STUDIA MATHEMATICA, T. LI. (1974)

s-Numbers of operators in Banach spaces

by

ALBRECHT PIETSCH (Jena)

Abstract. For each operator between Banach spaces one can define the sequence of aproximation numbers, Kolmogorov numbers, Gelfand numbers, etc. As an unification we present an axiomatic theory of the so-called s-numbers, and we discuss related ideals of operators.

As has been shown in the famous book of I. Z. Gochberg and M. G. Krejn [1] the s-numbers are an important tool in the spectral theory of Hilbert space (cf. [18]). The s-number $s_n(S)$ of a compact operator S from an infinite dimensional Hilbert space H into itself is defined as the nth eigenvalue of the operator $|S| := (S^*S)^{1/2}$.

Particularly, the s-numbers can be used to describe the ideals in the ring $\boldsymbol{L}(H, H)$ of operators. Let $\boldsymbol{S}_\infty(H, H)$ be the closed ideal of compact operators. Then the most interesting ideals discovered by J. v. Neumann and R. Schatten are defined by

$$\boldsymbol{S}_p(H, H) := \Big\{S \in \boldsymbol{S}_\infty(H, H) \colon \sum_1^\infty s_n(S)^p < \infty\Big\}, \quad 0 < p < \infty.$$

For $p = 1$ we obtain the trace class of operators, and $\boldsymbol{S}_2(H, H)$ is the ideal of Hilbert–Schmidt operators which are characterized by the inequality

$$\sum_{i,k} |(Se_i, f_k)|^2 < \infty$$

for arbitrary complete orthonormal systems (e_i) and (f_k).

The purpose of this paper is to present an axiomatic theory of s-numbers of operators in Banach spaces. Since we want to give a general survey, some known results for special s-numbers are reproduced.

Finally, I wish to thank my student F. Fiedler for his help by the elaboration of some proofs and H. Junek for a simpler version of the proof of Lemma 7.2.

0. Prerequisites. Let E be a real or complex Banach space with the closed unit ball U_E. The identity map of E is denoted by I_E.

A subspace is a closed linear subset. The embedding map of a subspace M into E is denoted by J_M^E, and the canonical map of E onto the quotient space E/M is denoted by Q_M^E.

An operator is a bounded linear map. Let $\boldsymbol{L}$ be the class of all operators between Banach spaces. The set of those operators which map E into F is denoted by $\boldsymbol{L}(E, F)$.

Let $\dim(M)$ be the dimension of the subspace M, and let $\operatorname{codim}(M) := \dim(E/M)$ be the codimension. If the operator S is of finite rank, then the dimension of the image is denoted by $\dim(S)$.

For $a_0 \in E'$ (dual Banach space) and $y_0 \in F$ let $a_0 \otimes y_0$ be the map: $x \to \langle x, a_0\rangle y_0$.

Next we state some important lemmas which are used in the following.

LEMMA 0.1. (*Principle of local reflexivity*, cf. [5]). *Let M be a finite dimensional subspace of E''. If $\varepsilon > 0$ then there exists $R \in \boldsymbol{L}(M, E)$ such that*

$$\|R\| \leqslant 1+\varepsilon \text{ and } RJ_E x = x \quad \text{for all } x \in E \cap J_E^{-1}(M)$$

where J_E denotes the canonical map of E into E''.

LEMMA 0.2. (Cf. [7], p. 199). *Let M and N be finite dimensional subspaces of E with $\dim(M) > \dim(N)$. Then there exists $x \in M$ such that*

$$\|Q_N^E x\| = \|x\| = 1.$$

LEMMA 0.3. (Cf. [6]). *Let M be a subspace of E with $\dim(M) = n$. Then there exists a projection $P \in \boldsymbol{L}(E, E)$ such that*

$$M = P(E) \quad \text{and} \quad \|P\| \leqslant n^{1/2}.$$

1. Axiomatic properties of s-numbers. For operators in Banach spaces there are several possibilities to define sequences of numbers which coincide with s-numbers in the case of Hilbert space. A report about such numbers was given by B. S. Mitiagin and A. Pełczyński [11] at the Moscow Congress in 1966 (cf. [12]).

In the sequel we deal with an axiomatic theory of s-numbers. A map

$$s: S \to (s_n(S))$$

from $\boldsymbol{L}$ into the set of sequences of non-negative numbers is called an *s-number function* if the following conditions are satisfied ($n = 1, 2, \ldots$):

(1) $\|S\| = s_1(S) \geqslant s_2(S) \geqslant \ldots \geqslant 0$ *for* $S \in \boldsymbol{L}$.

(2) $s_n(S+T) \leqslant s_n(S) + \|T\|$ *for* $S, T \in \boldsymbol{L}(E, F)$.

(3) $s_n(RST) \leqslant \|R\| s_n(S) \|T\|$ *for* $T \in \boldsymbol{L}(E_0, E)$, $S \in \boldsymbol{L}(E, F)$, $R \in \boldsymbol{L}(F, F_0)$.

(4) *If* $\dim(S) < n$ *then* $s_n(S) = 0$.

(5) *If* $\dim(E) \geqslant n$ *then* $s_n(I_E) = 1$.

The number $s_n(S)$ is said to be the *n-th s-number* of the operator S.

THEOREM 1.1. *The s-numbers are continuous functions since*

$$|s_n(S) - s_n(T)| \leqslant \|S - T\| \quad \text{for } S, T \in \boldsymbol{L}(E, F).$$

Proof. By (2) we have

$$s_n(S) \leqslant s_n(T) + \|S - T\|.$$

Futhermore the inverse statement of condition (4) is valid.

