$.

LEMMA 1. *Suppose* $A \in L(X, Y)$ *and* $\mathfrak{R}(A+T)$ *is closed for all* $T \in \Phi_c(X, Y)$. *Then* $A \in F_c(X, Y)$.

Proof. Let $T \in \Phi_c(X, Y)$. If $A+T$ had no index (i.e. if $\mathfrak{N}(A+T)$ was infinite-dimensional and $\mathfrak{R}(A+T)$ had infinite codimension in Y), by Theorem V.2.6 of [3] there would be a compact operator K with $\mathfrak{R}(A+T+K)$ not closed. But since $K(X, Y) \subseteq F_c(X, Y)$, $T+K \in \Phi_c(X, Y)$ so by hypothesis $\mathfrak{R}(A+T+K)$ must be closed. Thus $A+T$ has an index; similarly, for any scalar c, $cA+T$ has an index. By the constancy of the index under small perturbations ([3], Theorem V.1.6), $A+T$ has the same index as T, and hence is Fredholm.

LEMMA 2. *Suppose* $A \in L(X, Y)$ *and for every* $T \in \Phi_c(X, Y)$, $\mathfrak{N}(A+T)$ *is finite-dimensional. Then* $A \in F_c(X, Y)$.

Proof. In Theorem 23 of [13] it is shown that if $S \in L(X)$ and $\mathfrak{R}(S)$ is not closed, there is $K \in K(X)$ with $\mathfrak{N}(S-K)$ infinite-dimensional. The same proof is valid for $S \in L_c(X, Y)$ and $K \in K(X, Y)$. Thus if A is as above and $T \in \Phi_c(X, Y)$, $\mathfrak{N}(A+T+K)$ is finite-dimensional for all $K \in K(X, Y)$, so $\mathfrak{R}(A+T)$ is closed. By Lemma 1, $A \in F_c(X, Y)$.

LEMMA 3. $S_1(X, Y) \subseteq F_c(X, Y)$.

Proof. Let $A \in S_1(X, Y)$ and $T \in \Phi_c(X, Y)$. It suffices to show $\mathfrak{R}(A^*+T^*)$ is closed, for then $\mathfrak{R}(A+T)$ is closed ([3], Theorem IV.1.2) and we can apply Lemma 1. Suppose $\mathfrak{R}(A^*+T^*)$ were not closed. Take $Y = \mathfrak{R}(T) \oplus M$, M finite-dimensional, so $Y^* = \mathfrak{N}(T^*) \oplus M^\perp$, and $(A^*+T^*)\big(M^\perp \cap \mathfrak{D}(T^*)\big)$ is not closed since $\mathfrak{N}(T^*)$ is finite-dimensional and $\mathfrak{R}(A^*+T^*) = (A^*+T^*)\big(M^\perp \cap \mathfrak{D}(T^*)\big) + A^*\big(\mathfrak{N}(T^*)\big)$. As in [15] we obtain biorthogonal sequences $y_n^* \in M^\perp \cap \mathfrak{D}(T^*)$, $y_n \in Y$ such that $\|y_n^*\| = 1$, $\|y_n\| \leqslant 2^{2n-1}$, $\|(A^*+T^*)y_n^*\| \leqslant 2^{-3n}c$, where $c = \|(T^*_{|M^\perp})^{-1}\|^{-1}$. Now defining $B \in L(X, Y)$ by $Bx = \sum_{n=1}^{\infty} \langle x, (A^*+T^*)y_n^*\rangle y_n$, we find that B^* is compact with norm at most $\frac{1}{2}c$. Then $Z = M^\perp \cap \mathfrak{N}(T^*+A^*-B^*) = \big(M + \mathfrak{R}(T+A-B)\big)^\perp$ is weak-*closed in Y^*, and infinite-dimensional since each $y_n^* \in Z$. For $y^* \in Z$, $\|A^*y^*\| \geqslant \|T^*y^*\| - \|B^*y^*\| \geqslant \frac{1}{2}c\|y^*\|$, so A^* has a bounded inverse on Z. But this contradicts the characterization ([15], Lemma 1b) of strictly cosingular operators: $A \in S_1(X, Y)$ iff A^* does not have a bounded inverse on any weak-*closed infinite-dimensional subspace of Y^*.

