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Centered operators

by

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Abstract. An operator T on a Hilbert space is called a centered operator in case the sequence $\ldots T^2(T^*)^2, TT^*, T^*T, (T^*)^2T^2, \ldots$ consists of mutually commuting operators. In this paper, all centered operators are completely described up to unitary equivalence and criteria are given for deciding when one is irreducible. Roughly speaking, it is shown that the most general centered operator is a direct sum of unilateral weighted shifts (backward, forward, or truncated) with commuting operator weights and a weighted translation operator acting on a space of vector-valued functions.

§ 1. Introduction. A computation reveals that if T is a weighted shift (unilateral or bilateral, forward or backward), then the operators in the sequence $\ldots, T^2(T^*)^2, TT^*, T^*T, (T^*)^2T^2, \ldots$ are mutually commuting operators. Following [10], we shall take this property as the defining property of a class of operators called *centered operators* and, answering the question raised in [10], we shall establish the extent to which this property determines the class of weighted shifts.

In the next section we show that the partial isometry in the polar decomposition of a centered operator is a power partial isometry (i.e., all of its positive powers are partial isometries). This fact coupled with the work of Halmos and Wallen [5] enables us to show that a centered operator can be written as a direct sum whose summands are either weighted shifts (with operator weights) or quasi-invertible centered operators. (Recall that a quasi-invertible operator is one with zero kernel and dense range.) We then show, in Section 3, that every quasi-invertible centered operator may be written as the direct sum of operators which are essentially weighted translation operators on spaces of vector-valued functions. In Section 4, we exhibit a complete set of unitary invariants for centered operators, while in Section 5, we derive conditions for a centered operator to be irreducible. Our concluding Section 6 is devoted to questions for future investigation.

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All Hilbert spaces considered here are complex and all operators are bounded and linear. The restriction of an operator T or of a family of operators $\mathscr{T}$ to an invariant subspace $\mathscr{M}$ will be denoted by $T|\mathscr{M}$ and $\mathscr{T}|\mathscr{M}$, resp.

§ 2. Classification. By definition, the *initial space* of an operator T is the orthogonal complement of its kernel and the *final space* of T is the closure of its range. We shall denote the projection onto the initial (resp., final) space of T by $E(T)$ (resp., $F(T)$), but for $k = 2, 3, \ldots$ we shall denote the projection onto the initial (resp., final) space of T^k by $E_k(T)$ (resp., $F_k(T)$). The following relations among these projections are easily verified:

(2.1) $$E(T^*) = F(T); \qquad E(T) = F(T^*),$$

(2.2) $$E(T) = E(T^*T); \qquad F(T) = F(TT^*),$$

and

(2.3) $$\text{if } j \leqslant k, \text{ then } E_j(T) \geqslant E_k(T) \text{ and } F_j(T) \geqslant F_k(T).$$

The unique representation of an operator T as the product UP where P is the non-negative square root of T^*T and U is a partial isometry such that $E(T) = E(U)$, will be called the *polar decomposition* of T (cf. [4]; problem 105). If T is a centered operator with polar decomposition UP, then the following two assertions are consequences of (2.2) and the spectral theorem:

(2.4) $$E_j(T) \text{ commutes with } F_k(T), j, k = 1, 2, \ldots,$$

and

(2.5) $$\text{for } k = 1, 2, \ldots, \text{ both } E_k(T) \text{ and } F_k(T) \text{ commute with } P.$$

THEOREM I. *Let T be a centered operator on a Hilbert space $\mathscr{H}$ and let UP be the polar decomposition of T. Then* a) *U is a power partial isometry,* b) *the operators in the sequence $\{(U^*)^k P U^k\}_{k=1}^{\infty}$ commute with one another, and* c) *the polar decomposition of T^n is $U^n\big[P(U^*PU)\ldots\big((U^*)^{n-1}PU^{n-1}\big)\big]$,* $n = 1, 2, \ldots$

Proof. We use induction to prove c) and the following variants of a) and b): a′) for each positive integer n, U^k is a partial isometry for all $k \leqslant n$; and b′) for each positive integer n, the operators in the sequence $\{(U^*)^k P U^k\}_{k=0}^{n-1}$ commute with one another.

Since there is nothing to prove in case $n = 1$, we pass to the induction step and assume that assertions a′), b′), and c) all hold for some positive integer $n \geqslant 2$.

To see that a′) holds at $n+1$, we note that the induction hypothesis c) implies that $F_n(U) = F_n(T)$ and that (2.4) implies that $F_n(U)$ commutes with $E(T) = E(U)$. Applying Lemma 2 of [5], one has that $U^{n+1} = U\cdot U^n$ is a partial isometry.

