STUDIA MATHEMATICA, T. LI. (1974)

Inverse limits of compact convex sets and direct limits of spaces of continuous affine functions

by

PETER D. TAYLOR (Kingston, Ontario, Canada)

Abstract. Let (K_∞, φ_t) be the inverse limit of an inverse system (K_t, φ_t^s) of compact convex sets. Then it is proved that, in the category of Banach spaces and linear contractions, $(\mathscr{A}(K_\infty), \mathscr{A}(\varphi_t))$ is a direct limit of the direct system $(\mathscr{A}(K_t), \mathscr{A}(\varphi_t^s))$.

If K is a compact convex set, let $A(K)$ denote the space of affine continuous functions on K. We always assume that $A(K)$ separates points in K. The purpose of this note is to prove that if K_∞ is the inverse limit of an inverse system $\{K_t\}$ of compact convex sets, then $A(K_\infty)$ is the direct limit of the direct system $\{A(K_t)\}$. The terms are defined below.

Our result can probably best be stated in the language of categories. If **Compconv** denotes the category of compact convex sets and continuous affine maps, and $\mathbf{Ban}_1$ denotes the category of Banach spaces and linear contractions, then A: **Compconv** $\to \mathbf{Ban}_1$ is clearly a contravariant functor. Our purpose is to prove that A is inversely continuous, i.e., it transforms inverse limits into direct limits.

Let $\mathfrak{U}$ be a category. We recall that an inverse system (in $\mathfrak{U}$) is a family $(W_t)_{t\in T}$ of objects indexed by an upward directed set T, together with a family $(a_t^s)_{t\leqslant s}$ of morphisms a_t^s: $W_s \to W_t$ such that a_t^t is the identity and if $t \leqslant s \leqslant r$, then $a_t^r = a_t^s a_s^r$. An *inverse limit* of this system (in $\mathfrak{U}$) is an object W_∞ together with a family of morphisms a_t: $W_\infty \to W_t$ $(t \in T)$ such that (i) $a_t^s a_s = a_t$ for $t \leqslant s$ and (ii) for any object W and any family of morphisms b_t: $W \to W_t$ such that $a_t^s b_s = b_t$ for $t \leqslant s$, there exists a unique morphism b: $W \to W_\infty$ such that $a_t b = b_t$ for $t \in T$.

Dually, a *direct system* (in $\mathfrak{U}$) is a family $(W^t)_{t\in T}$ of objects indexed by an upward directed set T, together with a family $(a_t^s)_{t\leqslant s}$ of morphisms a_t^s: $W^t \to W^s$ such that a_t^t is the identity and if $t \leqslant s \leqslant r$, then $a_t^r = a_s^r a_t^s$. A *direct limit* of this system (in $\mathfrak{U}$) is an object W^∞ together with a family of morphisms a_t: $W^t \to W^\infty$ such that (i) $a_s a_t^s = a_t$ for $t \leqslant s$ and (ii) for any object W and any family of morphisms b_t: $W^t \to W$ such that $b_s b_t^s = b_t$ for $t \leqslant s$, there exists a unique morphism b: $W^\infty \to W$ such that $ba_t = b_t$ for $t \in T$.

It is clear that if an inverse (direct) limit exists it is unique up to unique commuting isomorphism.

In the category **Compconv** inverse limits always exist. If (K_t, φ_t^s) is an inverse system of compact convex sets, let $K_\infty = \{(k_t) \in \prod_t K_t: \varphi_t^s(k_s) = k_t \text{ for } t \leqslant s\}$, and let $\varphi_t: K_\infty \to K_t$ be the projection map. It is easy to check that (K_∞, φ_t), with the product topology and componentwise operations, is an inverse limit of the system. The same process constructs an inverse limit in the category of compact spaces. This particular realization of an inverse limit will be referred to as *the* inverse limit.

Before stating our theorem, we introduce some notation. If $\varphi: K_1 \to K_2$ is a continuous affine map between compact convex sets, we will denote by $\mathscr{A}(\varphi)$ the induced operator $\mathscr{A}(\varphi): \mathscr{A}(K_2) \to \mathscr{A}(K_1)$.

If B is a Banach space let us denote by O^*B the unit ball of the dual space B^*, with the weak* topology. Then O^*B is an object in the category **Compconv**. If $T: B_1 \to B_2$ is a linear contraction, then the induced map $O^*T: O^*B_2 \to O^*B_1$ is continuous and affine. Clearly, $O^*: \mathbf{Ban}_1 \to \mathbf{Compconv}$ is a contravariant functor.

