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A multiplier theorem for ultraspherical series

by

WILLIAM C. CONNETT* and ALAN L. SCHWARTZ* (St. Louis, Mo.)

Abstract. Multiplier operators on ultraspherical expansions of $f \in L^p(d\mu_\lambda)$ with $\lambda > 0$, $1 \leqslant p < \infty$, are studied by realizing these operators as a kernel of singular integral type. It then follows from the Calderón–Zygmund theory that such operators must be of strong type p-p for $1 < p < \infty$ and weak type 1-1.

1. Introduction. Let $\lambda > 0$ be fixed and define L_λ^p to be the collection of all f for which

$$\Big[\int_{-1}^{+1} |f|^p \, dm_\lambda\Big]^{1/p} < \infty,$$

where

$$dm_\lambda(x) = (1-x^2)^{\lambda-\frac{1}{2}}.$$

For $f \in L_\lambda^1$, define the ultraspherical series for f to be

$$f(x) \sim \sum_{n=0}^{\infty} c_n h_n^\lambda R_n^\lambda(x),$$

where

$$R_n^\lambda(x) = P_n^\lambda(x)/P_n^\lambda(1), \quad c_n = \int_{-1}^{+1} f(x) R_n^\lambda(x) \, dm_\lambda,$$

and

$$(h_n^\lambda)^{-1} = \int_{-1}^{+1} [R_n^\lambda(x)]^2 \, dm_\lambda(x),$$

where

$$\sum_{n=0}^{\infty} z^n P_n^\lambda(x) = (1-2xz+z^2)^{-\lambda}.$$

If the transformation M is defined by

$$Mf(x) \sim \sum_{n=0}^{\infty} m_n c_n h_n R_n^\lambda(x),$$

then M is called a *multiplier*.

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The purpose of this paper is to show how some abstract arguments from the theory of homogeneous spaces can be used to realize multiplier operators for ultraspherical polynomials as convolution operators with kernels of Calderón–Zygmund type. This realization allows us to conclude that M is of strong type p-p for $1 < p < \infty$ and weak type 1-1. (M is *of strong type* p-p if for any $f \in L^p$ $\|Mf\|_p \leqslant C\|f\|_p$; M is *of weak type* 1-1 if for any $f \in L^1_\lambda$, $|\{x \mid Mf(x) > \lambda\}| < C\lambda^{-1}\|f\|_1$.) There are other utilizations of this realization to which we will return at another time.

This technique was first utilized by Coifman and Weiss in their study of analysis on higher dimensional spheres [3]. Since $\Sigma_k \cong SO(k+1)/SO(k)$ a natural convolution structure for Σ_k can be derived from the group structure in $SO(k+1)$. Although we will prove some new results in multiplier theory, our major contribution is to show how this realization of M can be obtained when there is no group structure on which to base the convolution.

The multiplier theorems of Coifman and Weiss only apply to the case when 2λ is an integer. The same is true for the results of Strichartz [11]. Results for all λ but only some values of p, $1 < p < \infty$, were obtained by Muckenhoupt and Stein [9] and Bonami [1]. Another approach to this question is from approximation theory, where results for all $\lambda > 0$, all $p > 1$ were obtained by Butzer, Nessel, and Trebels [2]. However, the conditions they impose on the sequence $\{m_n\}$ are much more stringent than ours.

If $[\lambda]$ is the greatest integer function and $\langle\lambda\rangle = \lambda - [\lambda]$, then for all $p \geqslant 1$ and all $\lambda > 0$ we have the following

THEOREM. *If* $m = [\lambda+\frac{1}{2}]+1$ *and*

a) $m_n = O(1)$,

b) $\sum_{n=2^N}^{2^{N+1}} |\Delta^m m_n|^2 h_n = O(2^{-2N\gamma})$, $\gamma = 1-\langle\lambda+\frac{1}{2}\rangle$,

then M *is of strong type* p-p *for* $1 < p < \infty$ *and weak type* 1-1.

Remark 1. An interesting special case is the sequence $\{m_n\} = \{n^{ia}\}$ with $a > 0$.

See for example Muckenhoupt [8].

Remark 2. The above theorem is stated in its simplest form. A slightly better — but notationally more complicated — theorem can be proved using "fractional" differences in the style of Strichartz [11].

Remark 3. A simpler but equivalent statement of b) (based on the fact that $h_n \simeq C_\lambda n^{2\lambda}$ for some $C_\lambda > 0$ *vide infra* (2.4)) is that for any positive integer M

$$\sum_{n=M}^{2M} |\Delta^m m_n|^2 \leqslant KM^{-(1+2[\lambda+\frac{1}{2}])}.$$

The proof is based on ideas of Coifman and Weiss [4]. In order to use these it is necessary to have a convolution structure, $*$, for L^1_λ and an approximate identity φ_r such that

$$\lim_{r\to 0} \varphi_r * f = f \quad (f \in L^1_\lambda).$$

This identity must also satisfy certain smoothness conditions.

Fortunately, both of these exist. The convolution structure is described in § 2. The approximate identity is based on the Poisson kernel for ultraspherical polynomials, and its properties are discussed in § 3.

The approximate identity is used to "mollify" the multiplier transformation in the following sense. If $f(x) \in L^p_\lambda$, and by abuse of notation we write $\varphi_i(x) = \varphi_{2^{-i}}(x)$, then

$$f = \lim_{i\to\infty} f * \varphi_i$$

and also

$$f = \lim_{i\to\infty} f * \varphi_i * \varphi_i$$

so that

$$f = \lim_{n\to\infty} f*\varphi_n*\varphi_n = \lim_{n\to\infty} \Big\{\sum_{i=1}^{n-1} \varphi_{i+1}*\varphi_{i+1} - \varphi_i*\varphi_i)*f + \varphi_0*\varphi_0*f\Big\}.$$

We may assume that $\varphi_0 = 1$ and $\int f = 0$, so the last term dissappears, and by simple rearrangement we obtain

$$f = \sum_{i=1}^{\infty} (\varphi_{i+1}-\varphi_i)*(\varphi_{i+1}+\varphi_i)*f$$
$$= \sum_{i=1}^{\infty} \psi_i*(\varphi_{i+1}+\varphi_i)*f,$$

where $\psi_i = \varphi_{i+1}-\varphi_i$. And, speaking formally, we obtain

$$Mf = \lim_{n\to\infty} \sum_{i=0}^{n} M\psi_i*(\varphi_{i+1}+\varphi_i)*f$$
$$= \lim k_n * f.$$

We claim that under suitable conditions on M, k_n is an operator of strong type p-p for $1 < p < \infty$ and weak type 1-1, with the constants independent of the n.

Our method is to show that each k_n is a kernel of Calderón–Zygmund type since it satisfies a Hörmander condition (see Hörmander [7]). In § 5 the results of these computations are used to prove that the k_n satisfy the Hörmander condition uniformly in the n.

2. Ultraspherical series and convolution structures. The ultraspherical or Gegenbauer polynomials can also be described by

$$P_0^\lambda(x) = 1, \quad P_1^\lambda(x) = 2\lambda x,$$

and

$$(2.1)\quad nP_n^\lambda(x) = 2(n+\lambda-1)xP_{n-1}^\lambda - (n+2\lambda-2)P_{n-2}^\lambda(x), \quad n = 2, 3, 4, \ldots$$

In particular, for $\lambda = 1/2$ we obtain the Legendre polynomials $P_n = P_n^{1/2}$. A complete treatment of the subject can be found in Szegö's book [12].

