map 1 into 1, then it is very easy to see that $\mathscr{A}(K_\infty)$ is a direct limit of $(\mathscr{A}(K_t))$ in the category $\mathscr{A}K$. Our theorem is a much stronger statement: $\mathscr{A}(K_\infty)$ remains a direct limit in the larger category $\mathbf{Ban}_1$.

Remark 2. Compare our theorem with the following theorem of Semadeni.

THEOREM ([3], Theorem 1). *Let (X_∞, φ_t) be the inverse limit of the inverse system (X_t, φ_t^s) of compact topological spaces. Then $(\mathscr{C}(X_\infty), \mathscr{C}(\varphi_t))$ is the direct limit in* $\mathbf{Ban}_1$ *of the direct system $(\mathscr{C}(X_t), \mathscr{C}(\varphi_t^s))$.*

This theorem was proved earlier by Pełczyński ([2], p. 14), in the special case where all maps φ_t^s are surjections. Semadeni's theorem can, in fact, be obtained as a corollary of our theorem, but only with some additional work; one passes from compact spaces X to the simplex $\boldsymbol{P}(X)$ of Borel probability measures on X. One can identify $C(X)$ with $\mathscr{A}(\boldsymbol{P}(X))$, but one must show that if $X_\infty = \operatorname{inv}\lim X_t$, then $\boldsymbol{P}(X_\infty) = \operatorname{inv}\lim \boldsymbol{P}(X_t)$.

If one wants to obtain Semadeni's theorem by the methods of this paper, the most natural way is to imitate the proof of our theorem directly. The proof that $\lambda\colon O*\mathscr{C}(X_\infty)\to L$ is injective is similar: one uses the Weierstrass–Stone theorem to prove that $\bigcup \mathscr{C}(\varphi_t)(\mathscr{C}(X_t))$ is dense in $\mathscr{C}(X_\infty)$. The proof that λ is onto is somewhat simpler than in our case.

Remark 3. I am grateful to Professor Semadeni for bringing the problem to my attention and for acquainting me with the ideas from category theory.

Remark 4. I wish to thank the referee for providing me with some references and for suggesting some improvements in style.

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Received February 15, 1973 (660)

STUDIA MATHEMATICA, T. LI. (1974)

Continuity of operators on Saks spaces

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Abstract. In [13] Orlicz initiated the study of linear operators acting from so-called Saks spaces. His original investigations concerned operators taking values in Banach and Fréchet spaces. In this paper we extend the theory to the case of operators with values in locally convex and general topological vector spaces. Generally the proofs presented here are more direct than the original ones of Orlicz; some refinements of his classical statements and some new results have been obtained.

§ 0. Introduction. In the present paper we intend to give some generalizations of results contained in [13]. Since we will constantly refer to this fundamental work, we decide to preserve as far as possible its terminology, notation and conventions. There is, however, one important exception to this principle — an additive ([13], 2.1) continuous operator ν from a Saks space X_s into a topological vector space Y will be termed explicitly "*additive* (X_s, Y)*-continuous*", while Orlicz, according to the old terminology of Banach [1], calls ν in that case (X_s, Y)-linear. Moreover, in many situations the topology of Y is explicitly mentioned, i.e., $Y = (Y, \tau)$ for example; we will then say simply that $\nu\colon X_s\to Y$ is *τ-continuous*, or only that ν is τ-continuous, when no ambiguity about X_s and Y arises.

An operator $\nu\colon X_s\to Y$ will be said to be *linear* (cf. [13], 2.1) if for arbitrary $x_1, x_2 \in X_s$ and arbitrary scalars a_1, a_2, $a_1x_1+a_2x_2 \in X_s$ implies $\nu(a_1x_1+a_2x_2) = a_1\nu(x_1)+a_2\nu(x_2)$. Note that with this terminology it is obvious that an additive (X_s, Y)-continuous operator $=$ a linear (X_s, Y)-continuous operator.

Following Orlicz, a *Saks space* is defined as a closed unit ball of a fundamental normed space $(X, \|\cdot\|)$ on which another norm (in general non-homogeneous!), $\|\cdot\|^*$ say, defines the complete metric. It is clear, however, that instead of the unit ball of $(X, \|\cdot\|)$, a bounded, closed, convex balanced set of an arbitrary Hausdorff topological vector space could be taken. If this unit ball endowed with the metric induced by $\|\cdot\|^*$ is not complete, it is called a *Saks set.*

Denote by X_s a Saks space and by $K(x_0, r)$ the open sphere with centre x_0 and radius r in the space X_s. We shall consider Saks spaces satisfying the following conditions:

(Σ_1) Given any $x_0 \in X_s$ and $r > 0$, there exists a $d > 0$ such that every element $x \in X_s$ for which $\|x\|^* < d$ can be written in the form $x = x_1 - x_2$, with $x_1, x_2 \in K(x_0, r)$.

