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A note on wc_0-bases

by

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Abstract. A wc_0-basis which disproves a conjecture of P. Wojtaszczyk is constructed. A simple sufficient condition for $X+Y$ to have a wc_0-basis whenever X has a wc_0-basis and Y has a basis is given.

1. Introduction. Let X be a Banach space. A sequence $\{x_n\}$ in X is a basis if for every x in X, there exists a unique sequence $\{a_n\}$ of scalars such that $x = \sum_{n=1}^{\infty} a_n x_n$. A basis $\{x_n\}$ of X is wc_0 if, in addition, $0 < \inf \|x_n\| \leqslant \sup \|x_n\| < \infty$, and $\{x_n\}$ converges weakly to 0. A sequence $\{x_n\}$ is basic in X if it is a basis of $[x_n]$, the closed subspace of X spanned by $\{x_n\}$. It is well known that a sequence $\{x_n\}$ in a Banach space X is basic if and only if there exists a constant C such that $\|\sum_{n=1}^{m} a_n x_n\| \leqslant C \|\sum_{n=1}^{m+k} a_n x_n\|$ for all positive integers m and k, and all scalars $\{a_n\}$. The norm K of a basic sequence is the infimum of all numbers C for which the above inequality is satisfied. Finally, if X and Y are Banach spaces, $X+Y$ will denote the direct sum of X and Y.

In a recent paper by P. Wojtaszczyk [4], the following theorem was proved:

THEOREM 1. *Let Y be a Banach space with a basis and let X be a Banach space with a wc_0 basis $\{x_n\}$ satisfying the following condition:*

(∗) *there exist two sequences of natural numbers $\{p_r\}$ and $\{k_r\}$ such that $k_r < p_{r+1} < k_{r+1}$ and $\lim (k_r - p_r) = \infty$ and $\{\|\sum_{p_r}^{k_r} x_i\|^{-1} \sum_{p_r}^{k_r} x_i\}$ is weakly convergent to 0.*

Then $X+Y$ has a wc_0 basis.

Wojtaszczyk conjectured that condition (∗) is superfluous, since all the known wc_0 bases possessed this property. In this paper we will disprove this conjecture by constructing a wc_0 basis for which condition

(*) cannot be satisfied. We will show, however, that if X^* is separable, then condition (*) is indeed superfluous.

I like to thank P. Wojtaszczyk for sending me an advanced copy of his paper [4].

2. Counter example. We will construct a sequence in $C[0,1]$. Let $F_0 = [0,1]$, $F_1 = F_{1,1} \cup F_{1,2}$, where $F_{1,1} = [0, 1/4]$ and $F_{1,2} = [1/2, 3/4]$; and if $F_n = F_{n,1} \cup F_{n,2} \cup \ldots \cup F_{n,2^n}$, where $F_{n,i} = [a_i, b_i]$, $F_{n,i} \cap F_{n,j} = \emptyset$ $(i \neq j)$, and $a_i < a_{i+1}$ $(i = 1, \ldots, 2^n - 1)$, then $F_{n+1} = F_{n+1,1} \cup \ldots \cup F_{n+1,2^{n+1}}$, where $F_{n+1,2k-1} = [a_k, a_k + (b_k - a_k)/4]$, and $F_{n+1,2k} = [a_k + (b_k - a_k)/2, a_k + 3(b_k - a_k)/4]$ $(k = 1, \ldots, 2^n)$.

Let

$$x_1(t) = \begin{cases} 0 & t \in F_{1,1} \cup \{1\}, \\ 1 & t \in F_{1,2}, \\ \text{linear otherwise on } [0,1]. \end{cases}$$

For $n \geq 2$, let

$$x_n(t) = \begin{cases} 0 & t \in \complement F_{n-1}, \\ 0 & t \in F_{n,2k-1}\ (k = 1, \ldots, 2^{n-1}), \\ 1/j & t \in F_{n,2k}\ (k = 1, \ldots, 2^{n-1}), \\ & \text{and } j = \min\{m \in J^+ \mid \sum_{i=1}^{n-1} x_i(t) < \sum_{i=1}^{m} (1/i)\}, \\ \text{linear otherwise on } [0,1]. \end{cases}$$

PROPOSITION 2.1 *The sequence $\{x_n\}$ is basic in $C[0,1]$ and*

$$\Big\| \sum_{m}^{m+k-1} x_i \Big\| = \sum_{i=1}^{k} (1/i) \qquad (m = 1, 2, \ldots;\ k = 1, 2, \ldots).$$