LEMMA 4. *Let* $T \in \Phi_c(X, Y)$. *Then there is* $S \in L(Y, X)$ *such that for some bounded finite-dimensional operators* K_1, K_2, $TS = I + K_1$ *(defined on all* Y*) and* $STx = (I+K_2)x$ *for* $x \in \mathfrak{D}(T)$. (For bounded Fredholm operators this is due to Yood [17].)

Proof. Let P and Q be projections of X on $\mathfrak{N}(T)$ and Y on $\mathfrak{R}(T)$ respectively. Note P and $I-Q$ have finite rank. Let $T_1 \in L_c(\mathfrak{N}(P), \mathfrak{R}(T))$ be the restriction of T to $\mathfrak{N}(P) \cap \mathfrak{D}(T)$. Then T_1 has a bounded inverse.

Let $S = T_1^{-1}Q \in L(Y, X)$. Now $\mathfrak{R}(S) \subseteq \mathfrak{R}(T_1^{-1}) \subseteq \mathfrak{D}(T)$ and $Tx = T_1(I-P)x$ for $x \in \mathfrak{D}(T)$. So $TS = T_1(I-P)T_1^{-1}Q = Q$ and $STx = T_1^{-1}QT_1(I-P)x = (I-P)x$ for $x \in \mathfrak{D}(T)$.

In certain cases $F_c(X, Y)$ includes all of $L(X, Y)$. This obviously occurs if $S_0(X, Y) = L(X, Y)$ or $S_1(X, Y) = L(X, Y)$. There are also conditions on $L(Y, X)$ which imply $F_c(X, Y) = L(X, Y)$.

Theorem 1. *In any of the following cases, $F_c(X, Y) = L(X, Y)$:*

(a) $S_0(Y, X) = L(Y, X)$,

(b) $S_1(Y, X) = L(Y, X)$,

(c) *Y is separable and X is not separable.*

Proof. For any $T \in \Phi_c(X, Y)$ take S as in Lemma 4, and suppose $A \in L(X, Y)$. In cases (a) or (b), $S \in S_0(Y, X)$ or $S_1(Y, X)$ and so $AS \in S_0(Y)$ or $S_1(Y)$ respectively, so $AS \in F(Y)$. Then $(A+T)S = AS + I + K_1 \in \Phi(Y)$, and $\mathfrak{R}(A+T)$ contains $\mathfrak{R}\big((A+T)S\big)$, so is closed and of finite codimension in Y. By Lemma 1, $A \in F_c(X, Y)$. In case (c), $\overline{\mathfrak{R}(S)}$ must be separable and of finite codimension in X, which is impossible if X is not separable. Thus in this case $\Phi_c(X, Y)$ is empty and so trivially $F_c(X, Y) = L(X, Y)$.

One of the most important facts about $F(X)$ is that it is a closed ideal in $L(X)$. It is easy to show that $F_c(X, Y)$ is a closed subspace of $L(X, Y)$. The following two lemmas give a generalization of the multiplicative properties.

Lemma 5. *Let $A \in F_c(X, Y)$. Then for any $T_1 \in L(Y)$ and $T_2 \in L(X)$, T_1A and AT_2 are both in $F_c(X, Y)$.*

Proof. Let c be a nonzero scalar small enough so $\|cT_1\| < 1$, so that $I + cT_1$ is an isomorphism of Y onto Y. Since the properties defining Fredholm operators are preserved under isomorphisms, $(I + cT_1)A \in F_c(X, Y)$. Then since $F_c(X, Y)$ is a linear subspace of $L(X, Y)$, $T_1 A = c^{-1}\big((I + cT_1)A - A\big) \in F_c(X, Y)$. Similarly for T_2 acting on the right.