THEOREM 1.2. *If $s_n(S) = 0$ then $\dim(S) < n$.*

Proof. The assertion is an easy consequence of (3), (5) and

LEMMA 1.1. *Let $S \in \boldsymbol{L}(E, F)$. If $\dim(S) \geqslant n$ then there exist a Banach space G as well as operators $X \in \boldsymbol{L}(G, E)$ and $B \in \boldsymbol{L}(F, G)$ such that*

$$I_G = BSX \quad \text{and} \quad \dim(G) \geqslant n.$$

Proof. We choose $x_1, \ldots, x_n \in E$ such that $Sx_1, \ldots, Sx_n$ are linearly independent. Then by the Hahn–Banach theorem there are $b_1, \ldots, b_n \in F'$ with $\langle Sx_i, b_k\rangle = \delta_{ik}$. Let $G := l_2^n$,

$$X(\xi_i) := \sum_1^n \xi_i x_i \quad \text{for } (\xi_i) \in l_2^n,$$

and

$$By := (\langle y, b_i\rangle) \quad \text{for } y \in F.$$

2. s-Numbers of operators in Hilbert space. Now we show that s-numbers of operators in a Hilbert space H are determined uniquely by their axiomatic properties.

THEOREM 2.1. *Let $S \in \boldsymbol{L}(H, H)$, and let $P(\cdot)$ be the spectral measure of the positive operator $|S| := (S^* S)^{1/2}$. Then*

$$s_n(S) = \inf\{\sigma \geqslant 0 : \dim(P(\sigma, \infty)) < n\}$$

for each s-number function.

Proof. By the theorem of polar representation (cf. [1], p. 21; [17], p. 284) there is a partially isometric operator U such that

$$S = U|S| \quad \text{and} \quad S^* = U^* S.$$

Hence

$$s_n(S) = s_n(|S|).$$

We set $\sigma_n := \inf\{\sigma \geqslant 0 : \dim(P(\sigma, \infty)) < n\}$. If $\varepsilon > 0$ then from

$$|S| = \int_0^\infty \sigma P(d\sigma) = \int_0^{\sigma_n+\varepsilon} \sigma P(d\sigma) + \int_{\sigma_n+\varepsilon}^\infty \sigma P(d\sigma)$$

and

$$\dim\Big(\int_{\sigma_n+\varepsilon}^{\infty} \sigma P(d\sigma)\Big) < n$$

we obtain

$$s_n(|S|) \leqslant \Big\|\int_0^{\sigma_n+\varepsilon} \sigma P(d\sigma)\Big\| + s_n\Big(\int_{\sigma_n+\varepsilon}^{\infty} \sigma P(d\sigma)\Big) \leqslant \sigma_n+\varepsilon.$$

On the other hand, let $\sigma_n > \varepsilon > 0$. Then

$$P(\sigma_n-\varepsilon,\,\infty) = \Big(\int_0^{\infty} \sigma P(d\sigma)\Big)\Big(\int_{\sigma_n-\varepsilon}^{\infty} \sigma^{-1} P(d\sigma)\Big)$$

and

$$\dim\big(P(\sigma_n-\varepsilon,\,\infty)\big) \geqslant n$$

imply

$$1 = s_n\big(P(\sigma_n-\varepsilon,\,\infty)\big) \leqslant s_n(|S|)\Big\|\int_{\sigma_n-\varepsilon}^{\infty} \sigma^{-1} P(d\sigma)\Big\| \leqslant s_n(|S|)(\sigma_n-\varepsilon)^{-1}.$$

So we have

$$\sigma_n-\varepsilon \leqslant s_n(|S|) \leqslant \sigma_n+\varepsilon \quad \text{for all } \varepsilon > 0.$$

COROLLARY. *Let $S \in \mathbf{S}_\infty(H, H)$. Then $s_n(S)$ is the n-th eigenvalue of the positive operator $|S|$.*

3. Approximation numbers and isomorphism numbers. Now we present two examples of s-number functions.

For every operator $S \in \mathbf{L}(E, F)$ the *approximation numbers* are defined by

$$a_n(s) := \inf\{\|S-A\|\colon\ A \in \mathbf{L}(E, F),\ \dim(A) < n\}.$$

THEOREM 3.1. *The map*

$$\mathrm{app}\colon\ S \to \big(a_n(S)\big)$$

is an s-number function.

Proof. Since the other properties are trivial we prove the condition (5).

Let us assume $a_n(I_E) < 1$. Then there exists $A \in \mathbf{L}(E, E)$ such that $\|I_E-A\| < 1$ and $\dim(A) < n$. Consequently, $A = I_E-(I_E-A)$ is invertible by the Neumann series, and we have $\dim(A) \geqslant n$. Contradiction.

THEOREM 3.2. *The approximation numbers are the largest s-numbers.*

Proof. Let $S \in \mathbf{L}(E, F)$. Then for each s-number function and $A \in \mathbf{L}(E, F)$ with $\dim(A) < n$ we have

$$s_n(S) \leqslant s_n(A) + \|S-A\| = \|S-A\|.$$

Hence

$$s_n(S) \leqslant a_n(S) \quad \text{for all } S \in \mathbf{L}.$$

For every operator $S \in \mathbf{L}(E, F)$ the *isomorphism numbers* are defined as follows. If $\dim(S) < n$ we set $i_n(S) := 0$. If $\dim(S) \geqslant n$ then by Lemma 1.1 there exist a Banach space G as well as operators $X \in \mathbf{L}(G, E)$ and $B \in \mathbf{L}(F, G)$ such that

$$I_G = BSX \quad \text{and} \quad \dim(G) \geqslant n.$$

In this case let

$$i_n(S) := \sup\{\|B\|^{-1}\|X\|^{-1}\},$$

where the supremum is taken over all possibilities.

THEOREM 3.3. *The map*

$$\mathrm{iso}\colon\ S \to \big(i_n(S)\big)$$

is an s-number function.