To prove that b′) holds at $n+1$, we need to show that for $k = 0, \ldots, n-1$, $(U^*)^k P U^k$ commutes with $(U^*)^n P U^n$. For this, observe that the induction hypothesis c) together with (2.5) imply that for $k \leqslant n$, $F_k(T) = F_k(U)$ and that $F_k(U)$ commutes with P. Using the induction hypothesis b′), we obtain the following equation which is valid for $1 \leqslant k \leqslant n-1$.

$$\begin{aligned}[(U^*)^k P U^k][(U^*)^n P U^n] &= [(U^*)^k P][F_k(U)][(U^*)^{n-k} P U^{n-k}]\,U^k \\ &= [(U^*)^k F_k(U)][(U^*)^{n-k} P U^{n-k}] P U^k = (U^*)^k[(U^*)^{n-k} P U^{n-k}] P U^k \\ &= [(U^*)^n P U^{n-k}][P F_k(U)]\,U^k = [(U^*)^n P U^{n-k}][U^k (U^*)^k] P U^k \\ &= [(U^*)^n P U^n][(U^*)^k P U^k].\end{aligned}$$

To complete the induction argument for b′), we must show that P commutes with $(U^*)^n P U^n$; but for this, it suffices to show that UPU^* and $(U^*)^{n-1} P U^{n-1}$ commute since $P[(U^*)^n P U^n] = U^*(UPU^*)\times \times\big((U^*)^{n-1} P U^{n-1}\big)U$, whereas, $[(U^*)^n P U^n]P = U^*[(U^*)^{n-1} P U^{n-1}][UPU^*]\,U$. Next, observe that the induction hypothesis c) implies that $F_k(U) = F_k(T)$ for $0 \leqslant k \leqslant n$, and so, by (2.5), we have $[(U^*)^k P U^k]^2 = (U^*)^k P F_k(U) P U^k = (U^*)^k P^2 [F_k(U)\, U^k] = (U^*)^k P^2 U^k$, which shows that $(U^*)^k P U^k$ is the unique non-negative square root of $(U^*)^k P^2 U^k$. Similarly, UPU^* is the unique non-negative square root of $UP^2U^* = TT^*$. Thus, to complete the induction on b′) we need only show that TT^* commutes with $(U^*)^{n-1} P^2 U^{n-1}$. The induction hypothesis c) implies that

$$\begin{aligned}\big[(TT^*)\big((U^*)^{n-1} P^2 U^{n-1}\big)\big]\big((T^*)^{n-1} T^{n-1}\big) &= (TT^*)\big((T^*)^n T^n\big) \\ &= \big((T^*)^n T^n\big)(TT^*) = [(U^*)^{n-1} P^2 U^{n-1}][(T^*)^{n-1} T^{n-1}](TT^*) \\ &= [(U^*)^{n-1} P^2 U^{n-1}](TT^*)[(T^*)^{n-1} T^{n-1}],\end{aligned}$$

so that TT^* and $(U^*)^{n-1} P^2 U^{n-1}$ commute on the final space of $(T^*)^{n-1} T^{n-1}$. Next, we note that the induction hypothesis implies that $\ker T^n = \ker U^n$. Since $U^n f = 0$, $f \in \mathscr{H}$, if and only if $U^{n-1} f \in \ker U$, and since $\ker U = \ker P$, it follows that $\ker(T^n) = \ker(PU^{n-1})$. Upon taking orthogonal complements and recalling (2.2), we see that the initial space of $(U^*)^{n-1} P^2 U^{n-1}$ is the initial space of $(T^*)^n T^n$. Invoking (2.2) again and using (2.3) and the fact that $(T^*)^{n-1} T^{n-1}$ is Hermitian, we see that the initial space of $(U^*)^{n-1} P^2 U^{n-1}$ is contained in the final space of $(T^*)^{n-1} T^{n-1}$. This last inclusion implies that TT^* commutes with $(U^*)^{n-1} P^2 U^{n-1}$ on all of $\mathscr{H}$. Thus, b′) holds at $n+1$.

Finally, to show that c) holds at $n+1$, we need to show that $T^{n+1} = U^{n+1}\big[P(U^*PU)\ldots\big((U^*)^n P U^n\big)\big]$ and that the kernel of T^{n+1} is the kernel of U^{n+1}. Using the induction hypothesis c) together with the fact that b′) holds at $n+1$ we may conclude that

$$\begin{aligned}U^{n+1}[P(U^*PU)\dots((U^*)^{n-1}PU^{n-1})(U^*)^nPU^n] &= U^{n+1}[((U^*)^nPU^n)P(U^*PU)\dots((U^*)^{n-1}PU^{n-1})]\\ &= U(U^n(U^*)^n)P[U^nP(U^*PU)\dots((U^*)^{n-1}PU^{n-1})]\\ &= U[F_n(T)]P[U^nP(U^*PU)\dots((U^*)^{n-1}PU^{n-1})]\\ &= UP[F_n(T)]T^n\\ &= UPT^n = T^{n+1}.\end{aligned}$$

To complete the proof, we apply the induction hypothesis c) and the result of the preceding paragraph to obtain

$$\begin{aligned}&(T^*)^{n+1}T^{n+1} = (T^*)^nP^2T^n\\ &= [((U^*)^{n-1}PU^{n-1})\dots(U^*PU)P](U^*)^nP^2U^n[P(U^*PU)\dots((U^*)^{n-1}PU^{n-1})]\\ &= P^2(U^*PU)^2\dots((U^*)^{n-1}P^2U^{n-1})((U^*)^nP^2U^n)\\ &= ((T^*)^nT^n)((U^*)^nP^2U^n).\end{aligned}$$

Since the two factors in this last product are commuting Hermitian operators, an elementary application of the spectral theorem shows that the kernel of the product is the (closed) span of the kernels of $(T^*)^nT^n$ and $(U^*)^nP^2U^n$. But we have $\ker((T^*)^nT^n) = \ker T^n = \ker U^n$ by the induction hypothesis. Since $\ker U^n \subseteq \ker[(U^*)^nP^2U^n]$, it follows that the kernel of $(T^*)^{n+1}T^{n+1}$ is the kernel of $(U^*)^nP^2U^n$. Arguing as we did above, we find that $\ker((U^*)^nP^2U^n) = \ker(PU^n) = \ker U^{n+1}$. Thus c) holds at $n+1$ and the proof of Theorem I is complete.