THEOREM. *Let (K_∞, φ_t) be the inverse limit of an inverse system (K_t, φ_t^s) of compact convex sets. Then $\big(\mathscr{A}(K_\infty), \mathscr{A}(\varphi_t)\big)$ is a direct limit in the category* $\mathbf{Ban}_1$ *of the direct system $\big(\mathscr{A}(K_t), \mathscr{A}(\varphi_t^s)\big)$.*

The following lemmas are well known. The trick in Lemma 1 is to notice that any Banach space B can be thought of as the space of continuous affine functions on O^*B which vanish at zero. Lemma 3 is an immediate consequence of the Hahn–Banach Theorem.

LEMMA 1. *Suppose (B_t, ψ_t^s) is a direct system of Banach spaces and linear contractions, and suppose B is a Banach space and, for each t, $\psi_t: B_t \to B$ is a linear contraction such that $\psi_s\psi_t^s = \psi_t$ for $t \leqslant s$. Suppose that $(O^*B, O^*\psi_t)$ is an inverse limit of the inverse family $(O^*B_t, O^*\psi_t^s)$. Then (B, ψ_t) is a direct limit in* $\mathbf{Ban}_1$ *of (B_t, ψ_t^s).*

LEMMA 2 ([4], Chapter VIII Corollary 3.8). *Suppose (X_∞, a_t) is the inverse limit of an inverse family (X_t, a_t^s) of compact topological spaces. Then for every t, $a_t(X_\infty) = \bigcap_{s>t} a_t^s(X_s)$.*

LEMMA 3. *If X is a compact space let $M_1(X)$ denote the set of regular Borel (signed) measures on X of norm $\leqslant 1$. Suppose $a: X \to Y$ is a continuous map between compact spaces. If a is onto, then the induced map $\mu \mapsto \mu \circ a^{-1}$ from $M_1(X)$ to $M_1(Y)$ is onto.*

Proof of the theorem. According to Lemma 1, we have proved the theorem if we can show that $O^*\mathscr{A}(K_\infty)$ is an inverse limit of $\big(O^*\mathscr{A}(K_t)\big)$. For ease of notation, let us write $\bar{K}_t$ and $\bar{K}_\infty$ for $O^*\mathscr{A}(K_t)$ and $O^*\mathscr{A}(K_\infty)$, and $\bar{\varphi}_t^s$ and $\bar{\varphi}_t$ for $O^*\mathscr{A}(\varphi_t^s)$ and $O^*\mathscr{A}(\varphi_t)$. Let (L, λ_t) denote the inverse limit of $(\bar{K}_t, \bar{\varphi}_t^s)$. Since $\bar{\varphi}_t^s\bar{\varphi}_s = \bar{\varphi}_t$, there is a unique continuous affine map $\lambda: \bar{K}_\infty \to L$ such that $\lambda_t\lambda = \bar{\varphi}_t$. We will be done if we can show that λ is injective and onto.

First we show that λ is injective. Suppose $h, k \in \bar{K}_\infty$ and $h \neq k$. We must find some t for which $\bar{\varphi}_t(h) \neq \bar{\varphi}_t(k)$. Choose $f \in \mathscr{A}(K_\infty)$ such that $h(f) > k(f) + \varepsilon$ for some $\varepsilon > 0$. Now the subspace $\bigcup_t \mathscr{A}(\varphi_t)\big(\mathscr{A}(K_t)\big)$ of $\mathscr{A}(K_\infty)$ contains the constants and separates points of K_∞ (since the family $\{\varphi_t\}$ separates points of K_∞), so that by a standard result (see, for example, [1], p. 224) it is uniformly dense in $\mathscr{A}(K_\infty)$. So choose t and $g \in \mathscr{A}(K_t)$ such that $\|\mathscr{A}(\varphi_t)(g) - f\| < \varepsilon/2$. Then $\bar{\varphi}_t(h)(g) = h\big(\mathscr{A}(\varphi_t)(g)\big) > h(f) - \varepsilon/2 > k(f) + \varepsilon/2 > k\big(\mathscr{A}(\varphi_t)(g)\big) = \bar{\varphi}_t(k)(g)$, and we are finished.