We will have particular need for the following facts:

$$(2.2)\quad P_n^\lambda(1) = \Gamma(n+2\lambda)/\Gamma(2\lambda)\Gamma(n+1),$$

and if we define $dm_\lambda(x) = (1-x^2)^{\lambda-1/2}dx$,

$$(2.3)\quad \int_{-1}^{1} P_m^\lambda(x)P_n^\lambda(x)\,dm_\lambda(x) = \delta_{mn}\frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{\Gamma^2(\lambda)(n+\lambda)\Gamma(n+1)}.$$

We shall be interested in expansions in terms of the normalized polynomials

$$R_n^\lambda(x) = P_n^\lambda(x)/P_n^\lambda(1)$$

since $P_n(1) = 1$, $R_n^{1/2}(x) = P_n^{1/2}(x) = P_n(x)$.
By (2.2), (2.3), and Stirling's formula, we obtain

$$(2.4)\quad (h_n^\lambda)^{-1} = \int_{-1}^{1} (R_n^\lambda(x))^2\,dm_\lambda(x) = \frac{2^{1-2\lambda}\pi\Gamma^2(2\lambda)}{\Gamma^2(\lambda)}\,\frac{\Gamma(n+1)}{(n+\lambda)\Gamma(n+2\lambda)}$$

$$\simeq \frac{2^{1-2\lambda}\pi\Gamma^2(2\lambda)}{\Gamma^2(\lambda)}n^{-2\lambda} = C_\lambda n^{-2\lambda};$$

in particular $h_n^{1/2} = n+1/2$.

We will leave off the index λ whenever there seems to be no danger of confusion.

If

$$(2.5)\quad f(x) \sim \sum c_n h_n R_n^\lambda(x),$$

$$(2.6)\quad g(x) \sim \sum b_n h_n R_n^\lambda(x).$$

Then under appropriate hypotheses on f and g

$$(2.7)\quad \int f(x)g(x)\,dm_\lambda(x) = \sum b_n c_n h_n;$$

in particular if $f \in L^2$

$$(2.8)\quad \int_{-1}^{1} |f(x)|^2\,dm_\lambda(x) = \sum_{n=0}^{\infty} c_n^2 h_n.$$

In the special case $\lambda = 1/2$ we have the Legendre coefficient defined by

$$c_n = \int_{-1}^{1} f(x)P_n(x)\,dx$$

and the Legendre series of f is given by

$$(2.9)\quad f(x) \sim \sum_{n=0}^{\infty} c_n(n+\tfrac{1}{2})P_n(x).$$

We now describe the convolution structure for the ultraspherical series. For f and $g \in L^1$ we wish to define a function $f*g \in L^1$ such that

$$(2.10)\quad \|f*g\|_1 \leqslant \|f\|_1\|g\|_1$$

and

$$(2.11)\quad f*g(x) \sim \sum_{n=0}^{\infty} b_n c_n h_n R_n^\lambda(x)$$

if (2.5) and (2.6) hold. Proceeding formally, we replace b_n and c_n in (2.11) by their defining integrals and we arrive at

$$(2.12)\quad f*g(x) \sim \int_{-1}^{1}\int_{-1}^{1} f(y)g(z)D_\lambda(x,y,z)\,dm_\lambda(y)\,dm_\lambda(z),$$

where, at least formally,

$$(2.13)\quad D_\lambda(x,y,z) = \sum_{n=0}^{\infty} R_n^\lambda(x)R_n^\lambda(y)R_n^\lambda(z)h_n.$$

In fact the sum in (2.13) is known to converge to the value

$$D_\lambda(x,y,z) = \frac{2^{1-2\lambda}(1-x^2-y^2-z^2+2xyz)^{\lambda-1}}{\Gamma^2(\lambda)[(1-x^2)(1-y^2)(1-z^2)]^{\lambda-1/2}}$$

if $-1 \leqslant x, y, z \leqslant 1$ and $1-x^2-y^2-z^2+2xyz > 0$, and to zero otherwise (see Hirschman [5], [6]). D_λ has the following properties for $-1 \leqslant x, y, z \leqslant 1$:

$$(2.14)\quad D_\lambda(x,y,z) \geqslant 0,$$

$$(2.15)\quad \int_{-1}^{1} D_\lambda(x,y,z)R_n^\lambda(x)\,dm_\lambda(x) = R_n^\lambda(y)R_n^\lambda(z).$$

Setting $n = 0$ in (2.15) we obtain

$$(2.16)\quad \int_{-1}^{1} D_\lambda(x,y,z)\,dm_\lambda(x) = 1.$$

We can now define $f*g(x)$ by setting it equal to the right-hand side of (2.12), then (2.10) and (2.11) are easy consequences of (2.14), (2.15), and (2.16). The convolution can be extended in a natural way to give a Banach algebraic structure for the measures on $[-1,1]$. We will not be concerned with this except in one special case: if δ_x is the unit point mass concentrated on x, we define

$$f*\delta_x(y) = \int_{-1}^{1} f(z) D_\lambda(x,y,z)\,dm_\lambda(z). \tag{2.17}$$

(The reader can find details of this extension in an analogous situation with Bessel functions instead of ultraspherical polynomials in Schwartz's paper [10].)

We will often use the change of variables $x = \cos\theta$, $y=\cos\varphi$, $z=\cos\psi$ then we define

$$d\mu_\lambda(\theta) = dm_\lambda(\cos\theta) = \sin^{2\lambda}(\theta)\,d\theta \qquad (|x|\leqslant 1,\ 0\leqslant\theta\leqslant\pi),$$

and

$$\Phi_\lambda(\theta,\varphi,\psi) = D_\lambda(x,y,z),$$

then

$$\Phi_\lambda(\theta,\varphi,\psi) = 0 \quad \text{unless } |\varphi-\psi|<\theta<\varphi+\psi. \tag{2.18}$$

When there is no danger of ambiguity, we will delete the index λ. $\int\ldots d\mu$ and $\int\ldots dm$ denote integration over the intervals $[0,\pi]$ and $[-1,1]$ respectively. Σ denotes summation as all repeated indices range over $N=\{0,1,2,\ldots\}$. C denotes a constant which need not be the same at each occurrence.

3. The Poisson kernel. In this section the Poisson kernel is defined, and shown to be compatible with the convolution structure of Section 2. The main result is Proposition (3.1) which shows that the kernel satisfies: a) an integral Lipschitz condition; and b) has a good bound on its growth near the origin. The following is a very natural *definition*. The Poisson kernel for the ultraspherical polynomials of index λ is

$$P_s(\theta,\varphi) = \sum s^n h_n R_n^\lambda(\cos\theta) R_n^\lambda(\cos\varphi). \tag{3.1}$$

We have adopted the following conventions

$$r = 1-s,\qquad \varphi_r(\theta,\varphi) = P_{1-s}(\theta,\varphi),\qquad \varphi_r(\theta)=\varphi_r(\theta,0).$$

From the definition of convolution, it follows that

$$\varphi_r(\theta,\varphi) = \int \varphi_r(\psi)\Phi(\theta,\varphi,\psi)\,d\mu(\psi) \tag{3.2}$$

since both sides have the same ultraspherical series (cf. (2.15)).