(Σ_2) If $(x_n) \subset X_s$, $x_n \to 0$, $\varepsilon_n > 0$ and $\varepsilon_n \to 0$, then there exist an increasing sequence (n_k) of indices and a sequence $(\hat{x}_{n_k})$ such that:

(i) $\|\hat{x}_{n_k} - x_{n_k}\|^* < \varepsilon_{n_k}$ for $k = 1, 2, \ldots$,

(ii) $\sum_{k=1}^{\infty} \hat{x}_{n_k}$ is subseries-convergent in X_s (i.e. for each subset M of a set of natural numbers $\sum_{k \in M} \hat{x}_{n_k}$ is convergent).

These conditions were introduced by Orlicz in [13]. As to the origins of (Σ_1) see [13], p. 241; condition (Σ_2) is self-explainatory — it ensures the subseries convergence of $\sum_{k=1}^{\infty} \hat{x}_{n_k}$ in X_s, thus, as we shall see, a frequent application of the Orlicz–Pettis type theorems will be possible.

The use of the above conditions explains also the fact of deep analogies between vector measures and additive operators acting from Saks spaces. Actually, in many cases the vector measure is simply an additive operator from the so-called generalized Saks space ([13], 1.63). Many of the results presented below are motivated by the recent developments in the theory of vector measures, and have been obtained on the way "from a vector measure to an additive operator on a Saks space". Probably one could expect new results in measure theory from the opposite point of view.

In the sequel "let X_s be (Σ_1) or (Σ_2)" means "let X_s be a Saks space satisfying condition (Σ_1) or (Σ_2)".

In what follows the results are formulated for Saks spaces; it is clear, however, that every statement in which only condition (Σ_2) is assumed is valid for Saks sets also (i.e., the completeness of X_s is then superfluous).

For certain restrictions concerning the scope of "the theory of Saks spaces" see [12]. The reader is referred also to [14], [15] and [16].

We will denote by $\boldsymbol{N}$ the set of positive integers, and by ν, μ additive operators from Saks spaces; the field of scalars may be the real or complex field throughout.

Let Y be a Hausdorff locally convex (topological vector) space, and Y' its dual; $\sigma(Y, Y')$ will denote the weak topology, $\tau(Y, Y')$ the Mackey topology, $\beta(Y, Y')$ the strong topology, and $\varkappa(Y, Y')$ the topology of uniform convergence on weakly compact subsets of Y', $Y_\sigma = (Y, \sigma(Y, Y'))$ and similarly the notation Y_τ, $Y'_\varkappa$ etc. will be used.

The following fundamental lemma will frequently be used below:

0.1. Lemma. *Let ν be an vdditive operator from a Saks space X_s into a topological vector space Y;*

(a) *if X_s is (Σ_1) and ν has one point of continuity, then ν is uniformly (X_s, Y)-continuous;*

(b) *if X_s is arbitrary and ν has one point of continuity, x_0 say, such that $\|x_0\| < 1$, then ν is uniformly (X_s, Y)-continuous.*

This lemma is proved in [13] 2.2(A) when the arrival space Y is an F-space, but the completeness of Y is obviously superfluous, and repeating the reasoning of Orlicz with respect to each F-pseudo-norm separately, one obtains the lemma for an arbitrary topological vector space.

§ 1. Operators with values in general topological vector spaces.

1.1. Proposition. *Let X_s be (Σ_1) or (Σ_2), (Y, β) a separable linear topological vector space and α a linear topology on Y such that β has a basis of α-sequentially closed neighbourhoods at 0. If $\nu\colon X_s \to Y$ is α-continuous, then it is β-continuous.*