In [5], the authors show that every power partial isometry may be uniquely represented as the direct sum of a unitary operator, a pure isometry, a pure co-isometry, and a direct sum of nilpotent power partial isometries. This result, together with Theorem I, motivates the following definition.

DEFINITION 2.1. Let T be a centered operator with polar decomposition UP. Then T will be called a *type* I (resp., *type* II, *type* III, *type* IV) *centered operator* in case U is a pure isometry (resp., a pure co-isometry, a direct sum of nilpotent power partial isometries, a unitary operator). If U is a nilpotent power partial isometry of index n, then T will be called a type III_n centered operator.

Let $\mathscr{H}$ be a Hilbert space and let $\{A_k\}_{k=1}^{\infty}$ be a sequence of mutually commuting, quasi-invertible, non-negative operators, with $\sup\|A_k\| < \infty$. We define operators C_i, $i = 1, 2, 3$, by the following matrices acting

on suitable direct sums of copies of $\mathscr{H}$:

(2.6) $$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots\\ A_1 & 0 & 0 & 0 & \dots\\ 0 & A_2 & 0 & 0 & \dots\\ 0 & 0 & A_3 & 0 & \dots\\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix},$$

(2.7) $$C_2 = \begin{bmatrix} 0 & A_1 & 0 & 0 & 0 & \dots\\ 0 & 0 & A_2 & 0 & 0 & \dots\\ 0 & 0 & 0 & A_3 & 0 & \dots\\ \vdots & \vdots & \vdots & & \ddots & \end{bmatrix},$$

and

(2.8) $$C_3 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0\\ A_1 & 0 & 0 & \dots & 0 & 0\\ 0 & A_2 & 0 & \dots & 0 & 0\\ & & \ddots & \ddots & & \\ 0 & & & \ddots & \ddots & \\ \vdots & & \dots & 0 & A_{n-1} & 0 \end{bmatrix}.$$

Elementary computations reveal that C_1 (resp., C_2, C_3) is a type I (resp., type II, type III_n) centered operator. The next theorem shows that our classification of centered operators is exhaustive and that these examples are canonical.

THEOREM II. *Let T be a centered operator on a Hilbert space $\mathscr{H}$. Then $\mathscr{H}$ may be written uniquely as $\mathscr{H} = \mathscr{H}_{\mathrm{I}} \oplus \mathscr{H}_{\mathrm{II}} \oplus \mathscr{H}_{\mathrm{III}} \oplus \mathscr{H}_{\mathrm{IV}}$ where $\mathscr{H}_N$ reduces T and $T|\mathscr{H}_N$ is a type N centered operator, $N = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$. The space $\mathscr{H}_{\mathrm{III}}$ may be written uniquely as $\mathscr{H}_{\mathrm{III}} = \sum_{n=1}^{\infty} \oplus \mathscr{H}_{\mathrm{III}_n}$ where each $\mathscr{H}_{\mathrm{III}_n}$ reduces T and $T|\mathscr{H}_{\mathrm{III}_n}$ is a type III_n centered operator. Finally, if T is a type* I *(resp., type* II, *type* III_n*) centered operator, then T is unitarily equivalent to an operator of the form* (2.6) *(resp.,* (2.7), (2.8)*), where the A_k are commuting, quasi-invertible, non-negative operators.*

Proof. Let UP be the polar decomposition of T. By Theorem I, U is a power partial isometry satisfying $E_k(U) = E_k(T)$ and $F_k(U) = F_k(T)$, $k = 1, 2, 3, \dots$ According to the main result of [5], we may uniquely decompose $\mathscr{H}$ into the direct sum of at most four subspaces, $\mathscr{H} = \mathscr{H}_{\mathrm{I}} \oplus \mathscr{H}_{\mathrm{II}} \oplus \mathscr{H}_{\mathrm{III}} \oplus \mathscr{H}_{\mathrm{IV}}$, all of which reduce U, such that $U|\mathscr{H}_{\mathrm{I}}$ is a pure isometry, $U|\mathscr{H}_{\mathrm{II}}$ is a pure co-isometry, $U|\mathscr{H}_{\mathrm{III}}$ is a direct sum of nilpotent power partial isometries, and $U|\mathscr{H}_{\mathrm{IV}}$ is unitary.

The proof given in [5] actually shows that the orthogonal projections onto the subspace $\mathscr{H}_N$, $N = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, lie in the weakly closed algebra generated by the family $\{E_k(U), F_k(U)\}_{k=1}^{\infty}$, or, equivalently, the family $\{E_k(T), F_k(T)\}_{k=1}^{\infty}$. Since each of the projections in this family commutes with $P = (T^*T)^{1/2}$ (by (2.5)), it follows that $\mathscr{H}_N$ reduces T, $N = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$. Since the polar decomposition of $T|\mathscr{H}_N$ is $(U|\mathscr{H}_N)(P|\mathscr{H}_N)$, $N = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, it follows from the definition that $T|\mathscr{H}_N$ is a centered operator of type N, $N = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$. Noting that a subspace which reduces T must also reduce U, we observe that the uniqueness portion of the Halmos–Wallen decomposition implies that of our decomposition.

