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Compact, non-nuclear operators

by

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Abstract. If X is a super-reflexive Banach space, then there are nearly Euclidean subspaces E of dimension n in E and projections P of X onto E with $\|P\| = O(n^{\alpha})$, where $\alpha < 1/2$ does not depend on n. Thus, if X is super-reflexive and Y is arbitrary there are compact operators from X to Y which are non-p-summing for all p. It follows that if either X or Y is super-reflexive there are compact, non-nuclear operators from X to Y.

Grothendieck [6] asked, "*if all operators from X to Y are nuclear, must either X or Y be finite dimensional?*" We show that this is true if either X or Y is super-reflexive.

Notationally, all spaces are infinite dimensional Banach spaces unless specified otherwise. For the definitions and basic properties of p-summing and p-nuclear operators, the reader is referred to [5]. In this note we use only the result of Pietsch–Pełczyński (cf. [5]) that for each p, $1 \leqslant p < \infty$, $\pi_p(I_n) \geqslant C_p n^{1/2}$, where $\pi_p(I_n)$ is the p-summing norm of the identity operator on n-dimensional Euclidean space and C_p is a positive constant which depends only on p.

Super-reflexive Banach spaces were introduced by James [8], and Enflo [4] has shown that a space is super-reflexive if and only if it can be renormed to be uniformly convex. However, all we use here is the Gurarii–James theorem [7], [9]: If X is super-reflexive, then there are positive constants $A, B, 1 < p \leqslant 2 \leqslant q < \infty$ such that, if (x_i) is basic in X with constant $\leqslant 2$, $1 \leqslant \|x_i\| \leqslant 4$, then $A\left(\sum |a_i|^q\right)^{1/q} \leqslant \left\|\sum a_i x_i\right\| \leqslant B\left(\sum |a_i|^p\right)^{1/p}$ for all scalars $a_1, \ldots, a_n$. (A sequence (x_i) (possibly finite) is basic with basis constant $\leqslant K$ if for arbitrary $j < m$ and scalars $(t_i)_{i=1}^m$, $\left\|\sum_{i=1}^{j} t_i x_i\right\| \leqslant K \left\|\sum_{i=1}^{m} t_i x_i\right\|$.) Such inequalities and the Dvoretzky theorem [4], [13] allow us to construct rank n projections from super-reflexive spaces onto nearly Euclidean subspaces which have norm $< Cn^{1/2p}$ for arbitrarily large n. The existence of such projections allows us to show the non-equivalence of the uniform and p-summing norms of finite rank operators from X to Y when X is super-reflexive.

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LEMMA. *For any $\varepsilon>0$, $\delta>0$, and integer n, there is an integer $k=k(\varepsilon,\delta,n)$ such that if $(x_i)_{i=1}^k$ satisfy $\|x_i\|\leqslant 2$, $\|x_i-x_j\|\geqslant\delta$ for all $i\neq j$, then there are distinct indices $(n_i)_{i=1}^{2n}$ such that $(x_{n_{2i-1}}-x_{n_{2i}})_{i=1}^n$ has basis constant at most $1+\varepsilon$.*

Proof. We need the following elementary facts: If $a>0$ and p is an integer, then there is an integer $m=m(p,a)$ such that if $\dim U\leqslant p$, U a subspace of some Banach space X, then there exist norm one functionals $(f_i)_{i=1}^m$ in X^* such that $x\in U$ implies $\|u\|\leqslant(1+a)\max|f_i(u)|$. Also, for any $a>0$ and integer m, there exists an integer $s=s(m,a)$ and a $2^{-1}a$ net for the 2 ball of l_∞^m whose cardinality is at most s.

Let $a>0$ (smallness of $a=a(\varepsilon,\delta,n)$ to be specified later) and set $k=s(n-1,a)+2n$. Let $n_1=1$ and $n_2=2$ (so that $(x_{n_1}-x_{n_2})$ has basis constant 1).