Proof. For any finite set σ, let $|\sigma|$ denote the cardinality of σ. Then for any $1 \leq k \leq n$, and scalars $\{c_i\}$, $\| \sum_{i=1}^{n} c_i x_i \| = \max\{| \sum_{i=1}^{|\sigma|} (c_{n_i}/i) | \mid \sigma = \{n_1 < n_2 < \ldots < n_{|\sigma|}\} \subset \{1, 2, \ldots, n\}\} \geq \max\{| \sum_{i=1}^{|\sigma|} c_{n_i}/i | \mid \sigma = \{n_1 < \ldots < n_{|\sigma|}\} \subset \{1, \ldots, k\}\} = \| \sum_{i=1}^{k} c_i x_i \|$. Hence $\{x_n\}$ is basic. The second part follows from the above observation.

PROPOSITION 2.2. *The sequence $\{x_n\}$ converges weakly to 0 in $C[0,1]$.*

Proof. Since $\|x_i\| = 1$ for all i, it suffices to show that for all t in $[0,1]$, $\{x_i(t)\}$ converges to zero. But for $0 \neq t \in \bigcap_{n=1}^{\infty} F_n$, $\{x_i(t)\}$ is essentially the harmonic series (interrupted by finite strings of zeros), and for any other t, $x_i(t) = 0$ for all but a finite number of i's.

COROLLARY 2.1. *The sequence $\{x_n\}$ is a wc_0 basis of $[x_n]$.*

LEMMA 2.1. *Let $\{a_n\}$ be a divergent series of positive numbers. Then for any positive integer m, there exists a positive integer $N(m)$ such that $\sum_{m}^{m+n-1} a_i \geq (1/2) \sum_{i=1}^{n} a_i$ for all $n \geq N(m)$.*

The proof is trivial.

PROPOSITION 2.3. *There exist no sequences $\{p_r\}$ and $\{k_r\}$ of natural numbers such that $p_r \leq k_r < p_{r+1}$, $\overline{\lim}(k_r - p_r) = \infty$, and such that the sequence $z_r = \| \sum_{p_r}^{k_r} x_i \|^{-1} \sum_{p_r}^{k_r} x_i$ converges weakly to zero in $C[0,1]$.*

Proof. In the following, if m is a positive integer, then $N(m)$ will denote a positive integer having the property in Lemma 2.1, where $a_n = 1/n$. Let $r_1 = 1$. Since $\overline{\lim}(k_r - p_r) = \infty$, there exists $r_2 > r_1$ such that $k_{r_2} - p_{r_2} > N(k_{r_1} - p_{r_1} + 2)$. Similarly, there exists $r_3 > r_2$ such that $k_{r_3} - p_{r_3} > N(k_{r_1} - p_{r_1} + k_{r_2} - p_{r_2} + 4)$. Inductively, there exists $r_n > r_{n-1}$ such that $k_{r_n} - p_{r_n} > N(\sum_{i=1}^{n-1} (k_{r_i} - p_{r_i}) + 2(n-1))$ $(n = 2, 3, \ldots)$. For simplicity of notation, let $w_n = z_{r_n}$. We will show the existence of t_0 in $[0,1]$ such that $w_n(t_0) \geq 1/2$ for all n. Let

$$s_i = \begin{cases} 0 & \text{if } i \notin \bigcup_{j=1}^{\infty} [p_{r_j}, k_{r_j}] \cap J^+, \\ 1 & \text{if } i \in \bigcup_{j=1}^{\infty} [p_{r_j}, k_{r_j}] \cap J^+. \end{cases}$$

By the construction of F_i's, there exists a unique sequence $\{F_{n,i_n} \mid 1 \leq i_n \leq 2^n\}$ of closed intervals such that $F_{n,i_n} \supset F_{n+1,i_{n+1}}$ $(n = 1, 2, \ldots)$, and i_n is odd if $s_n = 0$, and i_n is even if $s_n = 1$. Let $\{t_0\} = \bigcap_{n=1}^{\infty} F_{n,i_n}$. From the definition of $\{x_n\}$, it is not hard to see that the subsequence of $\{x_n\}$ with subscripts $p_{r_1}, \ldots, k_{r_1};\ p_{r_2}, \ldots, k_{r_2};\ \ldots;\ p_{r_n}, \ldots, k_{r_n};\ \ldots$, when evaluated at t_0, is exactly the harmonic sequence. It follows from Proposition 2.1 and Lemma 2.1 that $w_n(t_0) \geq 1/2$.