Since the proof of the assertion concerning the decomposition of $\mathscr{H}_{\mathrm{III}}$ reduces to the corresponding result for power partial isometries, we shall omit it.

If T is type I, then U is a pure isometry by definition. Therefore, if $\mathscr{E} = (U\mathscr{H})^{\perp}$, then there is a Hilbert space isomorphism W from $\mathscr{H}$ onto $\mathscr{E}\oplus\mathscr{E}\oplus\mathscr{E}\oplus\ldots$ such that the matrix of WUW^{-1} with respect to the direct sum has $I_{\mathscr{E}}$'s on the first subdiagonal and zeros elsewhere. Since the projections onto the summand in $\mathscr{E}\oplus\mathscr{E}\oplus\ldots$ may be expressed as Boolean combinations of the projections $\{WF_k(U)W^{-1}\}_{k=1}^{\infty}$ and since each of these commutes with WPW^{-1} (Theorem I and (2.5)), it follows that the matrix of WPW^{-1} relative to this decomposition is a diagonal matrix with non-negative entries $\{A_k\}_{k=1}^{\infty}$. Hence, the matrix representation of T relative to this decomposition has the form (2.6). Since $E(P) = E(U) = \mathscr{H}$, each A_k is quasi-invertible. Finally, since the matrix of $W(U^nP(U^*)^n)W^{-1}$ is the matrix $\mathrm{diag}(0, \ldots, 0, A_1, A_2, \ldots)$ (n zeros), Theorem I implies that the A_k's commute.

We conclude the proof of Theorem II at this point because the remaining assertions are easily proved with minor variations of the proof just completed.

§ 3. Type IV centered operators. We omit the proof of the following lemma which is a variant of Theorem I.

LEMMA 3.1. *Let T be a quasi-invertible operator with polar decomposition UP. Then T is a (type IV) centered operator if and only if the operators $\{(U^*)^nPU^n\}_{n=-\infty}^{\infty}$ commute with one another.*

If T is a quasi-invertible operator with polar decomposition UP, then we shall denote the C^*-algebra (resp., von Neumann algebra) generated by $\{(U^*)^nPU^n\}_{n=-\infty}^{\infty}$ and the identity operator by $\mathscr{P}_T$ (resp., $\mathscr{P}_T''$). For $A \in \mathscr{P}_T''$ we define $\alpha(A) = UAU^*$. Then α is an automorphism of $\mathscr{P}_T''$ and $\alpha(\mathscr{P}_T) = \mathscr{P}_T$. We shall refer to α as the conjugation of $\mathscr{P}_T$ (or of $\mathscr{P}_T''$) effected by U. Note that by Lemma 3.1 $\mathscr{P}_T$ is commutative if and only if T is centered. This observation coupled with the analysis in [13] suggests a solution to the problem of representing type IV centered operators.

Recall that a commutative von Neumann algebra has *uniform multiplicity n* in case it is unitarily equivalent to an n-fold copy of a maximal abelian von Neumann algebra (cf. [11]).

DEFINITION 3.2. A type IV centered operator T will be called a *type* IV_n *centered operator* provided that $\mathscr{P}_T''$ has uniform multiplicity n.

LEMMA 3.3. *Let T be a type IV centered operator on a Hilbert space $\mathscr{H}$, let $\{E_i\}_{i\in\mathscr{I}}$ be the family of maximal projections of uniform multiplicity in $\mathscr{P}_T''$, and let n_i be the multiplicity of $\mathscr{P}_T''|E_i\mathscr{H}$. Then $E_i\mathscr{H}$ reduces T and $T|E_i\mathscr{H}$ is a type IV_{n_i} centered operator. If T_1 is a type IV centered operator on a Hilbert space $\mathscr{H}_1$, and if $\{F_j\}_{j\in\mathscr{J}}$ is the family of maximal projections of uniform multiplicity in $\mathscr{P}_{T_1}''$, then T and T_1 are unitarily equivalent if and only if there is a one-to-one function σ from $\mathscr{I}$ onto $\mathscr{J}$ such that for each $i \in \mathscr{I}$, $T|E_i\mathscr{H}$ is unitarily equivalent to $T_1|F_{\sigma(i)}\mathscr{H}_1$.*

Proof. Write UP for the polar decomposition of T and observe that since the projections E_i are unitary invariants for $\mathscr{P}_T''$, and since $U^*\mathscr{P}_T''U = \mathscr{P}_T''$, it follows that U, and hence T, is reduced by each $E_i\mathscr{H}$. Since the polar decomposition of $T|E_i\mathscr{H}$ is $(U|E_i\mathscr{H})(P|E_i\mathscr{H}$, it follows that $T|E_i\mathscr{H}$ is a type IV_{n_i} centered operator.

One half of the second part of the assertion is trivial. For the other half, let W be a Hilbert space isomorphism from $\mathscr{H}$ onto $\mathscr{H}_1$ such that $WTW^{-1} = T_1$. If $T_1 = P_1U_1$ is the polar factorization of T_1, then it follows easily that $WPW^{-1} = P_1$, and hence, that $WUW^{-1} = U_1$. Thus, $W\mathscr{P}_T''W^{-1} = \mathscr{P}_{T_1}''$, and since $\{E_i\}_{i\in\mathscr{I}}$ and $\{F_j\}_{j\in\mathscr{J}}$ are unitary invariants for $\mathscr{P}_T''$ and $\mathscr{P}_{T_1}''$, respectively, there is a one-to-one function σ from $\mathscr{I}$ onto $\mathscr{J}$ such that $WE_iW^{-1} = F_{\sigma(i)}$ for every $i \in \mathscr{I}$. It follows that $W|E_i\mathscr{H}$ effects a unitary equivalence between $T|E_i\mathscr{H}$ and $T_1|F_{\sigma(1)}\mathscr{H}_1$, and the proof of the lemma is complete.