The following properties of the kernel are well known (see for example Muckenhoupt and Stein [8], p. 27). Let

$$E = E(\theta,\varphi,s,t) = 1-2s(\cos\theta\cos\varphi+\sin\theta\sin\varphi\cos t)+s^2,$$

$$P_s(\theta,\varphi) = \frac{\lambda}{\pi}(1-s^2)\int E^{-\lambda-1}(\sin t)^{-1}\,d\mu(t), \tag{3.3}$$

$$\varphi_r(\theta,\varphi) \leqslant Cr\,\frac{\sin^{-\lambda}\theta\,\sin^{-\lambda}\varphi}{r^2+(\theta-\varphi)^2},\qquad 0<r\leqslant 1/2, \tag{3.4}$$

$$\varphi_r(\theta,\varphi)\leqslant C\,\frac{r}{[r^2+(\theta-\varphi)^2]^{\lambda+1}},\qquad 0<r\leqslant 1/2. \tag{3.5}$$

From (3.1) and the orthogonality of the ultraspherical polynomials it follows that

$$\int \varphi_r(\theta,\varphi)\,d\mu(\theta) = 1. \tag{3.6}$$

A simple computation gives

$$\begin{aligned} E(\theta,\varphi,s,t) &= [1-2s\cos(\theta-\varphi)+s^2]+2s\sin\theta\sin\varphi(1-\cos t)\\ &\geqslant C[(\theta-\varphi)^2+r^2]+4s\sin\theta\sin\varphi\sin^2 t/2\\ &\geqslant Cr|\theta-\varphi| \end{aligned} \tag{3.7}$$

which is non-negative, so

$$\varphi_r(\theta,\varphi)\geqslant 0. \tag{3.8}$$

Finally, if $|\theta-\varphi|\geqslant\eta>0$, $\varphi_r(\theta,\varphi)\leqslant r\eta^{-2(\lambda+1)}$ by (3.5). This, together with (3.6) and (3.8) implies that φ_r is an approximate identity as $r\to 0$.

Proposition 3.1. *There are constants C_1 and C_a such that*

a) $\displaystyle\int|\varphi_r(\theta,\varphi)-\varphi_r(\theta,\varphi_0)|\,d\mu(\theta)\leqslant C_1\frac{|\varphi-\varphi_0|}{r}$,

b) $\displaystyle\int\varphi_r(\theta,\varphi)\left(\frac{|\theta-\varphi|}{r}\right)^a d\mu(\theta)\leqslant C_a,\quad 0<a<1,$

where C_a depends only on the a.

The proof of this proposition is long and technical. The first step is to find a bound on the derivative of $P_s(\theta,\varphi)$.

Lemma 3.1.

$$\left|\frac{\partial}{\partial\varphi}P_s(\theta,\varphi)\right|\leqslant c\left[\frac{1}{r}+M(\varphi,\theta)\right]P_s(\theta,\varphi), \tag{3.9}$$

where

$$M(\varphi,\theta)\leqslant 1/\sin\varphi$$

and

$$M(\varphi,\theta)\leqslant c/r \quad \textit{if } \varphi\leqslant\theta/2.$$

Proof.

$$\left|\frac{\partial}{\partial\varphi}P_s(\theta,\varphi)\right| \leqslant \pi^{-1}\lambda(1-s^2)(\lambda+1)\int\frac{|\partial E/\partial\varphi|}{E}\,\frac{d\mu(t)}{E^{\lambda+1}\sin t}$$

$$\leqslant \pi^{-1}\lambda(1-s^2)\int\frac{C\sin|\theta-\varphi|}{E}\,\frac{d\mu(t)}{E^{\lambda+1}\sin t}+$$

$$+\pi^{-1}\lambda(1-s^2)\int\frac{C\sin\theta|\cos\varphi|(1-\cos t)}{E}\,\frac{d\mu(t)}{E^{\lambda+1}\sin t}.$$

Now

$$\frac{\sin|\theta-\varphi|}{E(\theta,\varphi,s,t)}\leqslant C\frac{\sin|\theta-\varphi|}{r|\theta-\varphi|}\leqslant c/r,$$

and

$$\frac{\sin\theta|\cos\varphi|(1-\cos t)}{E}\leqslant\frac{\sin\theta|\cos\varphi|(1-\cos t)}{2s\sin\theta\sin\varphi(1-\cos t)}\leqslant\frac{c}{\sin\varphi}$$

by (3.7); hence the first inequality of the lemma follows. To obtain the second one, observe that if $\varphi\leqslant\theta/2$, then $\theta-\varphi\geqslant\theta/2$ so

$$\frac{\sin\theta|\cos\varphi|(1-\cos t)}{E}\leqslant C\frac{\sin\theta|\cos\varphi|(1-\cos t)}{r|\theta-\varphi|}\leqslant C\frac{\sin\theta}{r\theta/2}$$

and the second relation follows as well.

Proof of Proposition (3.1). Without loss of generality it may be assumed that

(3.10) $$\varphi_0<\varphi \quad\text{and}\quad \varphi-\varphi_0<r.$$

It will be sufficient to prove the proposition for $\varphi_0<\pi/2$, since the proof for the other values of φ_0 follows by a simple change of variable.

By the mean value theorem and (3.9)

$$\int|\varphi_r(\theta,\varphi)-\varphi_r(\theta,\varphi_0)|\,d\mu(\theta)$$

$$\leqslant C\frac{|\varphi-\varphi_0|}{r}\Big[\int\varphi_r(\theta,\xi_\theta)\,d\mu(\theta)+\int rM(\xi_\theta,\theta)\varphi_r(\theta,\xi_\theta)\,d\mu(\theta)\Big],$$

where for each θ, $\varphi_0<\xi_\theta<\varphi$. To prove a), it is sufficient to show that

(3.11) $$\int\varphi_r(\theta,\xi_\theta)\,d\mu(\theta)<C$$

and

(3.12) $$\int rM(\xi_\theta,\theta)\varphi_r d_\mu(\theta)<C.$$

The boundedness of (3.11) is shown by considering two cases.

Case 1. If $\varphi_0<2r$, the integral in (3.11) is broken into two parts, and an estimate from (3.5) is used in both parts,

$$\int_0^{4r}\varphi_r(\theta,\xi_\theta)\,d\mu(\theta)\leqslant\int_0^{4r}Cr^{-2\lambda-1}\theta^{2\lambda}\,d\theta\leqslant C,$$

$$\int_{4r}^{\pi}\varphi_r(\theta,\xi_\theta)\,d\mu(\theta)\leqslant\int_{4r}^{\pi}Cr\theta^{2\lambda}/(\theta-\xi_\theta)^{2\lambda+2}\,d\theta$$

$$\leqslant Cr\int_{4r}^{\pi}\left(\frac{\theta}{\theta-\xi_\theta}\right)^{2\lambda+2}\frac{d\theta}{\theta^2}\leqslant Cr4^{2\lambda+2}\int_{4r}^{\pi}\frac{d\theta}{\theta^2}\leqslant C.$$

The penultimate inequality follows from the observation that

$$\frac{\theta}{\theta-\xi_\theta}=1+\frac{\xi_\theta}{\theta-\xi_\theta}\leqslant1+\frac{3r}{r}=4$$

since $\xi_\theta<3r$ and $\theta>4r$.