Proof. Since the other properties are trivial, we prove

$$(2) \qquad i_n(S+T) \leqslant i_n(S) + \|T\| \quad \text{for } S, T \in \mathbf{L}(E, F).$$

We may assume $i_n(S+T) > \|T\|$. If $0 < \varepsilon < i_n(S+T) - \|T\|$ then there exist a Banach space G as well as operators $X \in \mathbf{L}(G, E)$ and $B \in \mathbf{L}(F, G)$ such that

$$I_G = B(S+T)X, \quad \dim(G) \geqslant n, \quad \text{and} \quad \|B\|^{-1}\|X\|^{-1} \geqslant i_n(S+T) - \varepsilon > \|T\|.$$

Since $\|BTX\| < 1$, the operator

$$BSX = B(S+T)X - BTX = I_G - BTX$$

is invertible. From

$$I_G = (I_G-BTX)^{-1}BSX \quad \text{and} \quad \|(I_G-BTX)^{-1}\| \leqslant (1-\|BTX\|)^{-1}$$

it follows that

$$\begin{aligned} i_n(S) &\geqslant \|(I_G-BTX)^{-1}B\|^{-1}\|X\|^{-1} \\ &\geqslant (1-\|BTX\|)\|B\|^{-1}\|X\|^{-1} \\ &\geqslant \|B\|^{-1}\|X\|^{-1} - \|T\| \\ &\geqslant i_n(S+T) - \varepsilon - \|T\|. \end{aligned}$$

Consequently,

$$i_n(S+T) \leqslant i_n(S) + \|T\| + \varepsilon.$$

THEOREM 3.4. *The isomorphism numbers are the smallest s-numbers.*

Proof. Let $S \in \mathbf{L}(E, F)$, $X \in \mathbf{L}(G, E)$ and $B \in \mathbf{L}(F, G)$ such that

$$I_G = BSX \quad \text{and} \quad \dim(G) \geqslant n.$$

Then for each s-number function we have

$$1 = s_n(I_G) \leqslant \|B\|\,s_n(S)\,\|X\|.$$

Hence

$$i_n(S) \leqslant s_n(S) \quad \text{for all } S \in \mathbf{L}.$$

4. Injective s-numbers. An s-number function $\mathbf{s}$ is called *injective* if the following property is satisfied:

Let M be a subspace of F; then

$$s_n(J_M^F S) = s_n(S) \quad \text{for all } S \in \boldsymbol{L}(E, M).$$

In other words, injectivity means that the s-numbers $s_n(S)$ do not depend on the codomain of S.

For every operator $S \in \boldsymbol{L}(E, F)$ the *Gelfand numbers* are defined by

$$c_n(S) := \inf\{\|SJ_M^E\|: \operatorname{codim}(M) < n\}.$$

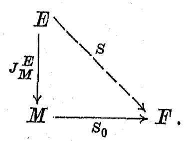
THEOREM 4.1. *The map*

$$\text{gel}: S \to (c_n(S))$$

is an injective s-number function.

The proof is left to the reader.

A Banach space F is said to have the *extension property* if for every operator S_0 mapping a subspace M of an arbitrary Banach space E into F there is an extension S from E into F such that $\|S\| = \|S_0\|$,



THEOREM 4.2. *If F has the extension property then*

$$c_n(S) = a_n(S) \quad \textit{for all} \quad S \in \boldsymbol{L}(E, F).$$

Proof. Let $S \in \boldsymbol{L}(E, F)$. Since $a_n(S)$ are the largest s-numbers, it is enough to show $a_n(S) \leqslant c_n(S)$.

If $\varepsilon > 0$ we choose a subspace M of E such that

$$\|SJ_M^E\| \leqslant c_n(S) + \varepsilon \quad \text{and} \quad \operatorname{codim}(M) < n.$$

Then there exists an extension $T \in \boldsymbol{L}(E, F)$ of SJ_M^E with $\|T\| = \|SJ_M^E\|$. We set $A := S - T$. Since $Ax = 0$ for all $x \in M$, we have $\dim(A) < n$. Hence

$$a_n(S) \leqslant \|S - A\| = \|T\| = \|SJ_M^E\| \leqslant c_n(S) + \varepsilon.$$

Every Banach space F is a subspace of a Banach space F^∞ which has the extension property. The embedding map of F into F^∞ is denoted by J_F^∞.

THEOREM 4.3. *Let $S \in \boldsymbol{L}(E, F)$; then*

$$c_n(S) = a_n(J_F^\infty S).$$

Proof. From the injectivity of the Gelfand numbers and Theorem 4.2 it follows

$$c_n(S) = c_n(J_F^\infty S) = a_n(J_F^\infty S).$$

THEOREM 4.4. *The Gelfand numbers are the largest injective s-numbers.*

Proof. Let $S \in \boldsymbol{L}(E, F)$. Then for each injective s-number function we have

$$s_n(S) = s_n(J_F^\infty S) \leqslant a_n(J_F^\infty S) = c_n(S).$$

Let $S \in \boldsymbol{L}(E, F)$. Then the *modulus of injectivity* is defined by

$$j(S) := \sup\{\varrho \geqslant 0: \|Sx\| \geqslant \varrho\|x\|\}.$$

Without proof we state the following lemmas.

LEMMA 4.1. *Let $S, T \in \boldsymbol{L}(E, F)$; then*

$$j(S+T) \leqslant j(S) + \|T\|.$$

LEMMA 4.2. *Let $T \in \boldsymbol{L}(E, F)$ and $S \in \boldsymbol{L}(F, G)$; then*

$$j(ST) \leqslant \|S\| j(T).$$

Moreover, if T is onto then

$$j(ST) \leqslant j(S)\|T\|.$$

For every operator $S \in \boldsymbol{L}(E, F)$ the *Bernstein numbers* are defined by

$$u_n(S) := \sup\{j(SJ_M^E): \dim(M) \geqslant n\}.$$

Remark. It is enough to take the supremum over all subspaces M with $\dim(M) = n$.

THEOREM 4.5. *The map*

$$\text{bern}: S \to (u_n(S))$$

is an injective s-number function.

Proof. We only show

(3) $\quad u_n(RST) \leqslant \|R\| u_n(S) \|T\| \quad$ for $T \in \boldsymbol{L}(E_0, E)$, $S \in \boldsymbol{L}(E, F)$, $R \in \boldsymbol{L}(F, F_0)$.