Suppose we have chosen $n_1,\ldots,n_{2p}$ so that for $1\leqslant q\leqslant p$, $\|\sum_{i=1}^q a_i(x_{n_{2i-1}}-x_{n_{2i}})\|\leqslant(1+\varepsilon)^{(p-1)/n}\|\sum_{i=1}^p a_i(x_{n_{2i-1}}-x_{n_{2i}})\|$ for arbitrary scalars $(a_1,\ldots,a_p)$. For $m=m(p,a)$, pick norm one functionals $(f_i)_{i=1}^m$ so that for each $u\in\operatorname{span}(x_{n_{2i-1}}-x_{n_{2i}})_{i=1}^p$, $\|u\|\leqslant(1+a)\max\limits_{1\leqslant i\leqslant m}|f_i(u)|$. Define $H\colon X\to l_\infty^m$ by $H(x)=\big(f_1(x),f_2(x),\ldots,f_m(x)\big)$. Now $k-2p>s(m,a)$, so there must be distinct indices n_{2p+1} and n_{2p+2}, different from those indices already chosen, which satisfy $\|Hx_{n_{2p+1}}-Hx_{n_{2p+2}}\|<a$. Thus, for arbitrary $u\in\operatorname{span}(x_{n_{2i-1}}-x_{n_{2i}})_{i=1}^p$ and scalar b, there exists i, $1\leqslant i\leqslant m$, such that $\|u+b(x_{n_{2p+1}}-x_{n_{2p+2}})\|\geqslant|f_i(u+b(x_{n_{2p+1}}-x_{n_{2p+2}}))|\geqslant(1+a)^{-1}\|u\|-|b|a\geqslant(1+a)^{-1}\|u\|-a\delta^{-1}\|b(x_{n_{2p+1}}-x_{n_{2p+2}})\|\geqslant(1+a)^{-1}\|u\|-a\delta^{-1}\|u\|-a\delta^{-1}\|u+b(x_{n_{2p+1}}-x_{n_{2p+2}})\|$. Thus if $a=a(\varepsilon,\delta,n)$ is chosen small enough, $\|u\|\leqslant(1+\varepsilon)^{1/n}\|u+b(x_{n_{2p+1}}-x_{n_{2p+2}})\|$.

Now for $1\leqslant q\leqslant p$ and scalars $a_1,\ldots,a_{p+1}$ we have

$$\Big\|\sum_{i=1}^q a_i(x_{n_{2i-1}}-x_{n_{2i}})\Big\|\leqslant(1+\varepsilon)^{(p-1)/n}\Big\|\sum_{i=1}^p a_i(x_{n_{2i-1}}-x_{n_{2i}})\Big\|$$
$$\leqslant(1+\varepsilon)^{(p-1)/n}(1+\varepsilon)^{1/n}\Big\|\sum_{i=1}^{p+1} a_i(x_{n_{2i-1}}-x_{n_{2i}})\Big\|.$$

This completes the inductive construction and hence the proof.

We are now in a position to produce nicely complemented l_2^n's in super-reflexive spaces. Recall that spaces X and Y are λ isometric if there exists a linear map $T\colon X\overset{\text{onto}}{\to}Y$ satisfying $\|T\|\cdot\|T^{-1}\|\leqslant\lambda$. Normalized basic sequences (x_n) and (y_n) are called λ equivalent provided the linear extension of the map $x_n\to y_n$ is a λ isometry from span (x_n) onto span (y_n).

PROPOSITION. *If X is super-reflexive, there exists $t>2$ such that for any $\varepsilon_n>0$ there are a sequence of subspaces (U_n) of X with $\dim U_n=n$, and projections $P_n\colon X\overset{\text{onto}}{\to}U_n$ such that U_n is $(1+\varepsilon_n)$ isometric to l_2^n and $\|P_n\|=O(n^{1/t})$.*

Proof. Fix n. Let $1>a>0$ and let $k=k(a,1,n)$ from the lemma. By Dvoretzky's theorem ([4], [13]) there is a subspace V of X which has a basis $(y_i)_{i=1}^k$ of unit vectors such that $(y_i)_{i=1}^k$ is $1+a$ equivalent to an orthonormal basis in l_2^k. Let $y_i^*\in V^*$ such that $y_i^*(y_j)=\delta_{ij}$, and let $g_i\in X^*$ be an arbitrary Hahn–Banach extension of y_i^* to all of X. Then $\|g_i\|\leqslant 1+a$ and for $i\neq j$, $\|g_i-g_j\|\geqslant(g_i-g_j)y_i=1$. By the lemma, choose $n_1,\ldots,n_{2n}$ so that $(h_i=g_{n_{2i-1}}-g_{n_{2n}})_{i=1}^n$ is basic with constant $1+a$. James [10] has shown that, since X^* is super-reflexive, there exists $r>1$ and a constant K so that if (u_i) is any basic sequence in X^* with basis constant $\leqslant 2$ and $\|u_i\|\leqslant 4$, then for all sequences (a_i) of scalars, $\|\sum a_iu_i\|\leqslant K(\sum|a_i|^r)^{1/r}$. Consider the projection $Px=\sum_{i=1}^n h_i(x)y_{n_{2i-1}}$. We have for x in X