Case 2. If $2r\leqslant\varphi_0\leqslant\pi/2$, then (3.11) is broken up into four parts.

$$\int_0^r+\int_r^{\varphi_0-r}+\int_{\varphi_0-r}^{\varphi_0+r}+\int_{\varphi_0+r}^{\pi}=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.$$

An argument similar to the one in case 1, gives

$$\mathrm{I}\leqslant\frac{C}{r^{2\lambda+1}}\int_0^r\theta^{2\lambda}\,d\theta=C.$$

An estimate derived from (3.4) yields

$$\mathrm{II}\leqslant\int_r^{\varphi_0-r}\frac{r}{(\theta-\xi_\theta)^2}\,\frac{\sin^{2\lambda}\theta\,d\theta}{\sin^\lambda\theta\sin^\lambda\xi_\theta}$$

$$\leqslant C\int_r^{\varphi_0-r}\frac{r}{(\theta-\xi_\theta)^2}\left(\frac{\sin\theta}{\sin\xi_\theta}\right)^{\lambda}d\theta\leqslant C\int_r^{\varphi_0-r}\frac{r}{(\theta-\xi_\theta)^2}\,d\theta\leqslant C.$$

Using (3.4) again we obtain

$$\mathrm{III}\leqslant C\int_{\varphi_0-r}^{\varphi_0+r}\frac{r}{r^2}\,\frac{\sin^{2\lambda}\theta}{\sin^\lambda\theta\sin^\lambda\xi_\theta}\,d\theta$$

$$=\frac{C}{r}\int_{\varphi_0-r}^{\varphi_0+r}\left(\frac{\sin\varphi}{\sin\xi_\theta}\right)^{\lambda}d\theta\leqslant\frac{C}{r}\left(\frac{\sin(\varphi_0+r)}{\sin\varphi_0}\right)^{\lambda}(2r)\leqslant C.$$

The estimates of II and III both use the fact that

$$\frac{\sin\theta}{\sin\xi_0} \leqslant C\frac{\sin(\varphi_0+r)}{\sin\varphi_0} = C\left[\cos r+\cos\varphi_0\frac{\sin r}{\sin\varphi_0}\right]\leqslant C.$$

In the final integral, $2E(\theta, \xi_0, s, t) \geqslant E(\theta, \varphi; s, t)$, from which it follows that

$$\mathrm{IV} \leqslant \int_{\varphi_0+r}^{\pi} C\varphi_r(\theta, \varphi)\,d\mu(\theta) \leqslant C.$$

We now turn to the integral in (3.12) which we treat in two cases. K will denote the value of the integral in (3.11).

Case 1. If $\varphi < 2r$, let $A = \{\theta\colon\ \xi_0 \leqslant \theta/2\}$ and $B = \{\theta\colon\ \xi_0 > \theta/2\}$. Then

$$\int rM(\xi_0, \theta)\varphi_r(\theta, \xi_0)\,d\mu(\theta) = \int_A + \int_B.$$

If $\theta \in A$, $rM(\xi_0, \theta)$ is bounded, hence $\int_A$ is comparable to K.

If $\theta \in B$, $rM(\xi_0, \theta) \leqslant r/\sin\xi_0 \leqslant cr/\sin\theta$, so we must estimate the integral

$$\int \frac{r}{\sin\theta}\varphi_r(\theta, \xi_0)\,d\mu(\theta) = \int_0^r + \int_r^{\pi-r} + \int_{\pi-r}^{\pi} = \mathrm{I}+\mathrm{II}+\mathrm{III}.$$

If $r \leqslant \theta \leqslant \pi-r$, $r/\sin\theta$ is bounded by a constant independent of r and θ, so II is comparable to K.

An estimate from (3.5) gives

$$\int_0^r + \int_{\pi-r}^{\pi} (r/\sin\theta)\varphi_r(\theta, \xi_0)\,d\mu(\theta) \leqslant 2r^{-2\lambda}\int_0^r \sin^{2\lambda-1}\theta\,d\theta \leqslant C \qquad (\lambda > 0).$$

Case 2. If $\varphi \geqslant 2r$, then $\xi_0 \geqslant r$, so $1/\sin\xi_0 \leqslant c/r$, and $rM(\xi_0, \theta)$ is bounded independent of r and θ and the integral (3.12) is once more comparable to K.

This concludes the proof of part (a) of the proposition. Part (b) is proven first for $\varphi = 0$,

$$\int_r^{\pi} \varphi_r(\theta)\left(\frac{\theta}{r}\right)^{\alpha} d\mu(\theta) \leqslant C\int_r^{\pi} r\theta^{-2\lambda-2}(\theta/r)^{\alpha}\theta^{2\lambda}\,d\theta \leqslant Cr^{1-\alpha}\int_r^{\pi}\theta^{\alpha-2}\,d\theta < C.$$

For general φ, use (3.2) to obtain

$$\int_{|\theta-\varphi|>r} \varphi_r(\theta, \varphi)\left(\frac{|\theta-\varphi|}{r}\right)^{\alpha} d\mu(\theta) = \iint_{|\theta-\varphi|>r} \varphi_r(\psi)\Phi(\theta, \varphi, \psi)\left(\frac{|\theta-\varphi|}{r}\right)^{\alpha} d\mu(\psi)\,d\mu(\theta).$$

By (2.18) the integrand is zero unless $|\theta-\varphi| < \psi$ so the above is bounded by

$$\int_{\psi>r} \varphi_r(\psi)\left(\frac{\psi}{r}\right)^{\alpha} d\mu(\psi) \leqslant C.$$

4. Difference computations. In this section we give two propositions, the proofs of which employ difference arguments of a highly technical nature. For a sequence $\{a_n\}$, we define

$$\Delta^0 a_n = a_n, \text{ and } \Delta^k a_n = \Delta^{k-1}a_{n+1} - \Delta^{k-1}a_n \text{ if } k = 1, 2, 3, \ldots$$

PROPOSITION 4.1. *Suppose $f \in L^2$, and $f \sim \sum c_n h_n R_n$; then there is a constant K depending only on λ and m such that*

$$\int (1-x)^m [f(x)]^2\,dm(x) \leqslant K\sum (\Delta^m c_n)^2 h_n. \tag{4.1}$$

We shall first prove (4.1) in the Legendre polynomial case $\lambda = 1/2$ with $m = 2$, since in general we will need (4.1) with $m = [\lambda+1/2]+1$; we will then extend the proof to general values of λ.