To prove that λ is onto, let us take $x \in L$. Then $x = (x_t)$ for elements $x_t \in \bar{K}_t$ for which $\bar{\varphi}_t^s(x_s) = x_t$ for $t \leqslant s$.

As a preliminary step we shall show that for each t there is an element y_t in $\bar{K}_\infty$ such that $\bar{\varphi}_t(y_t) = x_t$. First of all, for each s, use the Hahn–Banach Theorem to find a measure $\nu_s \in M_1(K_s)$ such that $\nu_s(f) = x_s(f)$ for every $f \in \mathscr{A}(K_s)$. Then for $s \geqslant t$, $\omega_s = \nu_s \circ [\varphi_t^s]^{-1}$ is a measure in $M_1(K_t)$ which lives on $\varphi_t^s(K_s)$. Suppose $\Gamma: \alpha \to s$ is a subnet such that $\{\omega_{\Gamma(\alpha)}\}$ converges (weak*) to $\mu_t \in M_1(K_t)$. Then μ_t lives on $\bigcap_{s>t} \varphi_t^s(K_s) = \varphi_t(K_\infty)$ by Lemma 2. Since $\varphi_t: K_\infty \to K_t$ maps onto the closed subset $\varphi_t(K_\infty)$' we can, by Lemma 3, choose a measure $\gamma_t \in M_1(K_\infty)$ such that $\mu_t = \gamma_t \circ \varphi_t^{-1}$. Then the map $f \to \gamma_t(f)$ for $f \in \mathscr{A}(K_\infty)$ is a member y_t of $\bar{K}_\infty$. It will be enough to show that $\bar{\varphi}_t(y_t)(f) = x_t(f)$ for $f \in \mathscr{A}(K_t)$. Indeed, if $\Gamma(\alpha) \geqslant t$, then

$$x_t(f) = \overline{\varphi_t^{\Gamma(\alpha)}}(x_{\Gamma(\alpha)})(f) = x_{\Gamma(\alpha)}(f \circ \varphi_t^{\Gamma(\alpha)})$$
$$= \nu_{\Gamma(\alpha)}(f \circ \varphi_t^{\Gamma(\alpha)}) = \omega_{\Gamma(\alpha)}(f).$$

Now as α grows large, these numbers converge to

$$\mu_t(f) = (\gamma_t \circ \varphi_t^{-1})(f) = \bar{\varphi}_t(y_t)(f).$$

Thus we must have $x_t = \bar{\varphi}_t(y_t)$.

Now we complete the proof that λ is onto. Suppose $\Lambda: \beta \to t$ is a subnet such that $\{y_{\Lambda(\beta)}\}$ converges in $\bar{K}_\infty$ to y. We will show that $\lambda(y) = x$. For arbitrary t and $\Lambda(\beta) \geqslant t$,

$$x_t = \overline{\varphi_t^{\Lambda(\beta)}}(x_{\Lambda(\beta)}) = \overline{\varphi_t^{\Lambda(\beta)}}\,\overline{\varphi_{\Lambda(\beta)}}(y_{\Lambda(\beta)}) = \bar{\varphi}_t(y_{\Lambda(\beta)}).$$

As β gets large, these expressions converge to

$$\bar{\varphi}_t(y) = \lambda_t\lambda(y).$$

Hence $x = \lambda(y)$.

Remark 1. The notion of direct limit depends heavily on the category in which it is taken. If we let $\mathscr{A}K$ denote the category of all spaces $\mathscr{A}(K)$ for K compact convex, with morphisms the positive linear operators which map 1 into 1, then it is very easy to see that $\mathscr{A}(K_\infty)$ is a direct limit of $(\mathscr{A}(K_t))$ in the category $\mathscr{A}K$. Our theorem is a much stronger statement: $\mathscr{A}(K_\infty)$ remains a direct limit in the larger category $\mathbf{Ban}_1$.

Remark 2. Compare our theorem with the following theorem of Semadeni.