Proof. Let (y_n) be a β-dense sequence in Y, and V an arbitrary β-neighbourhood of zero; then we can find an α-sequentially closed symmetric β-neighbourhood of zero U contained in V and we have $Y = \bigcup_{n=1}^{\infty} y_n + U = \bigcup_{n=1}^{\infty} U_n$, $\bigcup_{n=1}^{\infty} \nu^{-1}[U_n] = X_s$. Since X_s is metrizable, $\nu^{-1}[U_n]$ are closed in X_s; therefore by the Baire category theorem there is a ball $K(x_0, r) \subset X_s$ with $r > 0$ such that its image $\nu[K(x_0, r)] \subset y_m + U$ for a certain $m \in \boldsymbol{N}$. By condition (Σ_1) there exists an $s > 0$ such that $\nu[K(0, s)] \subset U + U \subset V + V$, i.e., ν is β-continuous at 0. Now by Lemma 0.1 it is β-continuous. Let condition (Σ_2) be fulfilled and suppose that ν is not β-continuous. Then there exist a sequence (x_n) and an α-sequentially closed β-neighbourhood U of zero in Y such that $\nu(x_n) \notin U$ (Lemma 0.1). U being α-seq. closed, $\nu^{-1}[\complement U]$ is open, and so we will find a sequence (a_n), $a_n > 0$, $a_n \to 0$, such that $\nu[K(x_n, a_n)] \subset \complement U$. By ($\Sigma_2$) there is an $(\hat{x}_{n_i})$ such that $\nu(\hat{x}_{n_i}) \in \complement U$ and $\sum_{i=1}^{\infty} \hat{x}_{n_i}$ is subseries-convergent in X_s. Then $\sum_{i=1}^{\infty} \nu(\hat{x}_{n_i})$ is α-subseries convergent, and hence, by [6], Theorem 1, β-subseries convergent[1]. This implies that $\nu(\hat{x}_{n_i}) \in U$ for $i \geqslant i_0$; a contradiction.

1.2. Theorem (compare [13], Theorem 1). *Let (Y, β) be a separable F-space, and α an arbitrary Hausdorff linear topology on Y coarser than β. Let X_s be (Σ_1) or (Σ_2); $\nu\colon X_s \to Y$ is β-continuous iff it is α-continuous.*

The proof is that of [6], Theorem 2.

Remark. If Y is locally convex Hausdorff, it is sufficient to suppose (Y, β) to be separable B_r-complete and α to be locally convex Hausdorff coarser than β.

The elimination of conditions (Σ_1) and (Σ_2) in the forthcoming theorem was suggested to the author by Dr. Drewnowski.

[1] In [6], α and β are supposed to be Hausdorff; this assumption, as easily seen, is superfluous (but in terms of measures we can only state that α-countable additivity implies β-exhaustivity).

1.3. THEOREM. *Let Y be a linear space, and α and β linear topologies on Y such that the set of partial sums of an α-subseries convergent series is β-bounded; $\nu\colon X_s \to Y$ α-continuous. Then*

(i) *ν transforms $\|\cdot\|^*$-bounded sets (i.e., subsets of X_s which are bounded in $(X, \|\cdot\|^*)$) into β-bounded sets;*

(ii) *if, moreover, the $\|\cdot\|^*$-norm is coarser than the $\|\cdot\|$-norm, then ν is β-bounded (i.e., $\nu[X_s]$ is a bounded subset of (Y, β)).*

Proof. Let $(x_n) \subset X_s$, $x_n \to 0$ and suppose that $(\nu(x_n))$ is not β-bounded. We then find a sequence (a_n) of numbers $0 < a_n < 1$, $a_n \to 0$ such that $a_n \nu(x_n) \not\to 0$. Let (a_{n_k}) be such a subsequence of (a_n) that $\sum_{k=1}^{\infty} \sqrt{a_{n_k}} < 1$ and $\sum_{k=1}^{\infty} \|\sqrt{a_{n_k}} x_{n_k}\|^* < \infty$. For every finite subset $M \subset \boldsymbol{N}$, $\sum_{k\in M} \sqrt{a_{n_k}} x_{n_k} \in X_s$ since X_s is convex-balanced in X. $\sum_{k=1}^{\infty} \sqrt{a_{n_k}} x_{n_k}$, being unconditionally Cauchy in X_s, is subseries-convergent by the completeness of X_s. Then $\sum_{k=1}^{\infty} \nu(\sqrt{a_{n_k}} x_{n_k})$ is α-subseries convergent in Y; thus by the assumption $(\nu(\sqrt{a_{n_k}} x_{n_k}))$ is β-bounded. But $\sqrt{a_{n_k}}\sqrt{a_{n_k}}\nu(x_{n_k}) \not\to 0$ in (Y, β); a contradiction. Now, let $E \subset X_s$ be $\|\cdot\|^*$-bounded, $(x_n) \subset E$, $1 > b_n \to 0$ and let $c_n \to 0$ be such that

$$1 > \left|\frac{b_n}{c_n}\right| \to 0.$$

We have $b_n \nu(x_n) = \dfrac{b_n}{c_n} \nu(c_n x_n) \to 0$, which proves (i). (i) implies (ii) since X_s is then $\|\cdot\|^*$-bounded.