From the recurrence relation (2.1), we obtain

$$(2x-2)(n+1/2)P_n(x) = a_n(n+3/2)P_{n+1}(x) - 2(n+1/2)P_n(x) + b_n(n-1/2)P_{n-1}(x)$$

with

$$a_n = (n+1)/(n+3/2) \quad\text{and}\quad b_n = n/(n-1/2), \quad n = 0, 1, 2, \ldots$$

Thus

$$\begin{aligned}(2x-2)f(x) &\sim \sum (a_{n-1}c_{n-1} - 2c_n + b_{n+1}c_{n+1})(n+1/2)P_n(x)\\ &= \sum [\Delta^2 c_{n-1} + (2n+1)^{-1}(\Delta c_n + \Delta c_{n-1})](n+1/2)P_n(x).\end{aligned}$$

Application of (2.8) yields

$$\int (1-x)^2 f^2(x)\,dm(x) \leqslant C_1\sum (\Delta^2 c_n)^2(n+1/2) + C_2\sum\left((n+1)^{-1}\Delta c_n\right)^2(n+1/2).$$

The first term is of the form desired in (4.1). The second term is only a little different. We claim

$$\sum\left(\frac{\Delta c_n}{n+1}\right)^2 h_n \leqslant C\sum (\Delta^2 c_n)^2 h_n. \tag{4.2}$$

This inequality will complete the proof of Proposition (4.1) in the Legendre case. Inequality (4.2) is actually a special case of the following

LEMMA 4.1. *Suppose $\sum |x_n|^2 h_n^{\lambda} < \infty$, then for every positive integer k*

$$\sum |(n+1)^{-k}x_n|^2 h_n^{\lambda} \leqslant M\sum |(n+1)^{-k+1}\Delta x_n|^2 h_n^{\lambda} \tag{4.3}$$

for some constant M.

Relation (4.2) is simply (4.3) with $k = 1$, $\lambda = 1/2$, and $x_n = \Delta c_n$.

Proof. We make use of the classical partial summation formula:

$$\sum_{n=0}^{q} a_n b_n = \sum_{n=0}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q, \quad \text{where } A_n = \sum_{i=0}^{n} a_i.$$

Putting

$$a_n = (n+1)^{-2k} h_n,$$

so that

$$A_n \leqslant K(n+1)^{-2k+1} h_n,$$

and

$$b_n = x_n^2$$

we obtain

$$\sum_{n=0}^{q} |(n+1)^{-k} x_n|^2 h_n \leqslant K \sum_{n=0}^{q-1} [(n+1)^{-k}(|x_n|+|x_{n+1}|)] \times [(n+1)^{-k+1}|\Delta x_n|] h_n + $$
$$+K(q+1)^{-2k+1}|x_q|^2 h_q.$$

The lemma can then be obtained by first letting q increase without bound, so that the last term vanishes, and then applying the Schwarz inequality in the form

$$\Big|\sum x_n y_n h_n\Big| \leqslant \{|x_n|^2 h_n\}^{1/2}\{|y_n|^2 h_n\}^{1/2}.$$

We now proceed to a proof of the proposition in the general case. From the recurrence (2.1), we obtain

$$(2x-2)h_n R_n(x) = a_n h_{n+1} R_{n+1}(x) - 2h_n R_n(x) + b_n h_{n-1} R_{n-1}(x),$$

where

$$a_n = (n+1)/(n+\lambda+1), \quad n = 0, 1, 2, 3, \ldots,$$

$$b_0 = 0, \quad \text{and} \quad b_n = (n+2\lambda-1)/(n+\lambda-1), \quad n = 1, 2, 3, \ldots$$

Thus

(4.4) $$(2x-2)f(x) \sim \sum (a_{n-1}c_{n-1} - 2c_n + b_{n+1}c_{n+1}) h_n R_n(x),$$
$$= \sum [\Delta^2 c_{n-1} + \lambda(n+\lambda)^{-1}(\Delta c_n + \Delta c_{n-1})] h_n R_n(x).$$

An expression for the ultraspherical coefficients of $(2x-2)^k f(x)$ in terms of differences of the c_n's can be obtained by reiterating (4.4). The next lemma gives a compact form of the expression, but first we make the following definition. If p and q are polynomials, and $r = p/q$, then we write $\deg r = \deg p - \deg q$.

Lemma 4.2. *Suppose $f \sim \Sigma c_n h_n R_n$; let*

$$c_n^{(k)} = \int (2x-2)^k f(x) R_n(x)\, dm(x).$$

Then

(4.5) $$c_n^{(k)} = \sum_{j=1}^{2k} \sum_{i=0}^{k} Q_{ij}^k(n)\, \Delta^j c_{n-i},$$

where Q_{ij}^k is a rational function in n of degree $j-2k$.

The proof is by induction. The coefficients in (4.4) are $c_n^{(1)}$, so (4.5) holds with $k = 1$. Let us assume (4.5) holds for a certain value of k. Then by (4.4)

(4.6) $$c_n^{(k+1)} = \Delta^2 c_{n-1}^{(k)} + \frac{\lambda}{n+\lambda}[\Delta c_n^{(k)} + \Delta c_{n-1}^{(k)}].$$

If we now use (4.5) for every occurrence of $c_n^{(k)}$ and $c_{n-1}^{(k)}$ in (4.6) and make use of the fact(¹) that $\deg \Delta Q_{ij}^k = (\deg Q_{ij}^k) - 1$, (4.5) will follow with k replaced by $k+1$. The computation is facilitated by the use of a Leibniz type difference formula $\Delta^2(a_n b_n) = (\Delta^2 a_n) b_n + 2(\Delta a_{n+1})(\Delta b_n) + a_{n+2}\Delta^2 b_n$.

We now use lemma (4.2) to obtain an estimate of $\int (1-x)^m (f(x))^2 dm(x)$ in terms of differences of the coefficients of f. It will then be a short step to the proof of the proposition.

Lemma 4.3. *If $f \in L^2$, then*

(4.7) $$\int (1-x)^m (f(x))^2 dm(x) \leqslant K\Sigma$$

for some constant K independent of f, where

$$\Sigma = \sum_{n=0}^{\infty} \sum_{j=1}^{m} [(n+1)^{j-m} \Delta^j c_n]^2 h_n.$$

Proof. If $m = 2k$ is an even integer, this follows by lemma (4.2) and (2.8).

If $m = 2k+1$, we let

$$g(x) = f(x)(1-x)^k \sim \sum c_n^{(k)} h_n R_n,$$

so

$$g(x)(1-x) \sim -\tfrac{1}{2} \sum \{\Delta^2 c_{n-1}^{(k)} + \lambda(n+\lambda)^{-1}[\Delta c_n^{(k)} + \Delta c_{n-1}^{(k)}]\} h_n R_n$$

by (4.4). Then by (2.7)

(4.8) $$\int |f(x)|^2 (1-x)^{2k+1} dm(x) = \int g(x)[g(x)(1-x)]\, dm(x)$$
$$= -\tfrac{1}{2}\sum c_n^{(k)} \Delta^2 c_{n-1}^{(k)} h_n - \tfrac{1}{2}\sum c_n^{(k)} \lambda(n+\lambda)^{-1}(\Delta c_n^{(k)} + \Delta c_{n-1}^{(k)}) h_n.$$

An application of the formula

$$\sum_{n=M}^{N} a_n B_n = \sum_{n=M}^{N} A_n b_n,$$

where

$$A_n = \sum_{i=n}^{N} a_i \quad \text{and} \quad B_n = \sum_{i=M}^{n} b_i,$$

(¹) We always understand ΔQ_{ij}^k to be the sequence with terms $Q_{ij}^k(n+1) - Q_{ij}^k(n)$.

with $a_n = \Delta^2 c_{n-1}^{(k)}$ and $B_n = c_n^{(k)} h_n$ shows that the first sum of (4.8) is bounded by

$$\sum (\Delta c_{n-1}^{(k)})^2 h_{n-1} + \sum (\Delta c_{n-1}^{(k)}) c_n^{(k)} \Delta h_{n-1}. \tag{4.9}$$

Now $\Delta h_{n-1} \leqslant C(n+1)^{-1} h_n$ so the expression in (4.9) is bounded by a multiple of Σ by lemma (4.2). Similarly the second sum in (4.8) is also bounded by a multiple of Σ, and so (4.7) follows.