Although there may be many ways in which one may express a type IV centered operator as a direct sum of type IV_n centered operators, the lemma provides one which is both canonical and a unitary invariant for the operator. Accordingly, we shall refer to the decomposition in Lemma 3.3 as the *canonical decomposition* of a type IV centered operator into a direct sum of type IV_n centered operators.

In order to avoid uninteresting technical complications we shall assume for the remainder of this paper that *all Hilbert spaces under consideration are separable.* The interested reader will find that with sufficient care, our analysis may be modified to handle the non-separable case as well.

Suppose that X is a compact Hausdorff space, that τ is a homeomorphism from X onto X, and that μ is a measure on X (all measures considered are assumed to be positive, regular, and Borel). Recall that μ is called *quasi-invariant* (with respect to τ) in case μ and $\mu\circ\tau$ have the same null sets. If $\mathscr{E}$ is a Hilbert space, then $L^2_{\mathscr{E}}(\mu)$ will denote the Hilbert space of all measurable $\mathscr{E}$-valued functions on X which are square-integrable with respect to μ. If $\varphi \in L^\infty(\mu)$, then M_φ will denote the multiplication operator on $L^2_{\mathscr{E}}(\mu)$ defined by $(M_\varphi f)(x) = \varphi(x)f(x)$, $f \in L^2_{\mathscr{E}}(\mu)$. An operator A on $L^2_{\mathscr{E}}(\mu)$ is called *decomposable* in case there is a bounded measurable function $A(x)$ from X into the algebra of operators on $\mathscr{E}$ such that $(Af)(x) = A(x)f(x)$ a.e. (μ), $f \in L^2_{\mathscr{E}}(\mu)$. Recall that the algebra of decomposable operators is the commutant of the algebra of multiplication operators.

THEOREM III. *Let T be a type IV_n centered operator on a Hilbert space $\mathscr{H}$, let UP be its polar decomposition, let X be the maximal ideal space of $\mathscr{P}_T$, and let Γ denote the Gelfand transform $\mathscr{P}_T$ onto $C(X)$. Then there exists* 1) *a homeomorphism τ of X onto X,* 2) *a finite quasi-invariant measure μ on X,* 3) *an n-dimensional Hilbert space $\mathscr{E}$,* 4) *a decomposable unitary operator Θ on $L^2_{\mathscr{E}}(\mu)$, and* 5) *a Hilbert space isomorphism W from $\mathscr{H}$ onto $L^2_{\mathscr{E}}(\mu)$ such that $WTW^{-1} = \Theta S M_{\Gamma(P)}$, where S is the unitary operator on $L^2_{\mathscr{E}}(\mu)$ given by*

$$(3.1)\qquad (Sf)(x) = f(\tau(x))\big(d(\mu\circ\tau)/d\mu\big)^{1/2}(x), \qquad f \in L^2_{\mathscr{E}}(\mu).$$

Proof. Recall that α, the conjugation effected by U, is an automorphism of $\mathscr{P}_T$. Therefore $\tilde{\alpha} = \Gamma\alpha\Gamma^{-1}$ is an automorphism of $C(X)$, and so there is a homeomorphism τ from X onto X such that $\tilde{\alpha}(\varphi) = \varphi\circ\tau$ for all φ in $C(X)$. Hence we have

$$(3.2)\qquad \Gamma(UAU^*) = [\Gamma(A)]\circ\tau, \qquad A \in \mathscr{P}_T.$$

By hypothesis, $\mathscr{P}_T''$ has uniform multiplicity n. Thus there is a finite measure μ on X, an n-dimensional Hilbert space $\mathscr{E}$, and a Hilbert space isomorphism W from $\mathscr{H}$ onto $L^2_{\mathscr{E}}(\mu)$ such that

$$(3.3)\qquad WAW^{-1} = M_{\Gamma(A)}, \qquad A \in \mathscr{P}_T.$$

Putting $\tilde{U} = WUW^{-1}$, (3.2) and (3.3) yield

$$(3.4)\qquad \tilde{U}M_\varphi\tilde{U}^* = M_{\varphi\circ\tau}, \qquad \varphi \in C(X).$$

Since μ is regular, (3.4) persists when $\varphi \in C(X)$ is replaced by a function in $L^\infty(\mu)$ and this implies that μ is quasi-invariant. It follows that S, defined by (3.1), is a well-defined unitary operator on $L^2_{\mathscr{E}}(\mu)$ satisfying

$$(3.5)\qquad SM_\varphi S^* = M_{\varphi\circ\tau}, \qquad \varphi \in L^\infty(\mu).$$

Setting $\Theta = \tilde{U}S^*$, while noting that (3.4) and (3.5) imply that Θ commutes with all multiplication operators, we may conclude that $WTW^{-1} = \tilde{U}M_{\Gamma(P)} = \Theta S M_{\Gamma(P)}$, and that the proof is complete.