1.3′. COROLLARY.

(A) (compare [13], Theorem 2). *Let α, β be two linear topologies on Y such that β has a basis of α-sequentially closed neighbourhoods at 0. Then* (i) *and* (ii) *hold.*

(B) *Let (Y, α) be a locally convex Hausdorff space. There exists a locally convex barelled topology β on Y, $\alpha \subset \beta$ such that* (i) *and* (ii) *hold.*

(C) *Let (Y, β) be a locally convex Hausdorff B_r-complete space and a any Hausdorff locally convex topology on X coarser than β; then* (i) *and* (ii) *hold.*

Proof. [11], Theorem 3 implies (A); [7], 2.4 implies (B); [7], 2.7 implies (C).

1.4. THEOREM (compare [13], Theorem 3). *Let X_s be (Σ_2); Y, α, β as in 1.3′ (A); (Y, β) an (O)-space*(2) *(i.e. if the set of finite partial sums of the series $\sum_{n=1}^{\infty} y_n$ is β-bounded, then the series is β-unconditionally Cauchy). Then if $\nu\colon X_s \to Y$ is α-continuous, it is β-continuous.*

Proof. If ν is not β-continuous, we find $x_n \to 0$ and an α-sequentially closed β-neighbourhood of 0 such that $\nu(x_n) \notin U$. By the argument used in 1.1 we find $\hat{x}_{n_i}$ such that $\sum_{i=1}^{\infty} \hat{x}_{n_i}$ is subseries-convergent in X_s and $\nu(\hat{x}_{n_i}) \notin U$. Then $\sum_{i=1}^{\infty} \nu(\hat{x}_{n_i})$ is α-subseries convergent; hence the set of all partial sums of this series is β-bounded. As (Y, β) is an (O)-space, our series is β-unconditionally Cauchy; thus $\nu(\hat{x}_{n_i}) \to 0$ in (Y, β); a contradiction.

1.5. THEOREM (compare [13], Theorem 1′). *Let Y be a locally convex Hausdorff space, and ν an additive (X_s, Y_σ)-continuous operator. Then each of the following conditions is sufficient for ν to be (X_s, Y_τ)-continuous:* (i) *X_s is (Σ_1) separable,* (ii) *X_s is (Σ_2).*

Proof. (i) X_s being separable, $\nu[X_s]$ is a separable subset in Y_σ. Let (y_n) be a dense sequence in $\nu[X_s]$. The linear $\sigma(Y, Y')$-closed span Y_0 of (y_n) is separable and contains $\nu[X_s]$. By the Mackey theorem Y_0 is also a $\tau(Y, Y')$-closed linear span of (y_n). The result follows by 1.1.

(ii) Let $x_n \to 0$ in X_s and suppose that $\nu(x_n) \not\to 0$ in Y_τ. There is then a $\sigma(Y, Y')$-closed neighbourhood of zero in Y_τ, say U, and (x_{n_k}) such that $\nu(x_{n_k}) \notin U$. By the same argument as in 1.1, we find $(\hat{x}_{n_i})$ such that $\sum_{i=1}^{\infty} \hat{x}_{n_i}$ is subseries-convergent and $\nu(\hat{x}_{n_i}) \notin U$. But $\sum_{i=1}^{\infty} \nu(\hat{x}_{n_i})$ is then subseries-convergent in Y_σ and hence in Y_τ by the Orlicz–Pettis theorem. Thus $\nu(\hat{x}_{n_i}) \to 0$ in Y_τ; a contradiction.

1.6. THEOREM. *Let (Y, τ) be an inductive limit of locally convex Hausdorff spaces (Y_i, τ_i), $i \in \boldsymbol{N}$, such that for each i:*

(j) *Y_i is sequentially closed in (Y_{i+1}, τ_{i+1});*

(jj) *there exists a linear topology τ' on Y which induces on Y_i its proper topology τ_i.*

Let X_s be (Σ_1), $\nu\colon X_s \to Y$ τ'-continuous. Then ν is τ-continuous.

Proof. In fact, $X_s = \bigcup_{n=1}^{\infty} \nu^{-1}[Y_n]$, and so by a standard application of the Baire category theorem and condition (Σ_1) we find a ball $K(0, s)$ such that $\nu[K(0, s)] \subset Y_i$ for a certain $i \in \boldsymbol{N}$. By definition τ induces on Y_i a topology which is coarser than τ_i. Hence ν is τ-continuous at 0. We apply Lemma 0.1.