$$\|Px\|^2\leqslant(1+a)\sum|h_i(x)|^2=(1+a)\Big(\sum\big(\operatorname{sgn}h_i(x)\big)h_i(x)h_i\Big)x$$
$$\leqslant(1+a)K\Big(\sum|h_i(x)|^r\Big)^{1/r}\|x\|$$
$$\leqslant(1+a)K\Big(\sum\|h_i\|^r\Big)^{1/r}\|x\|^2$$
$$\leqslant 2(1+a)^2K\|x\|^2\cdot n^{1/r}.$$

Thus, with $t=2r$, $\|P\|\leqslant\sqrt{2K}(1+a)n^{1/t}$ as desired.

We are now ready to prove the main result of this paper.

THEOREM. *If X is super-reflexive and Y is arbitrary, there is a compact operator from X to Y which fails to be p-absolutely summing for all p.*

Proof. We show that the p-summing and uniform norms fail to agree on the finite rank operators. By the proposition, let U have dimension n, be $3/2$ isometric to l_2^n and let P be a projection of X onto U having $\|P\|\leqslant K_Xn^{1/t}$. Let T_1 be an isomorphism of l_2^n onto U such that $\|T_1\|=1$, $\|T_1^{-1}\|\leqslant 3/2$. Let $V\subseteq Y$ be $3/2$ isometric to l_2^n and let $T_2\colon V\to l_2^n$ such that $\|T_2\|=1$, $\|T_2^{-1}\|\leqslant 3/2$. Then if I denotes the identity on l_2^n and if J is the embedding of V into Y, consider the map $S=JT_2^{-1}IT_1^{-1}P$ of X into Y. The restriction of this map to U is $JT_2^{-1}IT_1^{-1}$ and for any p between 1 and ∞, $\pi_p(S)\geqslant\pi_p(JT_2^{-1}IT_1^{-1})$. With $R=T_2^{-1}IT_1^{-1}$, $I=T_2RT_1$ and there exists C_p such that $C_p\sqrt{n}\leqslant\pi_p(I)\leqslant\|T_1\|\|T_2\|\pi_p(R)$, [3]. It follows that $\pi_p(S)\geqslant C_p\sqrt{n}$. However, $\|S\|\leqslant\|J\|\|I\|\|T_1^{-1}\|\|T_2^{-1}\|\|P\|\leqslant(\frac{3}{2})^2\|P\|\leqslant Kn^{1/t}$, where $t>2$. Thus $\pi_p(S)/\|S\|\geqslant A_pn^{1/2-1/t}$, where A_p depends only on X and p, and not on n. The p-absolutely summing norm is thus not equivalent to the operator norm on the finite rank operators, so by standard arguments, the compact operators which are p-absolutely summing for some p are category 1 in the compact operators from X to Y.

By a simple duality argument, we obtain the following

COROLLARY 1. *If Y is arbitrary and X is super-reflexive, there is a compact operator from Y to X whose adjoint fails to be p-summing for all p.*

Remark. The operator itself may, however, be 1-summing since every operator from a L_1 space to a Hilbert space is 1-summing.

COROLLARY 2. *If either X or Y is super-reflexive, there is a compact, non-nuclear operator from X to Y.*

Added in proof. We prove the following improvement on the Proposition of our paper ($d(X, Y)$ denotes $\inf\{\|T\|\cdot\|T^{-1}\|: T$ is an isomorphism from X onto $Y\}$. We follow exactly the notation of the paper).