Lemma 6. *Let $A \in F_c(X, Y)$. If $S_1 \in L(Z, X)$ and S_1T_1 is bounded for some $T_1 \in \Phi_c(X, Z)$, $AS_1 \in F_c(Z, Y)$. If $S_2 \in L(Y, Z)$ and for some $T_2 \in \Phi_c(Z, Y)$, T_2S_2 is bounded with $\mathfrak{R}(S_2) \subseteq \mathfrak{D}(T_2)$, $S_2A \in F_c(X, Z)$.*

Proof. Since S_1T_1 is bounded and $\mathfrak{D}(S_1T_1) = \mathfrak{D}(T_1)$ is dense in X, we can extend S_1T_1 uniquely to a member of $L(X)$, denoted $\overline{S_1T_1}$. By Lemma 5, $A\overline{S_1T_1} \in F_c(X, Y)$. Let $U_1 \in \Phi_c(Z, Y)$. Then by Theorem IV.2.7 of [3], $U_1T_1 \in \Phi_c(X, Y)$, its domain being contained in $\mathfrak{D}(T_1)$. Thus $(AS_1 + U_1)T_1 = A\overline{S_1T_1} + U_1T_1 \in \Phi_c(X, Y)$. Since $\mathfrak{R}\big((AS_1 + U_1)T_1\big) \subseteq \mathfrak{R}(AS_1 + U_1)$, the latter is closed and of finite codimension in Y. By Lemma 1, $AS_1 \in F_c(Z, X)$. T_2S_2 is bounded and defined on all Y, so $T_2S_2 \in L(Y)$ and by Lemma 5, $T_2S_2A \in F_c(X, Y)$. If $U_2 \in \Phi_c(X, Z)$, then $T_2U_2 \in \Phi_c(X, Y)$ so $T_2(U_2 + S_2A) = T_2U_2 + T_2S_2A \in \Phi_c(X, Y)$. Therefore $N(U_2 + S_2A)$ is finite-dimensional and by Lemma 2, $S_2A \in F_c(X, Z)$.

Now we investigate certain conditions under which $F_c(X, Y) = S_0(X, Y)$ or $S_1(X, Y)$. A number of cases are known where $F(X) = S_0(X)$: Pfaffenberger [12] proves this for X subprojective, and Mil'man [9] treats the cases L_1 and $C(S)$ (for S metric). Our basic tool in generalizing these results to $F_c(X, Y)$ is the following lemma.

Lemma 7. *Suppose X and Y are infinite-dimensional Banach spaces, with X separable and $Y^*_{\mathfrak{S}}$ having a countable total subset. Then $\Phi_c(X, Y)$ is nonempty.*

Proof. In [4] Goldberg and Kruse construct an operator $K \in K(Y, X)$ which is 1-1 and has dense range in X. Thus $K^{-1} \in L_c(X, Y)$ is 1-1 onto Y, so $K^{-1} \in \Phi_c(X, Y)$.

Lemma 8. *Given any closed subspace M of infinite codimension in a Banach space X, there is a closed subspace N of X such that $M \subseteq N$ and N/M and X/N are both infinite-dimensional.*

Proof. Let $p: X \to X/M$ be the quotient map. Choose biorthogonal sequences $x_j \in X/M$, $x_j^* \in (X/M)^*$ and let $N_1 = {}^{\perp}\{x_{2j}^*\}$. Then N_1 is closed and $x_{2j} \notin N_1$ while $x_{2j+1} \in N_1$. We can take $N = p^{-1}(N_1)$ and the conditions will be satisfied.

A Banach space X is said to be *subprojective* if every infinite-dimensional closed subspace of X contains an infinite-dimensional subspace which is complemented in X. It is said to be *superprojective* if every closed subspace of infinite codimension in X is contained in a complemented subspace of infinite codimension in X. Some examples and properties of these types of spaces are found in Whitley [16].