§ 4. Unitary invariants. Our objective in this section is to establish a complete set of unitary invariants for centered operators in terms of the parameters which enter into their canonical representation. First of all observe that two centered operators are unitarily equivalent if and only if the various summands (type I, type II, type III_n, and type IV_n) in the canonical decomposition of each are unitarily equivalent. Thus we may restrict our attention to centered operators of pure type. The following proposition is essentially due to Lambert [7] and so its proof will be omitted.

PROPOSITION 4.1. *For $i = 1, 2$, let T_i be a type* I (*resp., type* II, *type* III_n) *centered operator on the Hilbert space $\mathscr{H}_i$. Consider T_i in its matrix form* (2.6) (*resp.,* (2.7), (2.8)) *and let the non-zero entries for T_i be $A_k^{(i)}$. Then T_1 and T_2 are unitarily equivalent if and only if for each k, $A_k^{(1)}$ and $A_k^{(2)}$ are unitarily equivalent.*

Thus the problem of determining a complete set of unitary invariants for centered operators reduces to that for type IV_n centered operators. To present our solution, we need some additional terminology. Let X be a compact Hausdorff space, let τ be a homeomorphism of X onto X, let μ be a quasi-invariant measure on X, let $\mathscr{E}$ be a Hilbert space and let S be the unitary operator on $L^2_{\mathscr{E}}(\mu)$ defined by (3.1). A function $\Theta^{(n)}$ from the integers to the space of decomposable unitary operators on $L^2_{\mathscr{E}}(\mu)$ is called a *cocycle* on $L^2_{\mathscr{E}}(\mu)$ (relative to S or τ) in case $\Theta^{(n+m)} = \Theta^{(n)}\big(S^n\Theta^{(m)}(S^*)^n\big)$. A cocycle $\Theta^{(n)}$ is called a *coboundary* in case there is a decomposable unitary operator B on $L^2_{\mathscr{E}}(\mu)$ such that $\Theta^{(n)} = B\big(S^nB^*(S^*)^n\big)$. Finally, two cocycles $\Theta^{(n)}$ and $\Psi^{(n)}$ are called *cohomologous* in case there is a coboundary $B\big(S^nB^*(S^*)^n\big)$ such that $\Theta^{(n)} = B\Psi^{(n)}\big(S^nB^*(S^*)^n\big)$ for all n.

Unless $\dim\mathscr{E} = 1$, the set of cocycles does not form a group under operator multiplication. However, in any case the relation of being cohomologous is an equivalence relation on the set of cocycles; and, therefore, we shall refer to the equivalence class of a cocycle as its *cohomology class*. To understand cocycles a little better, observe that if $\Theta^{(n)}$ is one and if $U = \Theta^{(1)}S$, then $U^n = \Theta^{(n)}S^n$. Thus a cocycle is completely determined by its value at 1. Observe also that another cocycle $\Psi^{(n)}$ is cohomologous to $\Theta^{(n)}$ if and only if there is a decomposable unitary operator B with effects a unitary equivalence between $\Theta^{(1)}S$ and $\Psi^{(1)}S$. In particular, $\Theta^{(n)}$ is a coboundary if and only if there is a decomposable unitary operator which effects a unitary equivalence between $\Theta^{(1)}S$ and S. We refer the reader to Mackey's article [9] for further discussions about cocycles.

THEOREM IV. *For $i = 1, 2$, let T_i be a type IV_n centered operator on the Hilbert space $\mathscr{H}_i$, let U_iP_i be the polar decomposition of T_i, and let X_i, Γ_i, τ_i, etc., be the objects associated with T_i in Theorem* III. *Then T_1 is unitarily equivalent to T_2 if and only if there is a homeomorphism σ from*

X_2 onto X_1 such that 1) $\Gamma_2(P_2) = [\Gamma_1(P_1)]\circ\sigma$, 2) $\sigma\circ\tau_1 = \tau_2\circ\sigma$, 3) $\mu_1\circ\sigma$ *and μ_2 have the same null sets, and* 4) *if Θ_i is determined by the function $\Theta_i(x)$, $i = 1, 2$, and if $\tilde{\Theta}_i$ is the decomposable operator on $L^2_{\mathscr{E}_2}(\mu_2)$ determined by $V_0[\Theta_1(\sigma(x))]V_0^{-1}$, where V_0 is a Hilbert space isomorphism from $\mathscr{E}_1$ onto $\mathscr{E}_2$, then the cocycles determined by $\tilde{\Theta}_1$ and Θ_2 are cohomologous.*

Proof. Note that since each T_i is type IV_n, $\dim\mathscr{E}_1 = n = \dim\mathscr{E}_2$ by definition, and so 4) makes sense. Suppose W is a Hilbert space isomorphism from $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $WT_1W^{-1} = T_2$. Then for all n, $WU_1^nP_1(U_1^*)^nW^{-1} = U_2^nP_2(U_2^*)^n$, and we find that W effects a spatial isomorphism between $\mathscr{P}_{T_1}$ and $\mathscr{P}_{T_2}$. Arguing as in Section 3, there is a homeomorphism σ from X_2 onto X_1 such that $\Gamma_2(WAW^{-1}) = [\Gamma_1(A)]\circ\sigma$, $A\in\mathscr{P}_{T_1}$. This and (3.2) yield $[\Gamma_1(A)]\circ(\sigma\circ\tau_2) = [\Gamma_1(A)]\circ(\tau_1\circ\sigma)$, $A\in\mathscr{P}_{T_1}$. These two equations clearly yield 1) and 2).

