A real variable characterization of $H^p$

by

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Abstract. An explicit representation theorem for functions in $H^p(R)$ for $p < 1$, is given by means of a purely real variable construction. From this representation many of the classical results concerning these spaces, as well as representations of their duals, follow easily.

The purpose of this note is to give an explicit representation theorem for functions in $H^p(R)$ for $p < 1$. This is done by a purely real variable construction. Specifically, we let $0 < p < 1$ and define a $p$-atom as a function $b$ having support in an interval $I$ and satisfying:

$$|b(x)| \leq \frac{1}{|I|^p} \quad \text{and} \quad \int_{-\infty}^{\infty} b(x) x^k dx = 0$$

for all $0 \leq k \leq \left\lfloor \frac{1}{p} \right\rfloor - 1$ (the integral part of $k$). We then have:

**Theorem I.** A distribution $f$ is in $H^p(R)$ (1) if and only if there exist $a_i \in R$ and $b_i(x)$ $p$-atoms, $i = 0, 1, 2, \ldots$, such that

$$f(x) = \sum_{i=0}^{\infty} a_i b_i(x)$$

and

$$A \|f\|_p \leq \sum_{i=0}^{\infty} |a_i|^p \leq B \|f\|_p^p,$$

where $A, B$ depend only on $p$.

This representation theorem for $H^1$ functions was obtained by C. Herz in the martingale case (see [4]) and by C. Fefferman in our case. Fefferman observed that this result is an easy consequence of the duality between $H^1$ and B.M.O.; and, actually, is equivalent to the duality.

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(1) Our space $H^p(R)$ consists of boundary distributions of the real parts of the traditional Hardy spaces.

(2) All sums are to be interpreted in the sense of distributions.
The proof sketched here is an explicit decomposition of $f$ into $p$-atoms and yields simultaneously the duality results for all $0 < p \leq 1$. This decomposition is roughly the same as in the corresponding martingale case (see [4]).

We start as in [1] by defining the maximal functions.

$$ E^{0}(f) (a) = \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \left| t (1 - t)^{\alpha} f (x + et) dt \right|, \quad (1') $$

Simple integration by parts arguments show that

$$ E^{0}(f) (a) = c_{k} \sup_{t \in \mathbb{R}} \frac{1}{t^{k+1}} \int_{\mathbb{R}} \frac{1}{t^{k+1}} \left| \eta (x) \right| dx, $$

where $\eta$ has compact support and $k + 1$ continuous derivatives and

$$ \|\eta\|_{p} = \int_{\mathbb{R}} \int_{0}^{+\infty} \frac{1}{t^{k+1}} \left| \frac{\eta (x)}{t^{k+1}} \right| dx dt. $$

This realization of $E^{0}$ enables us to use it as in [3] to obtain Calderón-Zygmund type of decompositions from which the inequality

$$ \|f\|_{L^{p}} \leq A_{p} \|E^{0}(f) (a)\|_{p}, $$

follows. For $k > 1/p$ Fefferman and Stein [3] obtain the converse inequality by showing that the maximal non-tangential Poisson integral has $L^{p}$ norm equivalent to that of $E^{0}(f)$. Henceforth we will take $\|E^{0}(f)\|_{p}$ (where $k$ is the least integer $\geq 1/p$) as an equivalent norm on $H^{p}$.

In order to prove the left side of inequality (1) we let

$$ r_{k} (t) = \left\{ \begin{array}{ll} t (1 - t)^{k}, & 0 < t < 1, \\
0, & \text{otherwise} \end{array} \right. $$

and first show that

$$ \int \sup_{a} \left| \int_{a}^{1} \frac{1}{t^{k+1}} r_{k} \left( \frac{a - t}{t} \right) b (t) dt \right| \leq c_{p,h} $$

for all $p$-atoms and $k > 1/p$. We can obviously assume that $b (t)$ has support

in an interval $I = [-a, a]$ centered at 0, and let $I^{*} = [a, 5a]$. It is clear that

$$ E^{0}(b) (x) \leq \frac{c_{h}}{|I^{*}|^{\alpha}} $$

for all $a$; however, if $x$ lies outside $I^{*}$ it is enough to consider the integrals

$$ \int_{\mathbb{R}} \frac{1}{t^{k+1}} r_{k} \left( \frac{a - t}{t} \right) b (t) dt \leq \frac{c_{h}}{|I^{*}|^{\alpha}} |x| \frac{1}{2}. $$