Let $0 < \varepsilon < u_n(RST)$. Then there is a subspace M_0 of E_0 such that

$$u_n(RST) - \varepsilon \leqslant j(RSTJ_{M_0}^{E_0}) \quad \text{and} \quad \dim(M_0) \geqslant n.$$

Let $M := T(M_0)$, and let T_0 be the restriction of T to M_0 considered as a map into M. Then

$$RSTJ_{M_0}^{E_0} = RSJ_M^E T_0 \quad \text{and} \quad \|T_0\| \leqslant \|T\|.$$

Since by Lemma 4.1

$$0 < u_n(RST) - \varepsilon \leqslant j(RSJ_M^E T_0) \leqslant \|RSJ_M^E\| j(T_0),$$

we have $j(T_0) > 0$. Hence T_0 is one-to-one, and we obtain $\dim(M) \geqslant n$. Consequently, since T_0 is onto, Lemma 4.1 implies

$$u_n(RST) - \varepsilon \leqslant j(RSJ_M^E T_0) \leqslant \|R\| j(SJ_M^E)\|T_0\| \leqslant \|R\| u_n(S)\|T\|.$$

THEOREM 4.6. *The Bernstein numbers are the smallest injective s-numbers.*

Proof. Let $S \in \mathbf{L}(E, F)$. For each injective s-number function we show that $\dim(M) \geqslant n$ implies $j(SJ_M^E) \leqslant s_n(S)$. This proves

$$u_n(S) \leqslant s_n(S) \quad \text{for all } S \in \mathbf{L}.$$

We may assume $j(SJ_M^E) > 0$. Let $M_0 := S(M)$. Then the restriction S_0 of S to M considered as a map into M_0 is invertible, and we have

$$\|S_0^{-1}\| = j(SJ_M^E)^{-1}.$$

Now the conclusion follows from

$$1 = s_n(I_M) \leqslant s_n(S_0)\|S_0^{-1}\| = s_n(J_{M_0}^F S_0)\|S_0^{-1}\|$$
$$\leqslant s_n(SJ_M^E)\|S_0^{-1}\| \leqslant s_n(S) j(SJ_M^E)^{-1}.$$

5. Surjective s-numbers. An s-number function $\mathbf{s}$ is called *surjective* if the following property is satisfied:

Let E/N be a quotient space of E; then

$$s_n(SQ_N^E) = s_n(S) \quad \text{for all } S \in \mathbf{L}(E/N, F).$$

In other words, surjectivity means that the s-numbers $s_n(S)$ do not depend on the domain of S.

For every operator $S \in \mathbf{L}(E, F)$ the *Kolmogorov numbers* are defined by

$$d_n(S) := \inf\{\|Q_N^F S\|: \dim(N) < n\}.$$

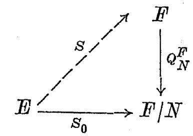
THEOREM 5.1 *The map*

$$\text{kol}: S \to (d_n(S))$$

is a surjective s-number function.

The proof is left to the reader (cf. [13]).

A Banach space E is said to have the *lifting property* if for every operator S_0 mapping E into a quotient space F/N of an arbitrary Banach space F, and for $\varepsilon > 0$, there is a lifting S from E into F such that $\|S\| \leqslant (1+\varepsilon)\|S_0\|$,



THEOREM 5.2. *If E has the lifting property then*

$$d_n(S) = a_n(S) \quad \textit{for all } S \in \mathbf{L}(E, F).$$

Proof. Let $S \in \mathbf{L}(E, F)$. Since $a_n(S)$ are the largest s-numbers, it is enough to show $a_n(S) \leqslant d_n(S)$.

If $\varepsilon > 0$ we choose a subspace N of F such that

$$\|Q_N^E S\| \leqslant d_n(S) + \varepsilon \quad \text{and} \quad \dim(N) < n.$$

Then there exists a lifting $T \in \mathbf{L}(E, F)$ of $Q_N^F S$ with $\|T\| \leqslant (1+\varepsilon)\|Q_N^F S\|$. We set $A := S - T$. Since $Ax \in N$ for all $x \in E$, we have $\dim(A) < n$. Hence

$$a_n(S) \leqslant \|S - A\| = \|T\| \leqslant (1+\varepsilon)(d_n(S)+\varepsilon).$$

Every Banach space E is a quotient space of a Banach space E^1 which has the lifting property. The canonical map of E^1 onto E is denoted by Q_E^1.

THEOREM 5.3. *Let $S \in \mathbf{L}(E, F)$; then*

$$d_n(S) = a_n(SQ_E^1).$$

Proof. From the surjectivity of the Kolmogorov numbers and Theorem 5.2 it follows

$$d_n(S) = d_n(SQ_E^1) = a_n(SQ_E^1).$$

THEOREM 5.4. *The Kolmogorov numbers are the largest surjective s-numbers.*

Proof. Let $S \in \mathbf{L}(E, F)$. Then for each surjective s-number function we have

$$s_n(S) = s_n(SQ_E^1) \leqslant a_n(SQ_E^1) = d_n(S).$$

Let $S \in \mathbf{L}(E, F)$. Then the *modulus of surjectivity* is defined by

$$q(S) := \sup\{\varrho \geqslant 0: S(U_E) \supset \varrho U_F\}.$$

LEMMA 5.1. *Let $S, T \in \mathbf{L}(E, F)$; then*

$$q(S+T) \leqslant q(S) + \|T\|.$$

Proof. We may assume $q(S+T) > \|T\|$. If $0 < \varepsilon < q(S+T) - \|T\|$ we set $\varrho := q(S+T) - \varepsilon$. Let $y \in U_F$. We choose inductively a sequence of elements $x_i \in E$ such that

$$Sx_1 + Tx_1 = (\varrho - \|T\|)y \quad \text{and} \quad \|x_1\| \leqslant \frac{\varrho - \|T\|}{\varrho},$$

$$Sx_2 + Tx_2 = Tx_1 \quad \text{and} \quad \|x_2\| \leqslant \frac{\|Tx_1\|}{\varrho},$$

$$\vdots$$

$$Sx_{n+1} + Tx_{n+1} = Tx_n \quad \text{and} \quad \|x_{n+1}\| \leqslant \frac{\|Tx_n\|}{\varrho},$$

$$\vdots$$

Then

$$\|x_n\| \leqslant \left(\frac{\|T\|}{\varrho}\right)^{n-1} \frac{\varrho - \|T\|}{\varrho} \quad \text{for} \quad n = 1, 2, \ldots$$

Since $\|T\| < \varrho$, it is possible to define

$$x := \sum_1^\infty x_n,$$

and we have

$$Sx = (\varrho - \|T\|)y \quad \text{and} \quad \|x\| \leqslant 1.$$

This proves $S(U_E) \supset (\varrho - \|T\|)U_F$. Consequently,

$$q(S) \geqslant \varrho - \|T\| = q(S+T) - \|T\| - \varepsilon.$$

Without proof we state

LEMMA 5.2. *Let $T \in \boldsymbol{L}(E, F)$ and $S \in \boldsymbol{L}(F, G)$; then*

$$q(ST) \leqslant q(S)\|T\|.$$

Moreover, if S is one-to-one then

$$q(ST) \leqslant \|S\| q(T).$$

For every operator $S \in \boldsymbol{L}(E, F)$ the *Mitiagin numbers* are defined by

$$v_n(S) := \sup\{q(Q_N^F S) : \operatorname{codim}(N) \geqslant n\}.$$

Remark. It is enough to take the supremum over all subspaces N with $\operatorname{codim}(N) = n$.