3. A sufficient condition. We conclude this note by giving a sufficient condition for condition (*).

PROPOSITION 3.1. *If X^* is separable, then condition (*) in Theorem 1 is superfluous.*

Proof. For each fixed positive integer k, consider the sequence $\{z_n^k\}_{n=1}^{\infty}$, where $z_n^k = \| \sum_{i=n}^{n+k} x_i \|^{-1} \sum_{i=n}^{n+k} x_i$. Since $\{x_n\}$ is a wc_0 basis, $\| \sum_{i=n}^{n+k} x_i \| \geq (1/C) \|x_n\| \geq C_1 > 0$, where C and C_1 are some positive constants independent of k and n; hence clearly $\{z_n^k\}_{n=1}^{\infty}$ converges weakly to zero also. Finally, because of the fact that the separability of X^* implies the metrizability of the unit sphere of X, we can use a diagonal process to obtain a sequence $\{z^k_{n_k}\}_{k=1}^{\infty}$ (with $n_k+k < n_{k+1}$) which converges weakly to zero. This completes the proof.

Remark. This result implies some results of P. Wojtaszczyk; if X is reflexive, or if X has a shrinking basis (for definition see [3]), or if X^* has a basis, Theorem 1 holds without condition (*), since in each case X^* is separable.

The simplified proof of Proposition 3.1 was pointed out by the referee who also informed us of the existence of a space X with a basis such that X^* is separable but does not possess a basis. (Namely, J. Lindenstrauss proved in [2], Corollary 3 and remark that such a space exists if there is a Banach space which does not have the approximation property; the existence of the latter is of course well known now [1].) Hence Proposition 3.1 is definitely an improvement of some of the results of P. Wojtaszczyk [4].

References

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Sur les équations d'évolution non linéaires 1

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Introduction. Dans ce travail nous essayons d'étendre au cas des opérations et des espaces non linéaires une partie des résultats établis dans el travail [1]. Le champ d'applications propre de ce travail étant, nous semble-t-il, les espaces dits espaces du type de Riemann–Hilbert, les applications de la théorie présentée ici seront exposées dans des travaux ultérieurs, où il sera question des éléments et des fondements de ces espaces.

1. Notions, notations et hypothèses

1.1. Soient X_t (avec t réel) des espaces métriques complets, avec la distance $\varrho(t;\, x, y)$ $(x, y \in X_t)$; les éléments de X_t différents seront désignés par les mêmes lettres x, y, etc. Faisons correspondre à tout couple $t \in \langle 0, \tau\rangle$, $\varepsilon \geqslant 0$ l'opération $S(t, \varepsilon) \in X_t \to X_{t+\varepsilon}$ et admettons les hypothèses suivantes:

(A) $\varrho\big(t+\varepsilon;\; S(t, \varepsilon)(x), S(t, \varepsilon)(y)\big) \leqslant (1+K\varepsilon)\varrho(t;\, x, y), \quad 0 \leqslant \varepsilon \leqslant \varepsilon,$

(B) à tout $t \in \langle 0, \tau\rangle$ correspond un ensemble fermé $Z_t \subset X_t$ tel que

$$S(t, \varepsilon) \in Z_t \to Z_{t+\varepsilon}, \tag{1}$$

(C) pour $x \in Z_t$, $\varepsilon, \delta \geqslant 0$,

$$\varrho\big(t+\varepsilon+\delta;\; S(t, \varepsilon+\delta)(x), S(t+\varepsilon, \delta)S(t, \varepsilon)(x)\big) \leqslant C\varepsilon\delta, \tag{2}$$

(D) $S(t, 0)(x) = x$ $(x \in Z_t)$.

Remarque. Pour éviter les difficultés typographiques nous écrirons dans la suite $S(t, \varepsilon)x$ au lieu de $S(t, \varepsilon)(x)$.

1.2. Définition et propriétés de l'opération $T(s, t)$. Soient: $s \in \langle 0, \tau\rangle$, π — une division de l'intervalle $\langle s, \tau\rangle$ telle que $0 \leqslant s = t_0 < t_1 < \ldots < t_p = \tau$. Tout comme dans [1], faisons correspondre à π l'opération $T(\pi, s, t) \in X_s \to X_t$ définie comme il suit:

DÉFINITION 1.

$$T(\pi, s, t) = S(t_n, t-t_n)S(t_{n-1}, \delta_{n-1}) \ldots S(t_0, \delta_0)$$

où $\delta_i = t_{i+1}-t_i$, $\quad t_n \leqslant t \leqslant t_{n+1}$.