For the rest of the proof, we shall regard W as a Hilbert space isomorphism from $L^2_{\mathscr{E}_1}(\mu_1)$ onto $L^2_{\mathscr{E}_2}(\mu_2)$. Then, from what has been shown so far, we have $WM_\varphi W^{-1} = M_{\varphi\circ\sigma}$, $\varphi\in C(X_1)$. But, as in § 3, this implies that $\mu_1\circ\sigma$ and μ_2 have the same null sets. This proves 3).

For 4), define V on $L^2_{\mathscr{E}_1}(\mu_1)$ by

$$Vf = [V_0(f\circ\sigma)]\big(d(\mu_1\circ\sigma)/d\mu_2\big)^{1/2}, \qquad f\in L^2_{\mathscr{E}_1}(\mu_1) \tag{4.1}$$

where V_0 is a Hilbert space isomorphism from $\mathscr{E}_1$ onto $\mathscr{E}_2$. From what has been shown so far, it follows that V is a Hilbert space isomorphism from $L^2_{\mathscr{E}_1}(\mu_1)$ onto $L^2_{\mathscr{E}_2}(\mu_2)$ which satisfies the following three equations:

$$VM_\varphi V^{-1} = M_{\varphi\circ\sigma}, \qquad \varphi\in C(X_1); \tag{4.2}$$

$$VS_1V^{-1} = S_2; \tag{4.3}$$

and

$$V\Theta_1V^{-1} = \tilde{\Theta}_1, \tag{4.4}$$

where $\tilde{\Theta}_1$ is defined in 4). Equations (4.1) and (4.2) imply that $B = WV^{-1}$ commutes with all multiplication operators on $L^2_{\mathscr{E}_2}(\mu_2)$ and so is decomposable. Since U_i is Θ_iS_i $(i = 1, 2)$ in the representation provided by Theorem III, $W\Theta_1S_1W^{-1} = \Theta_2S_2$. Hence (4.3) and (4.4) imply that

$$B\tilde{\Theta}_1S_2B^*S_2^* = (WV^{-1})(\tilde{\Theta}_1V)(V^{-1}S_2V)W^{-1}S_2^* = W(V^{-1}\tilde{\Theta}_1V)S_1W^{-1}S_2^*$$
$$= W(\Theta_1S_1)W^{-1}S_2^* = \Theta_2S_2S_2^* = \Theta_2;$$

and this verifies 4).

Conversely, suppose that there is a homeomorphism σ from X_2 onto X_1 and a Hilbert space isomorphism V_0 from $\mathscr{E}_1$ onto $\mathscr{E}_2$ satisfying 1)–4). If V is defined by (4.1), then V is a Hilbert space isomorphism from $L^2_{\mathscr{E}_1}(\mu_1)$ onto $L^2_{\mathscr{E}_2}(\mu_2)$ which satisfies (4.2), (4.3), and (4.4). By 4), there is a decomposable unitary operator B on $L^2_{\mathscr{E}_2}(\mu_2)$ such that $B\tilde{\Theta}_1S_2B^*S_2^* = \Theta_2$. Therefore, if $W = BV$, then W is a Hilbert space isomorphism from $L^2_{\mathscr{E}_1}(\mu_1)$ onto $L^2_{\mathscr{E}_2}$ such that $W\Theta_1S_1M_{\Gamma_1(P_1)}W^{-1} = \Theta_2S_2M_{\Gamma_2(P_2)}$; and this completes the proof.

§ 5. Irreducibility. To determine whether or not a centered operator is irreducible, it clearly suffices to restrict one's attention to operators of pure type. The analysis in [7] shows that a type I, II, or III_n centered operator is irreducible if and only if $\max(\dim(\ker T), \dim(\ker T^*)) = 1$. Thus it suffices to consider type IV centered operators. For these, we need some more terminology.

Let X be a compact Hausdorff space, let τ be a homeomorphism of X onto X, and let μ be a quasi-invariant measure. Suppose also that $\mathscr{E}$ is a Hilbert space which is written as the direct sum of two subspaces, $\mathscr{E} = \mathscr{E}_1\oplus\mathscr{E}_2$. Then, in the obvious way we consider $L^2_{\mathscr{E}_i}(\mu)$ as a subspace of $L^2_{\mathscr{E}}(\mu)$, and the equation $L^2_{\mathscr{E}}(\mu) = L^2_{\mathscr{E}_1}(\mu)\oplus L^2_{\mathscr{E}_2}(\mu)$ is valid. Next, let S (resp., S_1, S_2) be the unitary operator on $L^2_{\mathscr{E}}(\mu)$ (resp., $L^2_{\mathscr{E}_1}(\mu)$, $L^2_{\mathscr{E}_2}(\mu)$) defined by (3.1), and note that $S = S_1\oplus S_2$. Finally, let Θ_i be a decomposable unitary operator on $L^2_{\mathscr{E}_i}(\mu)$, $i = 1, 2$, and set $\Theta = \Theta_1\oplus\Theta_2$. Then Θ is a decomposable unitary operator on $L^2_{\mathscr{E}}(\mu)$ and we shall call the cocycle it determines the *direct sum* of the cocycles determined by Θ_1 and Θ_2. A cocycle cohomologous to the direct sum of two cocycles will be called *reducible* and a cocycle which is not reducible will be called *irreducible*. We note in passing that irreducible cocycles do indeed exist [6].(1)