THEOREM 5.5. *The map*

$$\text{mit}: S \to (v_n(S))$$

is a surjective s-number function.

Proof. We only show

$$(2) \qquad v_n(S+T) \leqslant v_n(S) + \|T\| \quad \text{for } S, T \in \boldsymbol{L}(E, F).$$

Let $\varepsilon > 0$. Then there exists a subspace N of F such that

$$q(Q_N^F(S+T)) \geqslant v_n(S+T) - \varepsilon \quad \text{and} \quad \operatorname{codim}(N) \geqslant n.$$

Using Lemma 5.1 we obtain

$$v_n(S+T) \leqslant q(Q_N^F(S+T)) + \varepsilon \leqslant q(Q_N^F S) + \|Q_N^F T\| + \varepsilon \leqslant v_n(S) + \|T\| + \varepsilon.$$

THEOREM 5.6. *The Mitiagin numbers are the smallest surjective s-numbers.*

The proof is similar to that of Theorem 4.6 and will be omitted.

6. Dual s-numbers. For each s-number function $\boldsymbol{s}$ a *dual s-number function* $\boldsymbol{s}^D$ can be defined by

$$s_n^D(S) := s_n(S') \quad \text{for all } S \in \boldsymbol{L}.$$

Without proof we state the trivial

THEOREM 6.1. *Let $S \in \boldsymbol{L}$; then*

$$a_n(S) \geqslant a_n(S').$$

Remark. Using the principle of local reflexivity, Miss C. V. Hutton [3] has proved that $a_n(S) = a_n(S')$ for every compact operator S, cf. Theorem 6.3. On the other hand she was able to compute the approximation numbers of the identity map I from l_1 into c_0

$$a_n(I) = 1 \quad \text{and} \quad a_n(I') = 1/2 \quad \text{for } n = 2, 3, \ldots$$

THEOREM 6.2. *Let $S \in \boldsymbol{L}$; then*

$$c_n(S) = d_n(S').$$

Proof. Let $S \in \boldsymbol{L}(E, F)$. By duality there is a one-to-one correspondence between subspaces M of E with $\operatorname{codim}(M) < n$ and subspaces N of E' with $\dim(N) < n$,

$$M \to N := \{a \in E' : \langle x, a\rangle = 0 \text{ for all } x \in M\},$$

$$N \to M := \{x \in E : \langle x, a\rangle = 0 \text{ for all } a \in N\}.$$

Now the assertion follows from

$$\|SJ_M^E\| = \|Q_N^{E'} S'\|.$$

THEOREM 6.3 *Let $S \in \mathbf{L}$ such that S is compact; then*

$$d_n(S) = c_n(S').$$

Proof. Using similar arguments as in the preceding proof we obtain $d_n(S) \geqslant c_n(S')$. To show the inverse inequality we need the compactness of S.

If $\varepsilon > 0$ then we find $x_1, \ldots, x_n \in U_E$ with

$$S(U_E) \subset \bigcup_1^n \{Sx_i + \varepsilon U_F\}$$

as well as a subspace N of F'' such that

$$\|Q_N^{F''} S''\| \leqslant d_n(S'') + \varepsilon \quad \text{and} \quad \dim(N) < n.$$

Then there is a finite dimensional subspace M of F'' with

$$N \subset M \quad \text{and} \quad J_F S x_i \in M \quad \text{for } i = 1, \ldots, n.$$

By Lemma 0.1 there exists $R \in \mathbf{L}(M, F)$ such that

$$\|R\| \leqslant 1 + \varepsilon \quad \text{and} \quad RJ_F Sx_i = Sx_i \quad \text{for } i = 1, \ldots, n.$$

We set $N_0 := R(N)$. Using the definition of the quotient norm on F''/N, we choose $z_i'' \in N$ with

$$\|S'' J_E x_i - z_i''\| \leqslant \|Q_N^{F''} S'' J_E x_i\| + \varepsilon \leqslant d_n(S'') + 2\varepsilon.$$

Let $z_i := Rz_i''$. Then $z_i \in N_0$, and therefore

$$\|Q_{N_0}^F Sx_i\| = \|Q_{N_0}^F RJ_F Sx_i\| = \|Q_{N_0}^F RS'' J_E x_i\| \leqslant \|RS'' J_E x_i - z_i\|$$
$$\leqslant \|R\| \, \|S'' J_E x_i - z_i''\| \leqslant (1+\varepsilon)\big(d_n(S'') + 2\varepsilon\big).$$

For each $x \in U_E$ with some index i_0 there holds

$$\|Sx - Sx_{i_0}\| \leqslant \varepsilon.$$

Consequently,

$$\|Q_{N_0}^F Sx\| \leqslant \|Q_{N_0}^F Sx_{i_0}\| + \varepsilon \leqslant (1+\varepsilon)\big(d_n(S'') + 2\varepsilon\big) + \varepsilon.$$

This proves

$$d_n(S) \leqslant d_n(S'') = c_n(S').$$

The proof of the following lemma is implicitly contained in [2], p. 62, or [21], p. 234.