Remark. With hypothesis (jj) the space (Y, τ) is in fact a strict inductive limit of (Y_i, τ_i); hence by [3], Ch. III, § 2, n° 4, Proposition 6, if X_s is $\|\cdot\|^*$-bounded, $\nu[X_s] \subset Y_n$ for a certain $n \in \boldsymbol{N}$.

(2) In [13] Orlicz calls such a space "fulfilling (Z) condition", but in recent literature the term "(O)-space" is used.

A locally convex Hausdorff space is said to be a *ϰ-space* if it can be represented as a union of countably many weakly compact subsets K_n. Since a balanced cover of a compact subset of a topological vector Hausdorff space is again compact, one can suppose K_n to be balanced. A closed vector subspace, the quotient (if it is Hausdorff), and a countable inductive limit (if it is Hausdorff) of $\varkappa$-spaces are $\varkappa$-spaces. It follows that a $\varkappa$-space need not be separable or metrizable.

The two topologies α and β are said to be consistent with each other [17] if, when x_1 and x_2 are any two distinct points, x_1 has an α-neighbourhood U_1 and x_2 a β-neighbourhood U_2 such that U_1 and U_2 are disjoint.

1.7. THEOREM. *Let X_s be (Σ_1) separable or $(\Sigma_1 \& \Sigma_2)$, Y a $\varkappa$-space and α a linear topology consistent with $\sigma(Y, Y')$, ν: $X_s \to Y$ α-continuous. Then ν is (X_s, Y)-continuous.*

Proof. In fact, α being consistent with $\sigma(Y, Y')$, there exists a Hausdorff linear topology $\gamma = \inf(\alpha, \sigma(Y, Y'))$. Of course, ν is γ-continuous and K_n, being γ-compact, are γ-closed. As γ and $\sigma(Y, Y')$ coincide on $K_n + K_n$, ν is (X_s, Y_σ)-continuous, since by a standard application of the Baire category theorem and condition (Σ_1) we can find a ball $K(0, s)$ such that $\nu[K(0, s)] \subset K_n + K_n$ for a certain $n \in \mathbf{N}$. Now, the result follows by 1.5.

§ 2. Operators with values in concrete function spaces. In the preceding section we gave some criteria which allow us to derive the "strong" continuity of operator ν: $X_s \to Y$ if its "weak" continuity is assumed, provided some additional hypotheses on the arrival space Y are satisfied. In this section we will give some examples of concrete (mainly vector-valued) function spaces Y which are not covered by the "general" theorems of § 1, and for which similar criteria on the continuity of operator ν: $X_s \to Y$ are still valid. As a consequence we will obtain some refinement of 1.5.

Let K be a topological space, Y a topological vector space. We denote by $C(K, Y)$ the space of continuous Y-valued functions on K, and by $C_s(K, Y)$ (resp. $C_d(K, Y)$, resp. $C_u(K, Y)$) the space $C(K, Y)$ endowed with the topology of pointwise (resp. pointwise in a set D dense in K, resp. uniform) convergence in K.

2.1. (A). THEOREM. *Let X_s be (Σ_1) or (Σ_2), Y a topological vector space, and ν_n: $X_s \to Y$ a sequence of continuous (additive) operators such that $\nu_n(x) \to \nu_0(x)$ $(n \to \infty)$ for each $x \in X_s$. Then ν_0 is (additive) (X_s, Y)-continuous and ν_n are uniformly (X_s, Y)-continuous (with respect to $n = 1, 2, \ldots$).*

There is an equivalent form of this theorem (see [8], 2.1):

2.1. (B). *Let X_s be (Σ_1) or (Σ_2), and μ: $X_s \to C_s(\mathbf{N}, Y)$ an (additive) continuous operator. Then μ: $X_s \to C_u(\mathbf{N}, Y)$ is continuous. Moreover, the limit operator μ_0: $X_s \to Y$, where $\mu_0(x) = \lim(\mu(x))(n)$ $(n \to \infty)$, is (additive) continuous.*

Proof. The case Σ_1. It is sufficient to prove the theorem for each F-pseudo-norm separately, and passing to the quotient we can suppose Y to be a metrizable topological vector space. Now, by the Mazur truncation method ([8], 2.1), μ: $X_s \to C_u(\mathbf{N}, Y)$ is of Baire's first class; hence it is continuous in a residual subset of X_s, and thus continuous by 0.1. We finish as in [8], 2.1.