Theorem 2. *Suppose X is separable and Y^* has a countable total subset.*

(a) *If Y is subprojective, $F_c(X, Y) = S_0(X, Y)$.*

(b) *If X is superprojective, $F_c(X, Y) = S_1(X, Y)$.*

Proof. (a) Suppose $A \in L(X, Y)$ is not strictly singular, and let M be a closed infinite-dimensional subspace of X such that $A|_M$ has a continuous inverse C defined on AM. Let M_1 be an infinite-dimensional closed subspace of infinite codimension in M. Let P be a projection of Y onto a closed infinite-dimensional subspace E of AM_1. Then CE is a closed infinite-dimensional subspace of M_1 which is the range of the projection CPA of X. Thus we can take $X = U \oplus CE$, $Y = V \oplus E$, where U and V are closed and infinite-dimensional since M/M_1 (and hence AM/AM_1) is infinite-dimensional. Note that U is separable and V^* has a countable total subset, so there is $T_1 \in \Phi_c(U, V)$. Now we define T with domain $D(T_1) + CE$ by $T(u + w) = T_1u + Aw$ for $u \in \mathfrak{D}(T_1)$, $w \in CE$. Then $T \in \Phi_c(X, Y)$, but $CE \subseteq \mathfrak{N}(T - A)$ is infinite-dimensional so $T - A$ is not Fredholm and $A \notin F_c(X, Y)$.

(b) Suppose $A \in L(X, Y)$ is not strictly cosingular, so that for some infinite-dimensional Banach space Z and $h \in L(Y, Z)$, hA is surjective. Let $\mathfrak{N}(hA) \subseteq M \subseteq X$, where M is closed and X/M and $M/\mathfrak{N}(hA)$ are infinite-dimensional. Let P be a projection of X on a subspace U of infinite codimension in X and containing M. Let $Q = I - P$ and $N = \mathfrak{R}(Q)$, so $X = U \oplus N$. Define $S\colon Z \to X$ by $Sz = Qx$, where $hAx = z$. Since hA is onto Z and $\mathfrak{N}(hA) \subseteq \mathfrak{N}(Q)$, S is well-defined and bounded. Moreover, $ShA = Q$. Now $AQSh$ is a projection and its range is $\mathfrak{R}(AQ) = AN$. So AN is closed and $Y = AN \oplus V$, where $V = \mathfrak{N}(AQSh)$. Note that V contains AM, which is infinite-dimensional since $M/\mathfrak{N}(hA)$ is infinite-dimensional. Now we can proceed as in part (a) to find $T \in \Phi_c(X, Y)$ which agrees with A on N, so $A \notin F_c(X, Y)$.

The method of Theorem 2(a) can be used to characterize $F_c(X, Y)$ for other types of Banach spaces, using more specialized properties. For example, we can use the following well-known properties of the space c_0: every closed infinite-dimensional subspace of c_0 contains a subspace isomorphic to c_0 [1], and in any separable Banach space a subspace isomorphic to c_0 is complemented [14].

THEOREM 3. *If X is isomorphic to a subspace of c_0 and Y is separable, then $F_c(X, Y) = S_0(X, Y)$.*

Proof. If $A \in L(X, Y)$ is not strictly singular, let M be a closed infinite-dimensional subspace of X such that A is an isomorphism of M on AM. Then we can take a closed subspace N of infinite codimension in M and isomorphic to c_0, so that N and AN are complemented in X and Y respectively. Then we can proceed as in Theorem 2 (a).

THEOREM 4. *Let S be a compact metric space (so $C(S)$ is separable) and let Y be separable. Then $F_c(C(S), Y) = S_0(C(S), Y)$.*

Proof. Pełczyński [11] shows that for $A \in L(C(S), Y)$, A is strictly singular if and only if A has no bounded inverse on subspaces isomorphic to c_0. Thus if A is not strictly singular, there is a subspace M of $C(S)$ isomorphic to c_0 such that A is isomorphism of M on AM. We can assume M and AM are of infinite codimension in $C(S)$ and Y respectively, since c_0 contains a subspace of infinite codimension isomorphic to c_0. Since M and AM are complemented in $C(S)$ and Y respectively (because these are separable), we can again use the method of Theorem 2 (a).