LEMMA 6.1. *Let $S \in \mathbf{L}$; then*

$$j(S) = q(S') \quad \text{and} \quad q(S) = j(S').$$

THEOREM 6.4. *Let $S \in \mathbf{L}$; then*

$$v_n(S) = u_n(S').$$

Proof. Let N be a subspace of F with $\operatorname{codim}(N) = n$, and let

$$M := \{b \in F' : \langle y, b\rangle = 0 \quad \text{for all } y \in N\}$$

be the corresponding subspace in F' with $\dim(M) = n$. Then by Lemma 6.1 we have

$$q(Q_N^F S) = j(S' J_M^{F'}).$$

Now the assertion follows using the same duality arguments as in the proof of Theorem 6.2.

Remark. It is unknown whether

$$u_n(S) = v_n(S')$$

holds for all $S \in \mathbf{L}$.

Finally we state the trivial

THEOREM 6.5. *Let $S \in \mathbf{L}$; then*

$$i_n(S) \leqslant i_n(S').$$

7. s-Numbers of diagonal operators. In this section we compute the s-numbers of diagonal operators S,

$$S(\xi_1, \ldots, \xi_m) := (\sigma_1 \xi_1, \ldots, \sigma_m \xi_m) \quad \text{with} \quad \sigma_1 \geqslant \ldots \geqslant \sigma_m > 0,$$

mapping l_p^m onto l_q^m. Since $s_n(S) = 0$ for $n > m$, in the following let $n = 1, \ldots, m$.

THEOREM 7.1. *If $1 \leqslant p = q \leqslant \infty$ then*

$$s_n(S) = \sigma_n$$

for each s-number function.

Proof. Since $\dim(A) < n$ for

$$A(\xi_1, \ldots, \xi_m) := (\sigma_1 \xi_1, \ldots, \sigma_{n-1}\xi_{n-1}, 0, \ldots, 0),$$

we have

$$s_n(S) \leqslant a_n(S) \leqslant \|S - A\| = \sigma_n.$$

One the other hand, let

$$J(\xi_1, \ldots, \xi_n) := (\xi_1, \ldots, \xi_n, 0, \ldots, 0),$$
$$Q(\xi_1, \ldots, \xi_n, \ldots, \xi_m) := (\xi_1, \ldots, \xi_n),$$

and

$$S_0(\xi_1, \ldots, \xi_n) := (\sigma_1 \xi_1, \ldots, \sigma_n \xi_n).$$

Then $S_0 = QSJ$, and therefore

$$1 = s_n(I_{l_p^n}) \leqslant s_n(S_0)\|S_0^{-1}\| \leqslant \|Q\| s_n(S) \|J\| \sigma_n^{-1} \leqslant s_n(S)\sigma_n^{-1}.$$

Consequently,

$$s_n(S) \geqslant \sigma_n.$$

To prove the following theorem we need two lemmas (cf. [9]).

LEMMA 7.1. *Let M be a subspace of l_∞^m with $\operatorname{codim}(M) < n$; then there exists $e = (\varepsilon_1, \ldots, \varepsilon_m) \in M$ with $\|e\|_\infty = 1$ such that the set*

$$K := \{k\colon\ |\varepsilon_k| < 1\}$$

has less than n elements.

Proof. We consider an extremum point e of U_M. Let us assume that K has at least n elements. If

$$N := \{x \in l_\infty^m\colon\ \xi_k = 0 \text{ for } k \notin K\}$$

then $\dim(N) \geqslant n$. Hence we find $y \in M \cap N$ with $\|y\|_\infty = 1$. Since

$$\varrho := \max\{|\varepsilon_k|\colon\ k \in K\} < 1,$$

we have

$$e \pm \delta y \in U_M \quad \text{for } 0 < \delta < 1 - \varrho.$$

So e cannot be an extremum point of U_M. Contradiction.

LEMMA 7.2. *If $0 < q < p < \infty$, $\mu_1, \ldots, \mu_{n+1} > 0$, and $|\xi_{n+1}| \leqslant |\xi_k|$ for $k = 1, \ldots, n$, then*

$$\frac{\Big\{\sum_1^{n+1} |\xi_k|^q \mu_k\Big\}^{1/q}}{\Big\{\sum_1^{n+1} |\xi_k|^p \mu_k\Big\}^{1/p}} \geqslant \frac{\Big\{\sum_1^{n} |\xi_k|^q \mu_k\Big\}^{1/q}}{\Big\{\sum_1^{n} |\xi_k|^p \mu_k\Big\}^{1/p}}.$$

Proof. We set

$$\alpha := \Big\{\sum_1^n |\xi_k|^p \mu_k\Big\}^{1/p} \quad \text{and} \quad \beta := \Big\{\sum_1^n |\xi_k|^q \mu_k\Big\}^{1/q}.$$

From

$$\left|\frac{\xi_k}{\xi_{n+1}}\right|^q \leqslant \left|\frac{\xi_k}{\xi_{n+1}}\right|^p \quad \text{for } k = 1, \ldots, n$$

it follows

$$\left|\frac{\xi_{n+1}}{\alpha}\right|^p \leqslant \left|\frac{\xi_{n+1}}{\beta}\right|^q.$$

Consequently,

$$\frac{\Big\{\sum_1^{n+1} |\xi_k|^q \mu_k\Big\}^{1/q}}{\Big\{\sum_1^{n+1} |\xi_p|^p \mu_k\Big\}^{1/p}} = \frac{\{\beta^q + |\xi_{n+1}|^q \mu_{n+1}\}^{1/q}}{\{\alpha^p + |\xi_{n+1}|^p \mu_{n+1}\}^{1/p}} = \frac{\beta\{1 + |\xi_{n+1}/\beta|^q \mu_{n+1}\}^{1/q}}{\alpha\{1 + |\xi_{n+1}/\alpha|^p \mu_{n+1}\}^{1/p}}$$

$$\geqslant \frac{\beta}{\alpha} = \frac{\Big\{\sum_1^{n} |\xi_k|^q \mu_k\Big\}^{1/q}}{\Big\{\sum_1^{n} |\xi_k|^p \mu_k\Big\}^{1/p}}.$$

THEOREM 7.2. *If $1 \leqslant q < p \leqslant \infty$ then*

$$a_n(S) = c_n(S) = d_n(S) = \Big\{\sum_n^m \sigma_k^r\Big\}^{1/r}$$

where $1/r := 1/q - 1/p$.