THEOREM ([3], Theorem 1). *Let (X_∞, φ_t) be the inverse limit of the inverse system (X_t, φ_t^s) of compact topological spaces. Then $(\mathscr{C}(X_\infty), \mathscr{C}(\varphi_t))$ is the direct limit in* $\mathbf{Ban}_1$ *of the direct system $(\mathscr{C}(X_t), \mathscr{C}(\varphi_t^s))$.*

This theorem was proved earlier by Pełczyński ([2], p. 14), in the special case where all maps φ_t^s are surjections. Semadeni's theorem can, in fact, be obtained as a corollary of our theorem, but only with some additional work; one passes from compact spaces X to the simplex $\boldsymbol{P}(X)$ of Borel probability measures on X. One can identify $C(X)$ with $\mathscr{A}(\boldsymbol{P}(X))$, but one must show that if $X_\infty = \operatorname{inv\,lim} X_t$, then $\boldsymbol{P}(X_\infty) = \operatorname{inv\,lim} \boldsymbol{P}(X_t)$.

If one wants to obtain Semadeni's theorem by the methods of this paper, the most natural way is to imitate the proof of our theorem directly. The proof that $\lambda\colon O*\mathscr{C}(X_\infty)\to L$ is injective is similar: one uses the Weierstrass–Stone theorem to prove that $\bigcup \mathscr{C}(\varphi_t)(\mathscr{C}(X_t))$ is dense in $\mathscr{C}(X_\infty)$. The proof that λ is onto is somewhat simpler than in our case.

Remark 3. I am grateful to Professor Semadeni for bringing the problem to my attention and for acquainting me with the ideas from category theory.

Remark 4. I wish to thank the referee for providing me with some references and for suggesting some improvements in style.

References

[1] F. Jellett, *Homomorphisms and inverse limits of Choquet Simplexes*, Math. Zeitschr. 103 (1968), pp. 219–226.

[2] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. (Rozprawy Mat.) 58 (1968).

[3] Z. Semadeni, *Inverse limits of compact spaces and direct limits of spaces of continuous functions*, Studia Math. 31 (1968), pp. 373–382.

[4] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton Mathematical Series 15, Princeton, N. J. 1952.

QUEEN'S UNIVERSITY
KINGSTON, ONTARIO
CANADA

Received February 15, 1973 (660)

STUDIA MATHEMATICA, T. LI. (1974)

Continuity of operators on Saks spaces

by

IWO LABUDA (Poznań)

Abstract. In [13] Orlicz initiated the study of linear operators acting from so-called Saks spaces. His original investigations concerned operators taking values in Banach and Fréchet spaces. In this paper we extend the theory to the case of operators with values in locally convex and general topological vector spaces. Generally the proofs presented here are more direct than the original ones of Orlicz; some refinements of his classical statements and some new results have been obtained.

§ 0. Introduction. In the present paper we intend to give some generalizations of results contained in [13]. Since we will constantly refer to this fundamental work, we decide to preserve as far as possible its terminology, notation and conventions. There is, however, one important exception to this principle — an additive ([13], 2.1) continuous operator ν from a Saks space X_s into a topological vector space Y will be termed explicitly "*additive* (X_s, Y)*-continuous*", while Orlicz, according to the old terminology of Banach [1], calls ν in that case (X_s, Y)-linear. Moreover, in many situations the topology of Y is explicitly mentioned, i.e., $Y = (Y, \tau)$ for example; we will then say simply that $\nu\colon X_s\to Y$ is τ*-continuous*, or only that ν is τ-continuous, when no ambiguity about X_s and Y arises.

An operator $\nu\colon X_s\to Y$ will be said to be *linear* (cf. [13], 2.1) if for arbitrary $x_1, x_2 \in X_s$ and arbitrary scalars a_1, a_2, $a_1x_1+a_2x_2 \in X_s$ implies $\nu(a_1x_1+a_2x_2) = a_1\nu(x_1)+a_2\nu(x_2)$. Note that with this terminology it is obvious that an additive (X_s, Y)-continuous operator = a linear (X_s, Y)-continuous operator.

Following Orlicz, a *Saks space* is defined as a closed unit ball of a fundamental normed space $(X, \|\cdot\|)$ on which another norm (in general non-homogeneous!), $\|\cdot\|^*$ say, defines the complete metric. It is clear, however, that instead of the unit ball of $(X, \|\cdot\|)$, a bounded, closed, convex balanced set of an arbitrary Hausdorff topological vector space could be taken.

If this unit ball endowed with the metric induced by $\|\cdot\|^*$ is not complete, it is called a *Saks set.*

Denote by X_s a Saks space and by $K(x_0, r)$ the open sphere with centre x_0 and radius r in the space X_s. We shall consider Saks spaces satisfying the following conditions: