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A real variable characterization of H^p

by

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Abstract. An explicit representation theorem for functions in $H^p(\boldsymbol{R})$ for $p < 1$, is given by means of a purely real variable construction. From this representation many of the classical results concerning these spaces, as well as representations of their duals, follow easily.

The purpose of this note is to give an explicit representation theorem for functions in $H^p(\boldsymbol{R})$ for $p \leqslant 1$. This is done by a purely real variable construction. Specifically, we let $0 < p \leqslant 1$ and define a *p-atom* as a function b having support in an interval I and satisfying:

$$|b(x)| \leqslant \frac{1}{|I|^{1/p}} \quad \text{and} \quad \int_{-\infty}^{\infty} b(x)x^k\,dx = 0$$

for all $0 \leqslant k \leqslant \left[\frac{1}{p}\right] - 1$ ($[x]$ denotes the integral part of x). We then have:

THEOREM I. *A distribution f is in $H^p(\boldsymbol{R})$* (¹) *if and only if there exist $a_i \in \boldsymbol{R}$ and $b_i(x)$ p-atoms, $i = 0, 1, 2, \ldots$, such that*

$$f(x) = \sum_{i=0}^{\infty} a_i b_i(x) \ (^2)$$

and

$$A\,\|f\|_{H^p}^p \leqslant \sum_{0}^{\infty} |a_i|^p \leqslant B\,\|f\|_{H^p}^p, \tag{1}$$

where A, B depend only on p.

This representation theorem for H^1 functions was obtained by C. Herz in the martingale case (see [4]) and by C. Fefferman in our case. Fefferman observed that this result is an easy consequence of the duality between H^1 and B.M.O.; and, actually, is equivalent to the duality.

(¹) Our space $H^p(\boldsymbol{R})$ consists of boundary distributions of the real parts of the traditional Hardy spaces.

(²) All sums are to be interpreted in the sense of distributions.

The proof sketched here is an explicit decomposition of f into p-atoms and yields simultaneously the duality results for all $0<p\leqslant 1$. This decomposition is roughly the same as in the corresponding martingale case (see [4]).

We start as in [1] by defining the maximal functions.

$$R^{(k)}(f)(x) = \sup_{\varepsilon\in\mathbf{R}}\left|\int_0^1 [t(1-t)]^k f(x+\varepsilon t)\,dt\right|. \quad (^3)$$

Simple integration by parts arguments show that

$$R^{(k)}(f)(x) = c_k \sup_{\eta\in C_0^{k+1}} \frac{\left|\int f(x-t)\eta(t)\,dt\right|}{\|\eta\|^{(k)}}$$

where η has compact support and $k+1$ continuous derivatives and

$$\|\eta\|^{(k)} = \int_{-\infty}^{\infty}\left|t^{2k+1}\frac{d^{k+1}}{dt^{k+1}}\left(\frac{\eta(t)}{t^k}\right)\right|dt.$$

This realization of $R^{(k)}$ enables us to use it as in [3] to obtain Calderón–Zygmund type of decompositions from which the inequality

$$\|f\|_{H^p}\leqslant A_p\|R^{(k)}(f)(x)\|_p$$

follows. For $k>1/p$ Fefferman and Stein [3] obtain the converse inequality by showing that the maximal non-tangential Poisson integral has L^p norm equivalent to that of $R^{(k)}(f)$. Henceforth we will take $\|R^{(k)}(f)\|_p$ (where k is the least integer $\geqslant 1/p$) as an equivalent norm on H^p.