Proof. Since $\dim(A) < n$ for

$$A(\xi_1, \ldots, \xi_m) := (\sigma_1\xi_1, \ldots, \sigma_{n-1}\xi_{n-1}, 0, \ldots, 0),$$

we have

(a) $$a_n(S) \leqslant \|S - A\| = \Big\{\sum_n^m \sigma_k^r\Big\}^{1/r}.$$

On the other hand, let M be an arbitrary subspace of l_p^m with $\operatorname{codim}(M) < n$. If

$$D(\xi_1, \ldots, \xi_m) := (\sigma_1^{-r/p}\xi_1, \ldots, \sigma_m^{-r/p}\xi_m)$$

is considered as a map from l_p^m onto l_∞^m by Lemma 7.1 there exists $e = (\varepsilon_1, \ldots, \varepsilon_m) \in D(M)$ with $\|e\|_\infty = 1$ such that

$$K := \{k\colon\ |\varepsilon_k| < 1\}$$

has less than n elements. We set $x := D^{-1}e$. Then from Lemma 7.2 it follows

$$\|SJ_M^{l_p^m}\| \geqslant \frac{\|Sx\|_q}{\|x\|_p} = \frac{\Big\{\sum_1^m |\varepsilon_k|^q \sigma_k^r\Big\}^{1/q}}{\Big\{\sum_1^m |\varepsilon_k|^p \sigma_k^r\Big\}^{1/p}} \geqslant \frac{\Big\{\sum_{k\notin K} |\varepsilon_k|^q \sigma_k^r\Big\}^{1/q}}{\Big\{\sum_{k\notin K} |\varepsilon_k|^p \sigma_k^r\Big\}^{1/p}} = \Big\{\sum_{k\notin K} \sigma_k^r\Big\}^{1/r} \geqslant \Big\{\sum_n^m \sigma_k^r\Big\}^{1/r}.$$

Consequently,

(c) $$c_n(S) \geqslant \Big\{\sum_n^m \sigma_k^r\Big\}^{1/r}.$$

By Theorem 6.2 we have

(d) $$d_n(S) = c_n(S') \geqslant \Big\{\sum_n^m \sigma_k^r\Big\}^{1/r}.$$

Finally, the assertion follows from (a), (c) and (d).

If $p = \infty$ the proof must be changed in an obvious way and we do not need Lemma 7.2.

In the case $1 \leqslant p < q \leqslant \infty$ the s-numbers $a_n(S)$, $c_n(S)$ and $d_n(S)$ seem to be unknown. A special result was proved by S. A. Smoljak (cf. [19]).

THEOREM 7.3. *If* $p = 1$ *and* $q = 2$ *then*

$$a_n(S) = d_n(S) = \max_{n \leqslant h \leqslant m} \left\{ \frac{h-n+1}{\sum_1^h \sigma_k^{-2}} \right\}^{1/2}.$$

Remark. Let I_m be the identity map of l_1^m onto l_2^m. If

$$M := \left\{ x \in l_1^m : \sum_1^m \xi_k = 0 \right\}$$

then $\operatorname{codim}(M) < 2$. Since

$$\|IJ_M^{l_1^m}\| = 1/\sqrt{2},$$

it follows

$$c_2(I) \leqslant 1/\sqrt{2}.$$

On the other hand, from Theorem 7.3 we obtain

$$a_2(I) = d_2(I) = \sqrt{(m-1)/m} > 1/\sqrt{2} \quad \text{for } m = 3, 4, \ldots$$

This proves that the s-numbers a_n, c_n and d_n are different in general.

Remark. In the next step one should try to compute the value of $a_n(I_m)$ for the identity map I_m of l_1^m onto l_∞^m. Using the operators

$$A_2 := \frac{1}{2} e \otimes e \quad \text{and} \quad A_m := I_m - \frac{1}{m} e \otimes e \quad \text{with } e = (1, \ldots, 1)$$

it can be proved that

$$a_2(I) = 1/2 \quad \text{and} \quad a_m(I) = 1/m.$$

From R. S. Ismagilov the author was informed about the following result:

$$a_n(I_m) = O\left(\frac{m^{1/2+\varepsilon}}{n}\right) \quad \text{for each } \varepsilon > 0.$$

There is some kind of duality between s-numbers.

LEMMA 7.3. *Let* $\dim(E) = \dim(F) = m$, *and let* $S \in \boldsymbol{L}(E, F)$. *If* S *is invertible then*

$$u_n(S) c_{m-n+1}(S^{-1}) = 1$$

and

$$v_n(S) d_{m-n+1}(S^{-1}) = 1.$$

Proof. Let M be a subspace of E with $\dim(M) \geqslant n$. If $N := S(M)$ then $\operatorname{codim}(N) < m-n+1$, and from

$$j(SJ_M^E)\|S^{-1}J_N^F\| = 1$$

we obtain

$$u_n(S) c_{m-n+1}(S^{-1}) = 1.$$

The proof of the other equality is analogous.

Remark. It is unknown whether, with the same assumption as in Lemma 7.3, there holds

$$i_n(S) a_{m-n+1}(S^{-1}) = 1.$$

Up to this time we can only prove an inequality. For this purpose, if $\varepsilon > 0$, we choose a Banach space G as well as operators $X \in \boldsymbol{L}(G, E)$ and $B \in \boldsymbol{L}(F, G)$ such that

$$i_n(S) - \varepsilon \leqslant \|B\|^{-1}\|X\|^{-1}, \quad I_G = BSX \quad \text{and} \quad \dim(G) \geqslant n.$$

Let $A := S^{-1} - XB$. Then from $\dim(X) \geqslant n$ and $A(F) \cap X(G) = \{0\}$ it follows $\dim(A) < m-n+1$. Consequently,

$$\big(i_n(S) - \varepsilon\big) a_{m-n+1}(S^{-1}) \leqslant \|B\|^{-1}\|X\|^{-1}\|S^{-1} - A\| \leqslant 1.$$

As an immediate consequence of Theorem 7.2 and Lemma 7.3 we obtain

THEOREM 7.4. *If* $1 \leqslant p < q \leqslant \infty$ *then*

$$i_n(S) = u_n(S) = v_n(S) = \left\{ \sum_1^n \sigma_k^{-r} \right\}^{-1/r}$$

where $1/r = 1/p - 1/q$.

