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Small ideals of operators

by

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Abstract. Let S_p^{app} be the ideal containing all operators S with $\sum_1^\infty a_n(S)^p < \infty$ where $a_n(S)$ is the nth approximation number and $0 < p < \infty$. It is shown that $S_p^{\mathrm{app}}(E, F) = L(E, F)$ implies $\dim(E) < \infty$ or $\dim(F) < \infty$.

Let L denote the class of all (bounded linear) operators between arbitrary Banach spaces. An ideal of operators A is called *small* if $A(E, F) = L(E, F)$ implies that at least one of the Banach spaces E and F is of finite dimension.

The purpose of this paper is to show that the ideals S_p^{app} which are deduced from the approximation numbers are small for $0 < p < \infty$. This result answers a question raised by the author at the Colloquium on Nuclear Spaces and Ideals in Operator Algebras held in Warsaw (June 1969, cf. [3]).

For some definitions and notations see [5] and [6].

If $S \in L(E, F)$ then the *approximation numbers* are defined by

$$a_n(S) := \inf\{\|S-A\|: A \in L(E, F), \dim(A) < n\}.$$

Let

$$S_p^{\mathrm{app}} := \Big\{S \in L: \sum_1^\infty a_n(S)^p < \infty\Big\}$$

and

$$\Sigma_p^{\mathrm{app}}(S) := \Big(\sum_1^\infty a_n(S)^p\Big)^{1/p};$$

then $[S_p^{\mathrm{app}}, \Sigma_p^{\mathrm{app}}]$ is a complete quasinormed ideal ([4], 8.2).

THEOREM 1. *The ideal S_p^{app} is small for $0 < p < \infty$.*

Proof. Let E and F be Banach spaces with

$$S_p^{\mathrm{app}}(E, F) = L(E, F).$$

Then by the closed graph theorem there exists $\varrho > 0$ such that

$$\Sigma_p^{\mathrm{app}}(S) \leqslant \varrho \|S\| \quad \text{for all } S \in \boldsymbol{L}(E, F).$$

If E and F are of infinite dimension by Dvoretzky's theorem ([1], [2]) for $m = 1, 2, \ldots$ we find quotient spaces E/N_m and subspaces M_m of F which can be mapped onto l_2^m by isomorphisms X_m and A_m such that $\|X_m\| \|X_m^{-1}\| \leqslant 2$ and $\|A_m\| \|A_m^{-1}\| \leqslant 2$. Now we use the following notations: $I_m := $ identity map of l_2^m, $Q_m := $ canonical map of E onto E/N_m, $J_m := $ embedding map of M_m into F, $d_n := $ Kolmogorov numbers, $u_n := $ Bernstein numbers. From the diagram

$$E \xrightarrow{Q_m} E/N_m \underset{X_m^{-1}}{\overset{X_m}{\rightleftarrows}} l_2^m \xrightarrow{I_m} l_2^m \underset{A_m}{\overset{A_m^{-1}}{\rightleftarrows}} M_m \xrightarrow{J_m} F,$$

the injectivity of u_n ([6], Theorem 4.5), the surjectivity of d_n ([6], Theorem 5.1), and the inequality $u_n \leqslant d_n \leqslant a_n$ ([6], Theorems 8.1 and 8.2) we obtain

$$\begin{aligned} 1 = u_n(I_m) &\leqslant \|A_m\| u_n(A_m^{-1} I_m X_m) \|X_m^{-1}\| = \|A_m\| u_n(J_m A_m^{-1} I_m X_m) \|X_m^{-1}\| \\ &\leqslant \|A_m\| d_n(J_m A_m^{-1} I_m X_m) \|X_m^{-1}\| = \|A_m\| d_n(J_m A_m^{-1} I_m X_m Q_m) \|X_m^{-1}\| \\ &\leqslant \|A_m\| a_n(J_m A_m^{-1} I_m X_m Q_m) \|X_m^{-1}\| \quad \text{for } n = 1, \ldots, m. \end{aligned}$$

Consequently,

$$\begin{aligned} m^{1/p} &\leqslant \|A_m\| \Sigma_p^{\mathrm{app}}(J_m A_m^{-1} I_m X_m Q_m) \|X_m^{-1}\| \leqslant \varrho \|A_m\| \|J_m A_m^{-1} I_m X_m Q_m\| \|X_m^{-1}\| \\ &\leqslant \varrho \|A_m\| \|A_m^{-1}\| \|X_m\| \|X_m^{-1}\| \leqslant 4\varrho. \end{aligned}$$

Contradiction.

With similar arguments one can prove

THEOREM 2. *The ideals $\boldsymbol{S}_p^{\mathrm{gel}}$ and $\boldsymbol{S}_p^{\mathrm{kol}}$ are small for $0 < p < \infty$.*

On the other hand we have the following general result.

THEOREM 3. *Let $[\boldsymbol{A}, A]$ be an injective and surjective complete quasi-normed ideal. If*

$$\lim_m A(I_m) = \infty$$

then $\boldsymbol{A}$ is small.

References

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