Proof of Proposition 4.1. By substituting $m-j$ for k and $\Delta^j c_n$ for x_n in Lemma 4.1 and reiterating the resulting relation $m-j$ times we conclude

$$\sum [(n+1)^{j-m} \Delta^j c_n]^2 h_n \leqslant C \sum [\Delta^m c_n]^2 h_n.$$

Thus Proposition 4.1 follows from Lemma 4.3.

The following proposition shows that the condition given on the m-th differences of $\{m_n\}$ implies certain conditions on the lower differences.

PROPOSITION 4.2. *Suppose $\{m_n\}$ is a bounded sequence such that*

$$\sum_{n=2^N}^{2^{N+1}} |\Delta^m m_n|^2 h_n = O(2^{-2N\gamma}), \tag{4.10}$$

where $m = [\lambda+\frac{1}{2}]+1$ and $\gamma = 1-\langle\lambda+\frac{1}{2}\rangle$, then

$$|\Delta^i m_n| = O(n^{-i}), \quad i = 0, 1, \ldots, m-1.$$

Proof. The proposition is trivial if $m = 1$, so without loss of generality, we assume $m > 1$. We will make repeated use of the following trivial inequality

$$\sum_{2^N}^{2^{N+1}} n^a \leqslant C_a 2^{N(a+1)} \tag{4.11}$$

which holds for all real a.

The idea of the proof is to find a polynomial P of degree at most $m-1$ which satisfies

$$|\Delta^i [m_n - P(n)]| = O(n^{-i}) \tag{4.12}$$

for $i = 1, 2, \ldots, m-1$. Then the boundedness of $\{m_n\}$ together with (4.12) for $i = 1$ imply that P is in fact constant so that (4.12) becomes equivalent to the conclusion of the lemma.

We proceed by first showing that $P = P_{m-1}$ can be chosen so that (4.12) holds for $i = m-1$, and secondly by showing that if for some $i_0 \geqslant 1$ there is a polynomial $P = P_{i_0}$ such that (4.12) holds for $i_0 \leqslant i \leqslant m-1$, then we can define a new $P = P_{i_0-1}$ such that (4.12) holds for $i_0-1 \leqslant i \leqslant m-1$.

Let $k < j$ and choose N such that $2^N \leqslant k < 2^{N+1}$; then

$$\begin{aligned} |\Delta^{m-1} m_j - \Delta^{m-1} m_k| &\leqslant \sum_{P=N}^{\infty} \sum_{n=2^P}^{2^{P+1}-1} |\Delta^m m_n| \sqrt{h_n} \sqrt{1/h_n} \\ &\leqslant \sum_{P=N}^{\infty} \Big[\sum_{n=2^P}^{2^{P+1}-1} (\Delta^m m_n)^2 h_n\Big]^{1/2} \times \Big[\sum_{n=2^P}^{2^{P+1}-1} 1/h_n\Big]^{1/2} \\ &\leqslant C \sum_{P=N}^{\infty} 2^{-P\gamma} \times 2^{P(-\lambda+1/2)} \leqslant Ck^{-(\gamma+\lambda-1/2)}, \end{aligned}$$

where we have used (4.10) to estimate the sum in the first pair of brackets, and (4.11) for that in the second pair.

We thus obtain

$$|\Delta^{m-1} m_j - \Delta^{m-1} m_k| \leqslant Ck^{-(m-1)}. \tag{4.13}$$

(4.13) implies that $\{\Delta^{m-1} m_k\}$ converges to some limit L_{m-1}. If we let $j \to \infty$ in (4.13), we obtain (4.12) with $P(k) = P_{m-1}(k) = L_{m-1} k^{m-1}/(m-1)!$, and $i = m-1$.

Now suppose (4.12) holds for some polynomial of the form $P(k) = P_{i_0}(k) = \sum_{i=i_0}^{m-1} L_i k^i / i!$ and $i = i_0, i_0+1, \ldots, m-1$.

An argument similar to the one above can be used to show

$$|\Delta^{i_0-1}[m_j - P_{i_0}(j)] - \Delta^{i_0-1}[m_k - P_{i_0}(k)]| \leqslant Ck^{-(i_0-1)}. \tag{4.14}$$

The argument is almost the same except we rely on (4.12) with $i = i_0$ and $P = P_{i_0}$ and (4.11), where we formerly used (4.10). Equation (4.14) implies that if $i_0 > 1 \{\Delta^{i_0-1}[m_k - P_{i_0}(k)]\}$ is a convergent sequence with limit, say, L_{i_0-1}. Thus, letting $j \to \infty$ in (4.14) we obtain

$$|\Delta^{i_0-1}[m_k - p_{i_0}(k)] - L_{i_0-1}| \leqslant Ck^{-(i_0-1)}.$$

Now define $P_{i_0-1}(k) = P_{i_0}(k) + L_{i_0-1} k^{i_0-1}/(i_0-1)!$, then (4.12) holds for $P = P_{i_0-1}$ and with $i_0-1 \leqslant i \leqslant m-1$.

Thus (4.12) holds for $P = P_1$ and $i = 1, 2, \ldots, m-1$. In particular $|\Delta[m_j - P(j)]| \leqslant K(j+1)^{-1}$, thus

$$\begin{aligned} |m_n - P(n) - m_0 + P(0)| &\leqslant \sum_{j=0}^{n-1} |\Delta[m_j - P(j)]| \\ &\leqslant \sum_{j=0}^{n-1} K(j+1)^{-1} \leqslant K \log n. \end{aligned}$$

Since $\{m_n\}$ is a bounded sequence, it follows that $P(n) = O(\log n)$ and is thus constant, whence $\Delta^i P = 0$ if $i = 1, \ldots, m-1$ and the proposition is proved.

5. The singular integral nature of the kernel. In this section, we show that the operator M can be given as the limit of a sequence of kernels which satisfy a version of Hörmander's condition with a uniform constant. We define φ_i, ψ_i and k_n as in the introduction, and introduce

$$a_r = M\psi_r * (\varphi_r + \varphi_{r/2}) = \big(M(\varphi_{r/2} - \varphi_r)\big) * (\varphi_r + \varphi_{r/2}),$$

and when $r = 2^{-i}$ we write a_i for a_r; finally

$$k_n(\theta, \varphi) = \sum_{i=0}^{n-1} a_i * \delta_\varphi(\theta).$$

The appropriate version of Hörmander's condition in our case is

$$\int_E |k_n(\theta, \varphi) - k_n(\theta, \varphi_0)|\, d\mu(\theta) \leqslant C,$$

where

$$E = \{\theta\colon\ |\theta - \varphi_0| > 2|\varphi - \varphi_0|\}.$$

The integral equals

$$\int_E |k_n * \delta_\varphi(\theta) - k_n * \delta_{\varphi_0}(\theta)|\, d\mu(\theta) \leqslant \sum_{i=0}^{n-1} \int_E |a_i * \delta_\varphi(\theta) - a_i * \delta_{\varphi_0}(\theta)|\, d\mu(\theta). \tag{5.1}$$

PROPOSITION 5.1. *This last sum is bounded by a constant C independent of n, φ, φ_0, where the m_n are bounded and*

$$\sum_{2^N}^{2^{N+1}} |\Delta^m m_n|^2 h_n = O(2^{-2N\gamma}) \tag{5.2}$$

for $\gamma = 1 - \langle \lambda + 1/2 \rangle$.