The case Σ_2. As above we can suppose Y to be metrizable. The topology of uniform convergence in $\mathbf{N}$ has a basis of neighbourhoods of zero which are closed in the topology of pointwise convergence in $\mathbf{N}$ (since the F-norm defining the uniform convergence is lower semi-continuous on $C_s(\mathbf{N}, Y)$). So by the argument applied in 1.1 we find $\hat{x}_{n_k}$ such that $\sum_{k=1}^{\infty} \hat{x}_{n_k}$ is subseries-convergent in X_s, and $\mu(\hat{x}_{n_k}) \not\to 0$ in $C_u(\mathbf{N}, Y)$. This is impossible by [8] 3.1 (B).

2.2. PROPOSITION.

(A) *Let X_s be (Σ_1) or (Σ_2), K a compact metric space, and ν: $X_s \to C_d(K, Y)$ continuous. Then ν: $X_s \to C_u(K, Y)$ is continuous.*

(B) *Let X_s be (Σ_1) or (Σ_2), K a sequentially compact space, and ν: $X_s \to C_s(K, Y)$ continuous. Then ν: $X_s \to C_u(K, Y)$ is continuous.*

Proof. Let $t \in K$ and let D be a dense subset of K determining the topology in $C_d(K, Y)$. One can find $(u_n) \subset D$, $u_n \to t$; then

$$\nu_n(x) = \nu(x)(u_n) \to \nu_0(x) = \nu(x)(t) \quad (n \to \infty).$$

Hence by the preceding theorem ν_0 is (X_s, Y)-continuous. This means also that ν: $X_s \to C_s(K, Y)$ is continuous. We complete the proof of (A) applying the proof of [8], 2.3, which at the same time proves (B).

Let Y be a Hausdorff locally convex space, Y' its dual, Y^* its algebraic dual, and $\mathscr{K}$ the family of subsets in Y' which are relatively sequentially compact in Y^*_σ.

2.3. COROLLARY. *Let X_s be (Σ_1) or (Σ_2). If ν is (X_s, Y_σ)-continuous, then ν: $X_s \to Y$ is γ-continuous, where γ is the topology of $\mathscr{K}$-convergence.*

It follows (compare [7], 1.4, 1.5), for instance, that if Y is separable (resp. strongly separable), the continuity of ν in the weak topology defined by the subset of Y' which is dense in every set of a compact cover $\mathscr{S}$ of Y'_σ (resp. in every set of a bounded cover(3) of Y'_σ) implies the $\varkappa(Y, Y')$-continuity (resp. $\beta(Y, Y')$-continuity) of ν.

(3) It is sufficient to suppose $\mathscr{S}$ to be merely a "linear compact (resp. bounded) cover" in Y'_σ, i.e. to suppose that each $E \in \mathscr{S}$ is compact (resp. bounded) in Y'_σ, and the linear span of $\bigcup \mathscr{S}$ equals Y'. A moment's reflection shows that a "linear compact cover" is a straightforward generalization of the notion of a fundamental set of functionals ([13], [4]) for Banach spaces.

2.4. THEOREM. *Let X_s be (Σ_1) separable or (Σ_2), K a compact space, D a sequentially dense subset of K, and $C_d(K, Y)$ a space $C(K, Y)$ endowed with the topology of pointwise convergence in D. If $\nu\colon X_s \to C_d(K, Y)$ is continuous, then $\nu\colon X \to C_u(K, Y)$ is continuous.*

Proof. Passing to the quotient of Y by the closure of zero if needed, we can suppose Y to be Hausdorff. By the reasoning applied in the proof of 2.2. (A) above, one finds that $\nu\colon X_s \to C_s(K, Y)$ is continuous. Then the proof of the case Σ_1 is the same as in [8] 2.5.

The case Σ_2. As in 2.1, the topology of $C_u(K, Y)$ has a basis of neighbourhoods which are closed in the topology of $C_s(K, Y)$. The continuity of $\nu\colon X_s \to C_s(K, Y)$ is obtained as above; then applying [8] 3.2 we complete the proof as in 2.1., the case Σ_2.

It follows (compare 1.5) that if X_s is (Σ_1) separable or (Σ_2), and Y a Hausdorff locally convex space, then the (X_s^*, Y_σ)-continuity of ν implies its $(X_s, Y_\varkappa)$-continuity.

Let Y be a Hausdorff locally convex space, T a locally compact space countable at infinity, and $\mathscr{K}(T, Y)$ the space of Y-valued continuous functions with compact support on T endowed with its natural inductive limit topology. We will denote by $\mathscr{K}_s(T, Y)$ a set $\mathscr{K}(T, Y)$ equiped with the topology of pointwise convergence in T.