By the definition of a $p$-atom, the absolute value of this last integral equals

$$ \frac{1}{e^{\alpha}} \left( \frac{1}{2} \right)^{\alpha} \left( \frac{1}{2} \right)^{\alpha} e^{\alpha} \left( \int_{|I^{*}|^{\alpha}} b (t) dt \right) \leq c_{h} |I^{*}|^{\alpha} \|b\|_{L^{p}} \leq c_{p,h}. $$

Thus,

$$ \int_{|I^{*}|^{\alpha}} |E^{0}(b) (x)|^{p} dx \leq c_{p,h}. $$

This, together with (3) implies (2). Moreover, any function of the form $\sum_{a} a_{b}$ clearly belongs to $H^{p}$ if $\sum_{a} a_{b}^{p} < \infty$.

In order to show the right hand side of inequality (1) we need the following refinement of the Calderón-Zygmund decomposition (see [2]):

**Lemma.** Let $f \in H^{p}$ and $\lambda > 0$, there exist $a_{1}, a_{1}'$ such that

$$ f = g_{1} + \sum_{a_{1}} a_{1}', $$

where $|a_{1}| < \xi$, $a_{1}'$ has support in $I_{1}$ and satisfies

$$ \int_{I_{1}} a_{1}' (x) u dx = 0 \quad \text{for} \quad k < \left( \frac{1}{p} \right)^{1 - \lambda}. $$

Moreover, the intervals $I_{1}$ are disjoint and

$$ \bigcup_{I_{1}} = \{ x : E^{0}(f) (a) > \lambda \}. $$

Theorem I follows by taking $\lambda = 2^{p}, k = 0, \pm 1, \pm 2, \ldots$, and decomposing $f$ as above: By letting

$$ f = g_{0} + \sum_{a_{0}} a_{0}, $$

where $|a_{0}| < \xi$, $a_{0}$ has support
it is clear that \( g_k(x) \to f(x) \) a.e. as \( k \to \pm \infty \) and \( g_k(x) \to 0 \) a.e. \( k \to \infty \); thus, we can write

\[
 f = \sum_{k=-\infty}^{\infty} (g_{k+1} - g_k) = \sum_k \int_{I_k} a^*_k \, dx.
\]

Since each interval \( I_{k+1} \) is contained in one of the intervals \( I_k \), we can define

\[
 \beta_k^* (x) = a_k^* (x) = \sum_{s \in S_{k+1} \cap I} a_s^* \beta_s^* (x).
\]

\( \beta_k^* (x) = g_{k+1}(x) - g_k(x) \) on \( I_k \), and hence, is bounded by \( c2^{k+1} \). It is also clear that

\[
 b_k^*(x) = \frac{1}{|I_k|} \int_{I_k} \beta_k^*(x) \, dx
\]

is a \( p \)-atom and \( f = \sum_k \int_{I_k} a_k^* \, dx \) where \( a_k^* \) is \( c2^{k+1} |I_k|^{1/p} \). But

\[
 \sum_k |a_k|^p = \sum_k |a_k^*|^p = \sum_m |a_m|^{2p} \left| \int |E_m^p(f)(x)|^2 \, dx \right| \leq \int |E_m^p(f)(x)|^p \, dx;
\]

which proves Theorem I for \( f \in L^{p \cap \mathcal{H}^p} \). (*)

Let \( P_1(f)(x) \) denote the polynomial of degree \( \left[ \frac{1}{p} \right] - 1 \), having the property

\[
 \int \left| f(x) - P_1(f)(x) \right|^{k} \, dx = 0 \quad \text{for} \quad i \leq \left[ \frac{1}{p} \right] - 1.
\]

The representation of the dual of \( \mathcal{H}^p \) as a Lipschitz space follows from

**Corollary.**

\[
 \int f(x) b(x) \, dx \leq c_p \left( \int |E_m^p(f)(x)|^p \, dx \right)^{1/p} \sup_k \frac{1}{|I_k|^{1/p}} \int |\beta_k^*(x) - P_1(b)| \, dx.
\]

This inequality follows immediately from Theorem I since it is trivial to check it for atoms having support in a fixed interval \( I \).

The duality result is obtained from the fact that

\[
 \sup_k \frac{1}{|I_k|^{1/p}} \int |\beta_k^*(x) - P_1(b)| \, dx
\]

is a norm equivalent to that of the Lipschitz space \( A_1 \), \( a = \frac{1}{p} - 1 \) (see [5]); if \( p = 1 \) this is the B.M.O. norm.