The proof of this will be given after the following chain of lemmas.

LEMMA 5.1. *If* (5.2), *then*

$$\int (1 - \cos\theta)^m |M\psi_r|^2\, d\mu(\theta) = O(r^{2\gamma}). \tag{5.3}$$

LEMMA 5.2. *If* (5.3) *and if* $0 < \eta < \gamma$, *then*

$$\int |M\psi_r(\theta)| \left(\frac{\theta}{r}\right)^\eta d\mu(\theta) \leqslant C. \tag{5.4}$$

LEMMA 5.3. *If* (5.4), *then*

$$\int_E |a_r * \delta_\varphi(\theta) - a_r * \delta_{\varphi_0}(\theta)|\, d\mu(\theta) \leqslant C \min\{[r/|\varphi - \varphi_0|]^\eta,\ |\varphi - \varphi_0|/r\}.$$

Proof of lemma (5.1). If $c_n = \left(1 - \frac{r}{2}\right)^n - (1-r)^n$, then $M\psi_r \sim \sim \sum c_n m_n h_n R_n$.

It was shown in proposition (4.1) that

$$\int (1 - \cos\theta)^m |M\psi_r|^2\, d\mu(\theta) \leqslant K \sum \big(\Delta^m(m_n c_n)\big)^2 h_n. \tag{5.5}$$

We proceed to estimate $D^m(m_n c_n)$. By a Leibniz formula for finite differences we obtain

$$\Delta^m(m_n c_n) = \sum_{j=0}^{m} \binom{m}{j} \Delta^j m_n \Delta^{m-j} c_{n+j} \tag{5.6}$$

and consequently the sum in (5.5) will be bounded by a constant multiple of the largest term in (5.6). For $j < m$ we estimate the factors separately. In the second factor

$$\Delta^{m-j}(1-r)^{n+j} = (1-r)^{n+j}(-r)^{m-j},$$

thus

$$|\Delta^{m-j} c_{n+j}| \leqslant C r^{m-j}(1-r)^{n+j}.$$

In Proposition 4.2 we estimated the $\Delta^j m_n$ and obtained $|\Delta^j m_n| = O(n^{-j})$ for $j < m$. Putting these together, we obtain

$$\sum |\Delta^j m_n \Delta^{m-j} c_{n+j}|^2 h_n \leqslant C \sum \{r^{m-j}(1-r)^{n+j}\}^2 (n^{-j})^2 n^{2\lambda}$$

$$\leqslant C r^{2m-2j} \sum n^{2\lambda-2j}(1-r)^{2n} \leqslant C r^{2m-2j}(r^{-2\lambda+2j-1}) = O(r^{2(m-\lambda)-1}) = O(r^{2\gamma}).$$

(Using here the fact that $\sum n^\beta (1-r)^n \leqslant \frac{c}{r^{\beta+1}}$.)

On the other hand if $j = m$

$$\sum_{N=0}^{\infty} \sum_{2^N}^{2^{N+1}-1} |c_n|^2 |\Delta^m m_n|^2 h_n \leqslant \sum_{N=0}^{\infty} \big(2^N r(1-r)^{2^N}\big)^2 \sum_{2^N}^{2^{N+1}-1} |\Delta^m m_n|^2 h_n$$

$$\leqslant c r^2 \sum_{N=0}^{\infty} 2^{2N}(1-r)^{2^{N+1}} 2^{-2N\gamma} = c r^2 \sum_{N=0}^{\infty} 2^{N(2-2\gamma)}(1-r)^{2^{N+1}} \leqslant c r^{2\gamma}.$$

The penultimate inequality follows from an argument reminiscent of the Cauchy condensation theorem. Let $1 \leqslant q$; then

$$\sum_0^\infty 2^{Nq}(1-r)^{2^{N+1}} = (1-r)^2 + 2^q(1-r)^4 + 4^q(1-r)^8 + \ldots$$

$$= (1-r)^2 + 2\big(2^{q-1}(1-r)^4\big) + 4\big(4^{q-1}(1-r)^8\big) + \ldots$$

$$\leqslant (1-r)^2 + 2^{q-1}(1-r)^3 + 3^{q-1}(1-r)^4 + 4^{q-1}(1-r)^5 + \ldots$$

$$= \sum_0^\infty N^{q-1}(1-r)^{N+1} \leqslant c r^{-q}.$$

A similar argument holds for $0 \leqslant q < 1$.

These two estimates together yield

$$\int (1-\cos\theta)^m |M\psi_r|^2 d\mu(\theta) = O(r^{2\gamma}).$$

Proof of lemma (5.2).

$$\int_0^\pi |M\psi_r(\theta)| \left(\frac{\theta}{r}\right)^\eta d\mu(\theta) = \int_0^r + \int_r^\pi.$$

The first integral is trivial since $\theta/r < 1$.

$$\int_r^\pi |M\psi_r(\theta)| \left(\frac{\theta}{r}\right)^\eta d\mu(\theta) = \frac{1}{r^\eta}\int_r^\pi |M\psi_r(\theta)|\, \theta^m \theta^{\eta-m} d\mu(\theta)$$

$$\leqslant r^{-\eta}\Big[\int_0^\pi |M\psi_r(\theta)|^2 \theta^{2m} d\mu(\theta)\Big]^{1/2} \times \Big[\int_r^\pi \theta^{2\eta-2m} d\mu(\theta)\Big]^{1/2}$$

$$\leqslant cr^{-\eta}\Big[\int_0^\pi |M\psi_r(\theta)|^2 (1-\cos\theta)^m d\mu(\theta)\Big]^{1/2} \times \Big[\int_r^\pi \theta^{2(\eta-m+\lambda)} d\theta\Big]^{1/2}$$

$$\leqslant cr^{-\eta}[r^{2\gamma}]^{1/2}[\pi^{2(\eta-\gamma)} - r^{2(\eta-\gamma)}]^{1/2} \leqslant C.$$

Proof of lemma (5.3).

$$(5.7)\qquad \int_E |a_r * \delta_\varphi(\theta) - a_r * \delta_{\varphi_0}(\theta)|\, d\mu(\theta) \leqslant C\|M\psi_r\|_1 \|\varphi_r * \delta_\varphi - \varphi_r * \delta_{\varphi_0}\|_1.$$

The first factor is bounded since

$$\int_0^\pi |M\psi_r|\, d\mu = \int_0^r + \int_r^\pi = \mathrm{I} + \mathrm{II},$$

$$\mathrm{I} \leqslant \Big[\int_0^r (M\psi_r(\theta))^2 d\mu(\theta)\Big]^{1/2}\Big[\int_0^r \sin^{2\lambda}\theta\, d\theta\Big]^{1/2} \leqslant C\|M\psi_r\|_2 r^{\lambda+1/2} \leqslant C\|\psi_r\|_2 r^{\lambda+1/2}.$$

But $\|\psi_r\|_2^2 \leqslant c\sum (1-r)^{2n} n^{2\lambda} \leqslant \dfrac{K}{r^{2\lambda+1}}$, so I is bounded.