2.5. THEOREM. *Let X_s be (Σ_1) separable or $(\Sigma_1 \& \Sigma_2)$. $\nu\colon X_s \to \mathscr{K}_s(T, Y)$ is continuous iff $\nu\colon X_s \to \mathscr{K}(T, Y)$ is continuous.*

If we take into account 1.6 and 2.4, the proof is the same as in [10], 2.2.

One could also obtain theorems which correspond exactly to [10], 2.3, 2.4.

Applying 1.4, one immediately infers that if X_s is (Σ_2), then $\nu\colon X_s \to L^p(\eta)$, $0 < p < \infty$, which is ω-continuous (where ω is the topology of convergence in η-measure on every set of finite η-measure), is continuous in the F-norm topology of $L^p(\eta)$ (in fact, we can even take as the arrival space a generalized Δ_2 Orlicz space, see [9], [11]).

However, we will present an alternative proof, using the truncation method of Turpin (cf. [9], Remark 1), firstly because this method is interesting in itself, and secondly because it will allow us to state the analog of the result mentioned above in the case of Bochner p-integrable functions; and these spaces are not (O)-spaces in general.

Let T be a set, $\mathscr{B}$ a σ-algebra of subsets of T, $(Y, \|\cdot\|)$ a Banach space, and $\eta\colon \mathscr{B} \to [0, \infty]$ a (countably additive) measure. The case $1 \leqslant p < \infty$ being well known, we recall only that we denote by $L^p(\eta; Y)$, $0 < p < 1$, a set of (Bochner) measurable Y-valued functions over $(T, \mathscr{B}, \eta)$ for which

$$\int \|f(t)\|^p d\eta = \|f\|_p < \infty.$$

One can easily show that the Lebesgue-dominated convergence theorem (LDCT) is still valid. If Y is a field of scalars we denote $L^p(\eta; Y)$ simply by $L^p(\eta)$. The topology ω of convergence in η-measure on every set of finite η-measure is coarser than the F-norm topology on $L^p(\eta; Y)$, $0 < p < \infty$.

2.6. THEOREM. *Let X_s be (Σ_1) or (Σ_2). $\nu\colon X_s \to L^p(\eta; Y)$ is continuous iff it is ω-continuous.*

Proof. The case Σ_1. (1) (Turpin truncation method.) Let η be finite, that is, $\eta(T) < \infty$. Put

$$f^N = \begin{cases} f(t) & \text{for } t \in \{t\colon \|f(t)\| \leqslant N\}, \\ 0 & \text{elsewhere}, \end{cases}$$

$$\nu^N(x) = (\nu(x))^N, \qquad N = 1, 2, \ldots$$

$\nu^N\colon X_s \to L^p(\eta; Y)$ is $\|\cdot\|_p$-continuous. In fact, take $x_n \to x$ in X_s. Then $\nu^N(x_n) \to \nu^N(x)$ $(n \to \infty)$ in η-measure, and since $\nu^N(x_n)$, $n \in \boldsymbol{N}$, are uniformly bounded by N, this sequence is norm convergent. On the other hand, $\nu^N(x)(t) \to \nu(x)(t)$ $(N \to \infty)$ for each $t \in T$, and so by the LDCT $\nu^N(x) \to \nu(x)$ (as functions in $L^p(\eta; Y)$) in the norm topology; and this is true for each $x \in X_s$. Thus $\nu\colon X_s \to L^p(\eta; Y)$ is of Baire's first class and hence continuous by Lemma 0.1.

(2) (Mazur truncation method.) Let η be σ-finite, i.e., there exists a sequence $(T_n) \subset \mathscr{B}$, $T_1 \subset T_2 \subset \ldots$, $\eta(T_n) < \infty$, $\bigcup_{n=1}^{\infty} T_n = T$. Put

$$\nu_N(x) = \nu(x)\chi(T_N), \qquad N = 1, 2, \ldots,$$

where $\chi(T_N)$ is the characteristic function of a set T_N. $\nu_N(x)(t) \to \nu(x)(t)$ $(N \to \infty)$ for each $t \in T$, and by the LDCT $\nu_N(x) \to \nu(x)$ (as functions in $L^p(\eta; Y)$) in the $\|\cdot\|_p$-norm topology. We can apply 2.1.

(3) Let $x_n \to x_0$. $\{t\colon \|\nu(x_n)(t)\| \neq 0\}$ is σ-finite, hence $E = \bigcup_{n=1}^{\infty} \{t\colon \|\nu(x_n)(t)\| \neq 0\}$ is σ-finite. Since $\{t\colon \|\nu(x_0)\| \neq 0\}$ is η-almost contained in E, we can suppose that (x_n), $n = 0, 1, 2, \ldots$, belongs to L^p over $(E, E \cap \mathscr{B}, \eta|_E)$. Thus we can suppose η to be σ-finite.