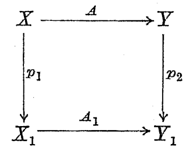
COROLLARY. *Let S be a compact Hausdorff space, X a separable quotient space of $C(S)$, and Y separable. Then every member of $F_c(X, Y)$ is weakly compact.*

Proof. An operator is said to be *unconditionally converging* [10] if it maps weakly unconditionally convergent sequences to unconditionally convergent sequences. Pełczyński [11] shows that any bounded linear operator that is not unconditionally converging has a bounded inverse on a subspace isomorphic to c_0. Thus, by the method of Theorem 2 (a), we have that if X and Y are separable, every member of $F_c(X, Y)$ is unconditionally converging. Pełczyński [10] shows that the property "V" (every unconditionally converging operator with domain X is weakly

compact) is possessed by $C(S)$ and by any quotient space of a space with the property.

THEOREM 5. *Suppose Y is $L_1(m)$ for some positive measure m. If X is separable and Y^* has a countable total subset, $F_c(X, Y) = S_1(X, Y)$.*

Proof. Pełczyński ([11], Part II, Theorem 1) shows that for $Y = L_1(m)$, $A \in L(X, Y)$ is strictly cosingular if and only if there is no commutative diagram



where p_1 and p_2 are projections, X_1 and Y_1 are isomorphic to l_1 and $A_1 = A|_{X_1}$ is an isomorphism onto Y_1. But if there is such a diagram it is clear by the method of Theorem 2(a) that $A \notin F_c(X, Y)$.

COROLLARY. *If X is $L_1(m)$ for some positive measure m then $F(X) = S_0(X) = S_1(X)$.*

Proof. Suppose $A \in L(X)$ is not strictly cosingular, so that there is a commutative diagram as given in Theorem 5 (with $X = Y$). Now $p_1 = A_1^{-1} p_2 A \notin F(X)$ since $\mathfrak{N}(I - p_1)$ has infinite codimension, so $A \notin F(X)$. Thus $F(X) = S_1(X)$. Also since $L_1(m)$ has the Dunford–Pettis property, Pełczyński's results [11] show that $S_1(X) = S_0(X)$.

One other type of Banach space in which we have $F(X) = S_0(X)$ is the class of "injective" Banach spaces. A Banach space X is injective if for any Banach spaces $Y \subset Z$ and any bounded linear operator $S\colon Y \to X$, there is an extension $T \in L(Z, X)$ of S.

THEOREM 6. *If X is injective, $F(X) = S_0(X)$.*

Proof. If $A \in L(X)$ is not strictly singular, the restriction of A to a closed infinite-dimensional subspace M of X has a bounded inverse C defined on AM. Since X is injective, C can be extended to a bounded linear operator $B\colon X \to X$, so that BA is the identity on M. Thus $M \subseteq \mathfrak{N}(I - BA)$, so $BA \notin F(X)$, and by Lemma 5, $A \notin F(X)$.

References

[1] S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, vol. 1, Warsaw 1932.

[2] I. C. Gohberg, A. S. Markus and I. A. Fel'dman, *Normally solvable operators and ideals associated with them*, Izv. Moldavsk. Fil. Akad. Nauk. SSSR 76, # 10 (1960), pp. 51–69; translated in Amer. Math. Soc. Translations 61, series 2 (1967), pp. 63–84.

[3] S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill, New York 1966.

[4] S. Goldberg and A. H. Kruse, *The existence of compact linear maps between Banach spaces*, Proc. Amer. Math. Soc. 13 (1962), pp. 808–811.

[5] M. I. Kadec, *On linear dimension of spaces L_p and l_q*, Uspekhi Mat. Nauk 13, # 6 (1958), pp. 95–98 (Russian).

[6] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. d'Analyse Math. 6 (1958), pp. 261–322.

[7] A. Lebow and M. Schechter, *Semigroups of operators and measures of non-compactness*, J. Funct. Anal. 7 (1971), pp. 1–26.

[8] V. D. Mil'man, *Some properties of strictly singular operators*, Functional Analysis and its Applications 3 (1969), pp. 77–78.

[9] — *Operators of classes C_0 and C_0^**, Teor. Funkcii, Funkcional Anal. i Prilozen. 10 (1970), pp. 15–26.

[10] A. Pełczyński, *Banach spaces in which every unconditionally converging operator is weakly compact*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 10 (1962), pp. 641–648.

[11] — *Strictly singular and strictly cosingular operators*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 13 (1965), pp. 31–41.