In order to prove the left side of inequality (1) we let

$$r_k(t) = \begin{cases} \big(t(1-t)\big)^k, & 0<t<1,\\ 0, & \text{otherwise}\end{cases}$$

and first show that

$$\int \sup_{\varepsilon}\left|\int\frac{1}{\varepsilon}r_k\left(\frac{x-t}{\varepsilon}\right)b(t)\,dt\right|^p dx\leqslant c_{p,k} \tag{2}$$

for all p-atoms and $k\geqslant 1/p$. We can obviously assume that $b(t)$ has support

(3) It can be shown that if $f\in H^p$ and $k>1/p$ the integrals can be defined as follows: it is enough to consider first integrable f in $H^p(\mathbf{R})$, prove duality estimates for this class (this is proved here) and then pass to the limit.

in an interval $I=[-a,a]$ centered at 0, and let $\bar I = 5I\equiv[-5a,5a]$. It is clear that

$$R^{(k)}(b)(x)\leqslant\frac{c_k}{|I|^{1/p}} \tag{3}$$

for all x; however, if x lies outside $\bar I$ it is enough to consider the integrals $\int\frac{1}{\varepsilon}r_k\left(\frac{x-t}{\varepsilon}\right)b(t)\,dt$ for $|\varepsilon|>\frac{|x|}{2}$. By the definition of a p-atom, the absolute value of this last integral equals

$$\left|\frac{1}{\varepsilon}\int_I\left(r_k\left(\frac{x-t}{\varepsilon}\right)-\sum_{j=0}^{[\frac{1}{p}]-1}\frac{1}{j!}r_k^{(j)}\left(\frac{x}{\varepsilon}\right)\left(\frac{t}{\varepsilon}\right)^j\right)b(t)\,dt\right|$$

$$\leqslant\frac{C_k}{\varepsilon}\int_I\left|\frac{t}{\varepsilon}\right|^{[\frac{1}{p}]}|b(t)|\,dt\leqslant\frac{C_k'|I|^{[\frac{1}{p}]+1-\frac{1}{p}}}{|x|^{[\frac{1}{p}]+1}}.$$

Thus,

$$\int_{|x|>8|I|}|R^{(k)}(b)(x)|^p\,dx\leqslant c_{p,k}.$$

This, together with (3) implies (2). Moreover, any function of the form $\sum\alpha_i b_i$ clearly belongs to H^p if $\sum|\alpha_i|^p<\infty$.

In order to show the right hand side of inequality (1) we need the following refinement of the Calderón–Zygmund decomposition (see [2]):

LEMMA. *Let $f\in H^p$ and $\lambda>0$, there exist g_λ, a_λ^i such that*

$$f = g_\lambda+\sum_i a_\lambda^i,$$

where $|g_\lambda|\leqslant c\lambda$, a_λ^i has support in I_λ^i and satisfies

$$\int a_\lambda^i(x)x^k\,dx = 0\quad\text{for } k\leqslant\left[\frac{1}{p}\right]-1.$$

Moreover, the intervals I_λ^i are disjoint and

$$\bigcup_i I_\lambda^i = \{x\colon R^{(k)}(f)(x)>\lambda\}.$$

Theorem I follows by taking $\lambda=2^k$, $k=0,\pm1,\pm2,\ldots$, and decomposing f as above: By letting

$$f = g_k+\sum_i a_k^i$$

it is clear that $g_k(x)\to f(x)$ a.e. as $k\to+\infty$ and $g_k(x)\to 0$ a.e. $k\to-\infty$; thus, we can write

$$f=\sum_{-\infty}^{\infty}(g_{k+1}-g_k)=\sum_k\Big(\sum_i a_k^i-\sum_j a_{k+1}^j\Big).$$

Since each interval I_{k+1}^j is contained in one of the intervals I_k^i we can define

$$\beta_k^i(x)=a_k^i(x)-\sum_{j:I_{k+1}^j\subseteq I_k^i}a_{k+1}^j.$$

$\beta_k^i(x)=g_{k+1}(x)-g_k(x)$ on I_k^i and, thus, is bounded by $c2^{k+2}$. It is also clear that

$$b_k^i(x)=\frac{1}{|I_k^i|^{1/p}c2^{k+2}}\beta_k^i(x)$$

is a p-atom and $f=\sum_k\sum_i \alpha_k^i b_k^i(x)$ where $\alpha_k^i=c2^{k+2}|I_k^i|^{1/p}$. But

$$\sum_k\sum_i|\alpha_k^i|^p=\sum_{-\infty}^{\infty}c^p2^{kp}|\{R^{(k)}(f)(x)>2^k\}|\leqslant\int[R^{(k)}(f)(x)]^p\,dx,$$

which proves Theorem I for $f\in L^1\cap H^p$.(4)