The case Σ_2. (1) By (3) above we can suppose η to be σ-finite, and then the norm $\|\cdot\|_p$ is lower semi-continuous over $(L^p(\eta; Y), \omega)$ (see [9]). If ν is not $\|\cdot\|_p$-continuous, by the standard argument (cf. 1.1) we find a series $\sum_{n=1}^{\infty} y_n$ of elements of $L^p(\eta; Y)$ which is ω-subseries convergent and not $\|\cdot\|_p$-subseries convergent. We will show that this is impossible.

(2) η is σ-finite; we shall prove what follows:

Let $\mathscr{R}$ be a σ-ring of sets, and $\mu\colon \mathscr{R} \to L^p(\eta; Y)$ a ω-countably additive set function. Then μ is $\|\cdot\|_p$-countably additive.

Suppose that the statement holds if η is finite; then if we define μ_N as in the case Σ_1 (2), $\mu_N\colon \mathscr{R}\to L^p(\eta;\ Y)$ are $\|\cdot\|_p$-countably additive set functions. Hence by the Nikodym theorem ([8] 3.1) μ is $\|\cdot\|_p$-countably additive. Thus we can suppose η to be finite.

(3) $\eta(T)<\infty$; let $|\cdot|$ be the F-norm defining ω; $\mu\colon \mathscr{R}\to\big(L^p(\eta;\ Y), |\cdot|\big)$ is countably additive. Thus the corresponding submeasure majorant $\bar{\mu}$ is order-continuous ([5], § 2, § 5); hence $(\mathscr{R}, \bar{\mu})$ is a Σ-space ([8], 1.4). Defining μ^N as in Σ_1 the case (1), we will prove by the argument employed there that $\mu\colon (\mathscr{R}, \bar{\mu})\to L^p(\eta;\ Y)$ is of Baire's first class, and hence continuous ([8], 1.2). This means precisely that μ is $\|\cdot\|_p$-countably additive since $\bar{\mu}$ is order-continuous.

Theorem 1.7 covers in particular $L^p(\eta)$ spaces for $1<p<\infty$ since, being reflexive, they are $\varkappa$-spaces. We will show, by a direct method, that this result is still valid for $L^1(\eta)$.

Let η be finite. It is known that if we take L^1 over $(T, \mathscr{B}, \eta)$, the correspondence between $f\in L^1$ and measures of the density f as elements of ca$(\mathscr{B})$ [2] is isometrically isomorphic. Therefore L^1 can be identified via this correspondence with a set of η-absolutely continuous measures on $\mathscr{B}$. Now, by an application of the criterion of Bartle, Dunford and Schwartz ([2], Theorem 1.3), $Y_N=\{f^N\colon f\in L^1\}$ (f^N are defined as in 2.6 the case Σ_1 (1)) is relatively weakly compact in L^1 (for Y_N is bounded and if f_i, $i\in I$, belong to Y_n, then $|\lambda_i(E)| = \big|\int_E f_i\,d\eta\big| \leqslant \int_E |f_i|\,d\eta \leqslant N\eta(E)\to 0$ with $\eta(E)\to 0$ uniformly in $i\in I$).

2.7. THEOREM. *Let X_s be (Σ_1) separable or (Σ_2), $(T, \mathscr{B}, \eta)$ an (arbitrary) positive measure space, and α a linear topology on $L^1(\eta)$ consistent with the weak topology of $L^1(\eta)$. If $\nu\colon X_s\to L^1(\eta)$ is α-continuous, then it is $\big(X_s, L^1(\eta)\big)$-continuous.*

Proof. Suppose first that $\eta(T)<\infty$ and let $L^1 = L^1(\eta)$, $L^1_\sigma = L^1(\eta)$ endowed with its weak topology σ. Since Y_N, $N=1,2,\ldots$, are relatively compact in L^1_σ, the topologies $\gamma=\inf(\alpha,\sigma)$ and σ coincide on Y_N Define ν^N as in 2.6, and by the argument used there $\nu^N(x)\to\nu(x)$ $(N\to\infty)$ in L^1, and hence in L^1_σ, for each $x\in X_s$. Therefore $\nu\colon X_s\to L^1_\sigma$ is of Baire's first class, and thus continuous (the reasoning as in 2.6 the case Σ_1 (1) with respect to each pseudo-norm separately). Theorem 1.5 implies that ν is (X_s, L^1)-continuous. Now, 2.6 implies the result for an arbitrary η.

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Received March 1, 1973 (656)