[12] W. Pfaffenberger, *On the ideals of strictly singular and inessential operators*, Proc. Amer. Math. Soc. 25 (1970), pp. 603–607.

[13] M. Schechter, *Riesz operators and Fredholm perturbations*, Bull. Amer. Math. Soc. 74 (1968), pp. 1139–1144.

[14] A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. 47 (1941), pp. 938–947.

[15] Ju. I. Vladimirskiĭ, *Strictly cosingular operators*, Sov. Math. Dokl. 8 (1967), p. 739.

[16] R. J. Whitley, *Strictly singular operators and their conjugates*, Trans. Amer. Math. Soc. 113 (1964), pp. 252–261.

[17] B. Yood, *Difference algebras of linear transformations on a Banach space*, Pacific J. Math. 4 (1954), pp. 615–636.

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Beurling algebras on locally compact groups, tensor products, and multipliers

by

J. EDWARD KERLIN (Lexington, Ky.)

Abstract. Let G, H, and K be locally compact groups and $\theta: K \to G$ and $\psi: K \to H$ be continuous homomorphisms. A $(\theta, p; \psi, q)$-multiplier is a bounded linear transformation T of $L^p(G)$ into $L^q(H)$ such that $T \circ \bar{L}_{\theta(z)} = L_{\psi(z)} \circ T$ for all z in K, where L_x is the left translation by x operator. Via tensor product theory, a representation of the Banach space of $(\theta, p; \psi, q)$-multipliers can be obtained by identifying the topological tensor product $L^p(G) \oplus_K L^{-q}(H)$. A fundamental step in this analysis is the representation of the tensor product $L^1(G) \otimes_K \bar{L}^{-1}(H)$ as $L^1(G \otimes_K H)$, where $G \otimes_K H$ is a locally compact homogeneous space (carrying a quasi-invariant measure) canonically related to G; H, K, θ and ψ. More generally, it is shown here that $L^1_\omega(G) \otimes_{L^1_\zeta(K)} L^1_\eta(H) \cong L^1_{\omega^* \otimes_\zeta \eta^*}(G \otimes_K H)$, where ω, η, and ζ are weight functions on G, H, and K defining the Beurling algebras $L^1_\omega(G)$, $L^1_\eta()$ and $L^1_\zeta(K)$. The analysis is effected by obtaining an extension of the isomorphism $L^1_\omega(G)/J^1_\omega(G, H) \cong L^1_{\dot\omega}(G/H)$ of Reiter (for closed *normal* subgroups H of G) to permit arbitrary closed subgroups H of G.

If G is a locally compact group, $L^p(G)$, for $1 \leqslant p \leqslant \infty$, denotes the usual Lebesgue space with respect to left Haar measure on G. For each $x \in G$, L_x denotes the left translation operator on $L^p(G)$ given by $L_x f(y) = f(x^{-1}y)$ for $f \in L^p(G)$ and $y \in G$. Let G, H, and K be locally compact groups, and let $\theta: K \to G$ and $\psi: K \to H$ be continuous group homomorphisms. Let $1 \leqslant p, q \leqslant \infty$. A *$(\theta, p; \psi, q)$-multiplier* is a bounded linear transformation T from $L^p(G)$ into $L^q(H)$ such that $T \circ \dot{L}_{\theta(z)} = L_{\psi(z)} \circ T$ for all $z \in K$. In this context the "multiplier problem" is to characterize the space $\mathrm{Hom}_K(L^p(G), L^q(H))$ of $(\theta, p; \psi, q)$-multipliers of $L^p(G)$ into $L^q(H)$. When $G = H = K$ and $\theta = \psi = \mathrm{id}_G$, the identity map on G, we recapture the classical multiplier problem of characterizing the bounded linear transformations of $L^p(G)$ into $L^q(G)$ which commute with left translation by the elements of G. When $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$, and $\frac{1}{q} + \frac{1}{q'} = 1$, the theory of tensor products of Banach modules introduced by Rieffel in [12] shows that

$$\mathrm{Hom}_K(L^p(G), L^{q'}(H)) \cong (L^p(G) \otimes_K \bar{L}^q(H))^*, \tag{0.1}$$