Let $p_I(f)(x)$ denote the polynomial of degree $\left[\frac{1}{p}\right]-1$, having the property

$$\int_I\big(f(x)-P_I(f)(x)\big)x^i=0\qquad\text{for } i\leqslant\left[\frac{1}{p}\right]-1.$$

The representation of the dual of H^p as a Lipschitz space follows from

COROLLARY.

$$\int f(x)b(x)\,dx\leqslant c_p\Big(\int|R^{(k)}(f)(x)|^p\,dx\Big)^{1/p}\sup_I\frac{1}{|I|^{1/p}}\int_I|b(x)-p_I(b)|\,dx.$$

This inequality follows immediately from Theorem I since it is trivial to check it for atoms having support in a fixed integral I.

The duality result is obtained from the fact that

$$\sup_I\frac{1}{|I|^{1/p}}\int_I|b(x)-p_I(b)|\,dx$$

is a norm equivalent to that of the Lipschitz space Λ_α, $\alpha=\frac{1}{p}-1$ (see [5]); if $p=1$ this is the B.M.O. norm.

(4) A limiting argument using the Corollary proves it in general.

Following a suggestion of C. Fefferman we would like to complete some results stated in [3]. These observations were made in a discussion with G. Weiss. We restrict our attention to $p>\frac{1}{2}$, the other cases being technically slightly more complicated but involving no new ideas.

We have

THEOREM II. *Let $k(x)$ be differentiable for $x\neq 0$ with support in $|x|\leqslant 1$ and satisfying*

$$|k'(x)|\leqslant\frac{1}{|x|^{2+\alpha}}\quad\textit{and}\quad|\hat{k}(\xi)|\leqslant\frac{1}{1+|\xi|^\beta}\,(5),$$

$0\leqslant\alpha,\beta$ *and* $\frac{1}{2}>\beta\geqslant\frac{1}{2}\frac{\alpha}{1+\alpha}$. *Then*

$$\|k*f\|_{H^p}\leqslant c\|f\|_{H^p}$$

for

$$\infty>p\geqslant p_0>\frac{1}{2}\quad\textit{where}\quad\frac{1}{p_0}-\frac{1}{2}=\frac{(\frac{3}{2}+\alpha)}{\alpha+\beta}\beta.$$

EXAMPLE. The operator

$$T_\lambda(f)=\lim_{\varepsilon\to 0}\int_{\varepsilon<|y|\leqslant 1}f(x-y)\frac{e^{i|y|^{a'}}}{|y|^{1+\lambda}}\,dy$$

satisfies the conditions of the theorem for $\alpha=\lambda-a'$ and $\beta=\lambda(1-a)+\frac{a}{2}$, where $\frac{1}{a}+\frac{1}{a'}=1$. The result reads

$$\|T_\lambda(f)\|_{H^p}\geqslant c_p\|f\|_{H^p}\quad\text{for } p\geqslant p_0$$

when

$$\frac{1}{p_0}-\frac{1}{2}=\frac{\beta}{a}.$$

We observe here that, if $p_0<1$ $\frac{b}{a}>\frac{1}{2}$ and $1+\lambda<1$, the kernel of T_λ belongs to L^1. The result we have is, however, best possible (i.e. it is false for $p<p_0$).

We now sketch the proof of Theorem II. It is obviously enough to show that H^p is mapped into L^p (this will show that both $k*f$ and $(k*f)^\sim=k*\tilde{f}$ belong to L^p).

We consider first a p-atom b supported in I centered at 0, where $|I|<1$, and suppose $\gamma=\dfrac{2p_0-1}{2p_0-1+\alpha p_0}<1$.

(5) $\hat{k}$ denotes the Fourier transform of the principal value distribution defined by k.