$$\mathrm{II} \leqslant \int_r^\pi |M\psi_r(\theta)| \left(\frac{\theta}{r}\right)^\eta d\mu(\theta) \leqslant c$$

by Lemma 5.2. The second factor of (5.7) is bounded by $|\varphi - \varphi_0|/r$ by Proposition 3.1, so we obtain

$$\int |a_r * \delta_\varphi(\theta) - a_r * \delta_{\varphi_0}(\theta)|\, d\mu(\theta) \leqslant c\frac{|\varphi-\varphi_0|}{r}.$$

The other estimate is more involved. If $|\theta - \varphi_0| > 2|\varphi - \varphi_0|$, then

$$|\theta-\varphi_0| \leqslant |\theta-\varphi| + |\varphi-\varphi_0| \leqslant |\theta-\varphi| + \frac{1}{2}|\theta-\varphi_0|,$$

so

$$|\theta-\varphi_0| \leqslant 2|\theta-\varphi| \quad\text{and}\quad |\varphi-\varphi_0| \leqslant |\theta-\varphi|.$$

Hence

$$\int_E |a_r * \delta_\varphi(\theta)|\, d\mu(\theta) \leqslant \left(\frac{r}{|\varphi-\varphi_0|}\right)^\eta \int |a_r * \delta_\varphi(\theta)| \left(\frac{|\varphi-\varphi_0|}{r}\right)^\eta d\mu(\theta)$$

$$\leqslant c\left(\frac{r}{|\varphi-\varphi_0|}\right)^\eta \int |a_r * \delta_\varphi(\theta)| \left(\frac{|\theta-\varphi|}{r}\right)^\eta d\mu(\theta) \leqslant c\left(\frac{r}{|\varphi-\varphi_0|}\right)^\eta.$$

This last inequality follows from the nature of the support of the convolution kernel.

Consider

$$\int |a_r * \delta_\varphi(\theta)| \left(\frac{|\varphi-\theta|}{r}\right)^\eta d\mu(\theta)$$

$$= \iiiint |M\psi_r(\xi)| \{\varphi_r(\varepsilon) + \varphi_{r/2}(\varepsilon)\} \Phi(\xi, \varepsilon, \psi)\Phi(\varphi, \psi, \theta) \times$$

$$\times \left(\frac{|\varphi-\theta|}{r}\right)^\eta d\mu(\xi)\, d\mu(\varepsilon)\, d\mu(\psi)\, d\mu(\theta).$$

The integrand is supported only in the region $|\varphi-\theta| \leqslant \psi \leqslant \xi+\varepsilon$, so the integral is bounded by

$$\iint |M\psi_r(\xi)| (\varphi_r(\varepsilon) + \varphi_{r/2}(\varepsilon)) \left(\frac{\xi+\varepsilon}{r}\right)^\eta d\mu(\xi)\, d\mu(\varepsilon)$$

$$\leqslant C\int |M\psi_r(\xi)|\, d\mu(\xi) \int \{\varphi_r(\varepsilon) + \varphi_{r/2}(\varepsilon)\} \left(\frac{\varepsilon}{r}\right)^\eta d\mu(\varepsilon) +$$

$$+ C\int \{\varphi_r(\varepsilon) + \varphi_{r/2}(\varepsilon)\}\, d\mu(\varepsilon) \int M\psi_r(\xi) \left(\frac{\xi}{r}\right)^\eta d\mu(\xi).$$

The first term is bounded by Proposition 3.1; the second by Lemma 5.2.

The conclusion of the proof of Proposition 5.1 requires only that we add the terms together.

$$\sum_1^n \int_E |a_i * \delta_\varphi(\theta) - a_i * \delta_{\varphi_0}(\theta)|\, d\mu(\theta) \leqslant \sum_{i=1}^N 2^i |\varphi-\varphi_0| + \sum_{i=N}^n \left(\frac{1}{2^i|\varphi-\varphi_0|}\right)^\eta$$

where N is chosen so that

$$2^N|\varphi-\varphi_0| \leqslant 1 \leqslant 2^{N+1}|\varphi-\varphi_0|.$$

Then

$$\sum_{i=1}^N 2^i|\varphi-\varphi_0| \leqslant |\varphi-\varphi_0|(2^{N+1}-1) \leqslant 2,$$

$$\sum_{i=N}^\infty \left(\frac{1}{2^i|\varphi-\varphi_0|}\right)^\eta = \left(\frac{1}{2^N|\varphi-\varphi_0|}\right)^\eta \sum_{i=0}^\infty 2^{-i\eta} \leqslant C.$$

Thus each operator k_n satisfies the Hörmander condition in a uniform manner.

That M is of strong type p-p for all $1 < p < \infty$, and weak type 1-1 follows from the methods of Coifman and Weiss ([4], pp. 71–75) and the observation that there is a constant A independent of x and r such that

$$S(x, r) \leqslant AS(x, r/2) \qquad (-1 \leqslant x \leqslant 1, r > 0),$$

where

$$S(x, r) = m\{[x-r, x+r] \cap [-1, 1]\}.$$

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Ersetzbarkeit von konvergenztreuen Matrixverfahren

von

JOHANN BOOS (Tübingen)

Auszug. Die Ersetzbarkeit von konvergenztreuen Matrixverfahren wurde von Wilansky, Chang, Macphail, Snyder und Bennett in den Arbeiten [1], [4], [6] und [7] betrachtet. In der folgenden Arbeit werden weitere notwendige und hinreichende Bedingungen für die Ersetzbarkeit gegeben.

1. Einleitung und Bezeichnungen. Wie üblich bezeichnen wir mit ω, m, c, c_0 bzw. l den Raum aller (komplexwertigen) Folgen, der beschränkten Folgen, der konvergenten Folgen, der Nullfolgen bzw. den Raum der absolut summierbaren Folgen. Weiter bezeichnen wir mit e die Folge mit 1 an jeder Stelle und mit e^k $(k \in \boldsymbol{N})$ die Folgen mit 1 an der k-ten Stelle und null sonst.

Die Elemente aus ω fassen wir als (unendliche) Spalten auf. Ist $A = (a_{nk})_{n,k=1}^{\infty}$ eine unendliche Matrix mit komplexen Koeffizienten und $x = (x_k)_{k=1}^{\infty} \in \omega$, so definiert das „Matrixprodukt" $Ax := \left(y_n = \sum_{k=1}^{\infty} a_{nk}x_k\right)_{n=1}^{\infty}$ eine lineare Abbildung von $d_A := \{x \in \omega : y_n \text{ existiert für alle } n \in \boldsymbol{N}\}$ in ω; A als Transformationsmatrix nennen wir *Matrixverfahren*. Ebenso ist für $x, s \in \omega$ in natürlicher Weise das „Matrixprodukt"

$$\bar{s}x := \sum_{k=1}^{\infty} s_k x_k$$

definiert, falls die Reihe existiert und $\bar{s}$ die zu s transponierte Folge ist. Bezeichnen wir mit

$$c_A := \left\{x \in d_A : \lim\nolimits_A x := \lim_{n\to\infty} y_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}x_k \text{ existiert}\right\}$$

das *Wirkfeld von* A, so heißt A *konvergenztreu*, wenn $c \subseteq c_A$ gilt. A heißt *absolut konvergenztreu*, wenn

$$l \subseteq l_A := \{x \in d_A : y = Ax \in l\} \quad \text{gilt.}$$