THEOREM V. *Let T be a type IV_n centered operator. In the notation of Theorem* III, *write T as $\Theta SM_{\Gamma(P)}$ acting on $L^2_{\mathscr{E}}(\mu)$. Then T is irreducible if and only if μ is ergodic and the cocycle determined by Θ is irreducible.*

Proof. Since the multiplication operator determined by an invariant function commutes with $\Theta SM_{\Gamma(P)}$, ergodicity is clearly necessary for T to be irreducible. Likewise, since the cohomology class of the cocycle determined by Θ is a unitary invariant for T, the irreducibility of the cocycle is clearly necessary for T to be irreducible. Conversely, suppose μ is ergodic and that the cocycle determined by Θ is irreducible, and let Q be a non-zero projection which commutes with $\Theta SM_{\Gamma(P)}$. Then Q commutes with ΘS and all multiplication operators on $L^2_{\mathscr{E}}(\mu)$ and so is decomposable. Moreover, if Q is determined by the projection-valued function $Q(x)$, then

$$Q(x) = [(\Theta S)Q(S^*\Theta^*)](x) = \Theta(x)Q\big(\tau(x)\big)\Theta^*(x) \qquad \text{a.e. } (\mu)$$

where $\Theta(x)$ is the unitary-valued function which determines Θ. This equation shows that the measurable functions which assign to each x

(1) Added in proof. Recently, S. C. Bagchi, Joseph Mathew and M. G. Nadkarni (*On systems of imprimitivity on locally compact abelian groups with dense actions*, Indian Statistical Institute Technical Report No. Math–Stat /21/73) have exhibited a general method for constructing irreducible cocycles with values on Hilbert spaces of arbitrary dimension.

the dimension and the condimension of the space $Q(x)\mathscr{E}$ are invariant; and since μ is ergodic, they must be constant. Hence there are orthogonal subspaces $\mathscr{E}_1$ and $\mathscr{E}_2$ such that $\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$, and there is a decomposable unitary operator B on $\boldsymbol{L}^2_{\mathscr{E}}(\mu)$ (determined by the unitary function $B(x)$) such that $B(x)\big(Q(x)\mathscr{E}\big) = \mathscr{E}_1$ and $B(x)\big(Q(x)\mathscr{E}\big)^{\perp} = \mathscr{E}_2$ a.e. (μ). It follows that the cocycle determined by $B\Theta SB^*S^*$ is the direct sum of cocycles on $\boldsymbol{L}^2_{\mathscr{E}_1}(\mu)$ and $\boldsymbol{L}^2_{\mathscr{E}_2}(\mu)$. If $\mathscr{E}_2$ is not the zero space, then we have contradicted the hypothesis on Θ. Hence we may conclude that $\mathscr{E}_2 = \{0\}$, Q is the identity, and that the proof is complete.

§ 6. Concluding remarks.

(6.1). If T is an operator on a Hilbert space $\mathscr{H}$, let $\mathscr{A}_T$ be the weakly closed algebra generated by T, let $\mathscr{A}'_T$ denote the commutant of T and let $\mathscr{A}''_T$ denote the commutant of $\mathscr{A}'_T$. In [12] Shields and Wallen showed that if T is a unilateral shift (with non-zero weights), then $\mathscr{A}_T = \mathscr{A}'_T$. On the other hand, Lambert and Turner [8] have exhibited type I centered operators T such that $\mathscr{A}_T \neq \mathscr{A}''_T$. It is therefore natural to ask: For which centered operators T does $\mathscr{A}_T = \mathscr{A}'_T$ or $\mathscr{A}''_T$?

(6.2). In both [3] and [10], the authors have shown (independently) that every hyponormal operator T such that TT^* and T^*T commute has a proper invariant subspace. In particular, every hyponormal centered operator has invariant subspaces. On the other hand, the Bishop operator is centered and is one of the leading candidates for an operator without a proper invariant subspace.(2) Therefore the following question may be difficult: Which centered operators have proper invariant subspaces?

(6.3). Here is an analogy which may be worth pursuing: Weighted shifts are to centered operators what diagonal operators are to normal operators. We call an operator which is the direct sum of diagonal operators (forward and backward), unilateral weighted shifts, and bilateral weighted shifts a *basic weighted shift*. The following question is due to P. R. Halmos: Are the centered operators the closure (in the uniform operator topology) of the basic weighted shifts?

(6.4). It would be interesting to characterize the intersection of the class of centered operators with other, more familiar, classes of operators. For example, an analysis due to Bastian [2] shows that the only irreducible, hyponormal, type IV, centered operators are bilateral weighted shifts.

(6.5). Finally, observe that if T is a quasi-invertible operator on a Hilbert space $\mathscr{H}$ with polar decomposition UP, then, as we noted in § 3, U effects an automorphism of $\mathscr{P}_T$ via conjugation. Thus, if T is actually invertible, the C^*-algebra generated by T contains U and P and may be regarded as a covariant representation of the covariance algebra generated by $\mathscr{P}_T$ and the automorphism [1, 13]. This remark, albeit a bit banal, puts into evidence certain structure which has not been exploited heretofore in the analysis of an operator in terms of the C^*-algebra it generates, and which may prove useful in future investigations.

(2) Added in proof. Recently A. M. Davie (*Invariant subspaces for Bishop's operator*, preprint) has sh wn that a large number of Bishop's operators have invariant subspaces.

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