PROPOSITION A. *The following are equivalent conditions on X.*

(1) *X does not contain l_1^n uniformly for large n (i.e., $d(E_n, l_1^n)\to\infty$ for any sequence (E_n) of subspaces of X).*

(2) *There are $t>2$ and functions $N=N(n)$ and $K=K(\lambda)$ so that if E is a subspace of X with $d(E, l_2^N)\leqslant\lambda$, then E has an n dimensional subspace which is $Kn^{1/t}$ complemented in X.*

From Proposition A we deduce (cf. proof of the Theorem of the paper):

THEOREM B. *Suppose X does not contain l_1^n uniformly for large n and Y is arbitrary. Then there is a compact operator T from X to Y (respectively, from Y to X) so that T (respectively, T^*) is not p-summing for any p.*

In particular, the T of Theorem B is not nuclear.

Given an unconditional basis (x_i), the U-constant of (x_i) is the smallest constant K for which $\|\sum_{i\in A} a_i x_i\|\leqslant K\|\sum_{i\in B} a_i x_i\|$ whenever $A\subseteq B$ and (a_i) are scalars. Brunel and Sucheston ([1], Proposition 1 and [2], Lemma 2.3) proved that if (x_n) is a bounded sequence in a Banach space and $\|x_i-x_j\|\geqslant\delta>0$ for all $i\neq j$, then for each $\varepsilon>0$ and integer n, there exists a subsequence (y_i) of (x_i) so that every length n subsequence of $(y_{2i-1}-y_{2i})$ has U-constant less than $1+\varepsilon$. This result and a standard limiting argument yields that the conclusion of the Lemma in our paper can be improved to "$(x_{n_{2i-1}}-x_{n_{2i}})_{i=1}^n$ has U-constant at most $1+\varepsilon$".

Suppose that X does not contain l_1^n uniformly. Then neither does X^*, so by a result of Pisier [13] there is $r>1$ so that $\inf\|\sum\varepsilon_i a_i u_i\|\leqslant 2\left(\sum|a_i|^r\right)^{1/r}\sup\|u_i\|$, where the inf is over all choices $\varepsilon_i=\pm1$ of signs. Thus if (u_i) has U constant less than 2, then $\|\sum a_i u_i\|\leqslant 4\left(\sum|a_i|^r\right)^{1/r}\sup\|u_i\|$.

The proof of (1)$\Rightarrow$(2) in Proposition A is identical to the proof of the Proposition of our paper, except that we use the improved version of the Lemma to guarantee that $(h_i)_{i=1}^k$ has U-constant less than $1+\alpha$ and use Pisier's result in place of the Guararii–James theorem. Of course, implication (2)$\Rightarrow$(1) in Proposition A is a well-known consequence of Grothendieck's inequality (cf., e. g., [11], p. 298).

References

[1] A. Brunel and L. Sucheston, *On B-convex Banach spaces*, Math. Systems Theory 7 (1973).

[2] — *On J-convexity and some ergodic super-properties of Banach spaces* (to appear).

[3] A. Dvoretzky, *Some results on convex bodies and Banach spaces*, Proc. Int. Symp. Lin. Spaces Jerusalem (1961), pp. 123–160.

[4] P. Enflo, *Spaces which can be given an equivalent uniformly convex norm*, Israel J. Math. 13 (1972), pp. 281–288.

[5] D. J. H. Garling and Y. Gordon, *Relations between some constants associated with finite dimensional Banach spaces*, Israel J. Math. 9 (1971), pp. 346–361.

[6] A. Grothendieck, *Produits tensoriels topologiques et espaces nucleaires*, Mem. Amer. Math. Soc. 16 (1955).

[7] V. I. Gurarii and N. I. Gurarii, *On bases in uniformly convex and uniformly smooth Banach spaces*, Izv. Akad. Nauk SSSR, Ser. Mat. 35 (35 (1971), pp. 216–223 [Russian].

[8] R. C. James, *Super-reflexive spaces*, Canad. J. Math. (to appear).

[9] — *Super-reflexive spaces with bases*, Pacific J. Math. 41 (1972), pp. 409–419.

[10] M. I. Kadec and M. G. Snobar, *Certain functionals on the Minkowski compactum*, Mat. Zametki 10 (1971), pp. 453–457 [Russian].

[11] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications*, Studia Math. 29 (1968), pp. 275–326.

[12] V. D. Milman, *New proof of the theorem of A. Dvoretzky on intersections of convex bodies*, Funkcional. Anal. i Priložen. 5 (1971), pp. 28–37 [Russian].

[13] G. Pisier, *Sur les espaces de Banach qui ne contiennent pas uniformément de l_n^1* (to appear).

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