Then, for $|x| > 2|I|^\gamma$,

$$|k*b(x)| = \left|\int k(x-y)\,b(y)\,dy\right| = \left|\int [k(x-y)-k(x)]\,b(y)\,dy\right| \leqslant c\,\frac{|I|^{2-1/p_0}}{x^{2+\alpha}}$$

and

$$\int\limits_{|x|>2|I|^\gamma} |k*b(x)|^{p_0}\,dx \leqslant c.$$

If $|x| < 2|I|^\gamma$ we write $\hat{k}(\xi) = |\xi|^{-\beta}|\xi|^{\beta}\hat{k}(\xi)$ and using standard results about fractional integrals we see that the operator induced by k maps L^q into L^2, for $\dfrac{1}{q} = \dfrac{1}{2}+\beta$. Using this and Hölder's inequality we have

$$\int\limits_{|x|\leqslant 2|I|^\gamma} |k*b|^{p_0}\,dx \leqslant c\,|I|^{\gamma\left(1-\frac{p_0}{2}\right)}\left(\int |k*b|^2\right)^{\frac{p_0}{2}} \leqslant c\,|I|^{\gamma\left(1-\frac{p_0}{2}\right)}\left(\int |b|^q\right)^{\frac{p_0}{2}} \leqslant c.$$

If $|I| > 1$ then $(k*b)(x) = 0$ for $|x| \geqslant 2|I|$ and

$$\int\limits_{|x|<2|I|} |k*b|^{p_0}\,dx \leqslant c\,|I|^{\left(1-\frac{p_0}{2}\right)}\left(\int |k*b|^2\right)^{\frac{p_0}{2}} \leqslant c\,|I|^{\left(1-\frac{p_0}{2}\right)}\left(\int |b|^2\right)^{\frac{p_0}{2}} \leqslant c.$$

References

[1] R. Coifman and Guido Weiss, *Maximal functions and H^p spaces defined by Ergodic transformations*, Proc. Nat. Acad. Sci. USA 70 (1973), pp. 1761–1763.

[2] C. Fefferman, N. M. Rivière, Y. Sagher, *Interpolation between H^p spaces. The real method*, to appear.

[3] — E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), pp. 137–193.

[4] C. Herz, *H^p spaces of martingales*, $0 < p \leqslant 1$, to appear in Zeit. für Wahrscheinlichkeitstheorie.

[5] G. N. Myers, *Mean oscillation over cubes and Hölder continuity*, Proc. A. M. S. 15(1964), pp. 717–721.

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Cerrection to "Convergence of Baire measures" Studia Mathematica 27(1966), pp. 251-268

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In my paper [2], Theorem 4 is false. It was quoted from [5], p. 223 (English), p. 96 (Russian). There exist separable metric spaces S on which every Borel probability measure is tight, but such that there are relatively compact sets of probability measures (for the usual weak topology) which are not uniformly tight. One example was given by R. Davies [1]. D. Preiss [4] showed that there are many such spaces, in particular the space of rational numbers, or any separable metric space which is a Borel subset, but not a G_δ, in its completion.

Theorem 4 was used in [2] only for a weakly convergent sequence, where it is true (LeCam [3], p. 222, Theorem 4). The proof is not difficult.

Another correction: the notation used near the end of p. 267 in [2] disagrees with the usual notations as in [3], [5]; σ should be replaced by τ.

References

[1] Roy O. Davies, *A non-Prohorov space*, Bull. London Math. Soc. 3 (1971), pp. 341–342.

[2] R. M. Dudley, *Convergence of Baire measures*, Studia Math. 27 (1966), pp. 251–268.

[3] Lucien LeCam, *Convergence in Distribution of Stochastic Processes*, Univer. of Calif. Publs. in Statistics 2 no. 11, 1957, pp. 207–236.

[4] D. Preiss, *Metric spaces in which Prohorov's theorem is not valid*, Z. Wahrscheinlichkeitsth. 27 (1973), pp. 109–116.

[5] V. S. Varadarajan, *Measures on topological spaces*, Amer. Math. Soc. Transl. 2 (48), pp. 161–228; Mat. Sbornik 55 (97) (1961), pp. 35–100.