(*) A limiting argument using the Corollary proves it in general.

Following a suggestion of C. Fefferman we would like to complete some results stated in [5]. These observations were made in a discussion with G. Weiss. We restrict our attention to \( p > \frac{1}{2} \), the other cases being technically slightly more complicated but involving no new ideas.

We have

**Theorem II.** Let \( k(x) \) be differentiable for \( x \neq 0 \) with support in \( |x| < 1 \) and satisfying

\[
 |k'(x)| \leq \frac{1}{|x|^{1+\epsilon}} \quad \text{and} \quad |k'(\xi)| \leq \frac{1}{1+|\xi|^p} \quad (\ast),
\]

\( 0 < \alpha, \beta \) and \( \frac{1}{2} > \beta > \frac{1}{2} \frac{\alpha}{1+\alpha} \).

Then

\[
 \|k \ast f\|_{L^p} \leq c_p \|f\|_{L^p}
\]

for

\[
 \infty > p > p_0 > \frac{1}{2} \quad \text{where} \quad \frac{1}{p} - \frac{1}{2} = \frac{\alpha}{\alpha+1+\alpha}. \quad \text{(*)}
\]

**Example.** The operator

\[
 T_1(f) = \lim_{\lambda \to \infty} \int \frac{f(x-y)}{|y|^{\lambda+1}} \, dy
\]

satisfies the conditions of the theorem for \( a = \lambda - 1 \) and \( b = \lambda(1-a) + \frac{a}{a+1} \), where \( \frac{1}{a} + \frac{1}{a'} = 1 \). The result reads

\[
 \|T_1(f)\|_{L^p} \geq c_p \|f\|_{L^p} \quad \text{for} \quad p > p_0
\]

when

\[
 \frac{1}{p} - \frac{1}{2} = \beta = \frac{\alpha}{\alpha+1}. \quad \text{(*)}
\]

We observe here that, if \( p_0 < 1 \frac{b}{a} < 1 \) and \( 1+\lambda < 1 \), the kernel of \( T_1 \) belongs to \( L^1 \). The result we have is, however, best possible (i.e. it is false for \( p < p_0 \)).

We now sketch the proof of Theorem II. It is obviously enough to show that \( \mathcal{H}^p \) is mapped into \( L^p \) (this will show that both \( k \ast f \) and \( (k \ast f)' = k \ast \tilde{f} \) belong to \( L^p \)).

We consider first a \( p \)-atom \( b \) supported in \( I \) centered at 0, where

\[
 |I| < 1, \quad \text{and} \quad \sup \gamma = \frac{2p_0 - 1}{2p_0 - 1 + \gamma p_0} < 1.
\]

\( \ast \) denote the Fourier transform of the principal value distribution defined by \( k \).
Then, for $|z| > 2|I|^{\gamma}$,

$$|k * b(z)| = \left| \int k(x-y) b(y) dy \right| = \int |k(x-y) - k(x)| b(y) dy \leq c \frac{|I|^{1-\gamma}}{a^{\delta + \alpha}}$$

and

$$\int_{|x| > 2|I|^{\gamma}} |k * b(x)|^{p_2} dx \leq c.$$

If $|z| < 2|I|^{\gamma}$ we write $k(x) = |x|^{-\gamma} |x|^{\gamma} \hat{k}(\xi)$ and using standard results about fractional integrals we see that the operator induced by $k$ maps $L^2$ into $L^1$, for $\frac{1}{q} = \frac{1}{2} + \beta$. Using this and Hölder's inequality we have

$$\int_{|x| > 2|I|^{\gamma}} |k * b(x)|^{p_2} dx \leq c |I|^{1-\gamma} \left( \int |k * b(x)|^{\frac{p_2}{2}} dx \right)^{\frac{2}{p_2}} \leq c |I|^{1-\gamma} \left( \int |k(x)|^{\frac{p_2}{2}} dx \right)^{\frac{2}{p_2}} \leq c.$$

If $|z| > 1$ then $(k * b)(z) = 0$ for $|z| \geq 2|I|$ and

$$\int_{|z| \leq 2|I|} |k * b(x)|^{p_2} dx \leq c |I|^{1-\gamma} \left( \int |k * b(x)|^{\frac{p_2}{2}} dx \right)^{\frac{2}{p_2}} \leq c.$$

References


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Received August 29, 1973 (714)