8. Relations between some s-numbers. As a consequence of the preceding results (Theorems 3.2, 3.4, 4.4, 4.6, 5.4, and 5.6) we have

THEOREM 8.1. *Let* $S \in \boldsymbol{L}$; *then*

$$a_n(S) \geqslant c_n(S) \geqslant u_n(S) \geqslant i_n(S)$$

and

$$a_n(S) \geqslant d_n(S) \geqslant v_n(S) \geqslant i_n(S).$$

The following statement is well-known (cf. [8]).

THEOREM 8.2. *Let* $S \in \boldsymbol{L}$; *then*

$$d_n(S) \geqslant u_n(S).$$

Proof. Let $S \in \boldsymbol{L}(E, F)$. Since

$$d_n(S) := \inf\{\|Q_N^F S\| : \dim(N) < n\}$$

and

$$u_n(S) := \sup\{j(SJ_M^E) : \dim(M) \geqslant n\},$$

it is enough to show

$$\|Q_N^F S\| \geqslant j(SJ_M^E).$$

We may assume $j(SJ_M^E) > 0$. If $M_0 := S(M)$ then $\dim(M_0) \geqslant n$. Consequently, by Lemma 0.2 there exists $x \in M$ such that

$$\|Q_N^F Sx\| = \|Sx\| = 1.$$

Now the inequality which we want to prove follows from

$$1 = \|Sx\| \geqslant j(SJ_M^E)\|x\| \quad \text{and} \quad 1 = \|Q_N^F Sx\| \leqslant \|Q_N^F S\|\,\|x\|.$$

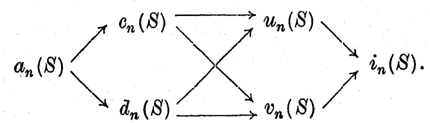
THEOREM 8.3. *Let $S \in \boldsymbol{L}$; then*

$$c_n(S) \geqslant v_n(S).$$

Proof. Using Theorems 6.2 and 6.4 we have

$$c_n(S) = d_n(S') \geqslant u_n(S') = v_n(S).$$

The results are represented in the following diagram where the arrows point from the larger s-numbers to the smaller ones,



THEOREM 8.4. *Let $S \in \boldsymbol{L}$; then*

$$a_n(S) \leqslant \varrho n^{1/2} d_n(S) \quad \text{and} \quad a_n(S) \leqslant \varrho n^{1/2} c_n(S)$$

where ϱ is a positive constant.

Proof. Let $S \in \boldsymbol{L}(E, F)$. For $\varepsilon > 0$ we choose a subspace N of F such that

$$\|Q_N^F S\| \leqslant d_n(S) + \varepsilon \quad \text{and} \quad \dim(N) < n.$$

Then by Lemma 0.3 there exists a projection $P \in \boldsymbol{L}(F, F)$ with $N = P(F)$ and $\|P\| \leqslant (n-1)^{1/2}$. Next, by

$$J(y+N) := y - Py$$

we define an operator $J \in \boldsymbol{L}(F/N, F)$. Then

$$\|J\| \leqslant \|I_F - P\| \leqslant 1 + (n-1)^{1/2} \leqslant \varrho n^{1/2}$$

where $\varrho = \sqrt{2}$. From

$$S - PS = (I_F - P)S = JQ_N^F S$$

we obtain

$$a_n(S) \leqslant \|S - PS\| \leqslant \|J\|\,\|Q_N^F S\| \leqslant \varrho n^{1/2}\big(d_n(S) + \varepsilon\big).$$

The proof of the other inequality is similar and will be omitted.

Remark. It is unknown whether

$$a_n(S) \leqslant \varrho_\alpha n^\alpha d_n(S)$$

holds for an exponent $\alpha < 1/2$.

THEOREM 8.5. *Let $S \in \boldsymbol{L}$; then*

$$u_n(S) \leqslant n^{1/2} i_n(S) \quad \text{and} \quad v_n(S) \leqslant n^{1/2} i_n(S).$$

Proof. Let $S \in \boldsymbol{L}(E, F)$. If $0 < \varepsilon < u_n(S)$ we choose a subspace M of E such that

$$u_n(S) - \varepsilon < j(SJ_M^E) \quad \text{and} \quad \dim(M) = n.$$

Let $N := S(M)$. Since $j(SJ_M^E) > 0$, the restriction S_0 of S to M considered as a map onto N is invertible, and we have

$$j(SJ_M^E) = \|S_0^{-1}\|^{-1}.$$

Let $P \in \boldsymbol{L}(F, N)$ such that $PJ_N^F = I_N$ and $\|P\| \leqslant n^{1/2}$. Then

$$I_M = S_0^{-1} PSJ_M^E.$$

Consequently,

$$i_n(S) \geqslant \|S_0^{-1}P\|^{-1}\|J_M^E\|^{-1} \geqslant n^{-1/2}\|S_0^{-1}\|^{-1} = n^{-1/2}\big(u_n(S) - \varepsilon\big).$$

The proof of the other inequality is similar and will be omitted.

The next statement was proved by B. S. Mitiagin and G. M. Henkin [10].

THEOREM 8.6. *Let $S \in \boldsymbol{L}$; then*

$$c_n(S) \leqslant n^2 v_n(S) \quad \text{and} \quad d_n(S) \leqslant n^2 u_n(S).$$

Remark. Probably there holds

$$c_n(S) \leqslant n v_n(S) \quad \text{and} \quad d_n(S) \leqslant n u_n(S).$$

A smaller exponent of n as $\alpha = 1$ is impossible since for the identity map I of l_1 into l_∞ we have

$$u_n(I) = v_n(I) = 1/n \quad \text{and} \quad c_n(I) = d_n(I) \geqslant 1/2, \quad \text{cf. [3].}$$

As an immediate consequence of the preceding results we obtain

THEOREM 8.7. *Let $S \in \boldsymbol{L}$; then*

$$a_n(S) \leqslant \varrho n^3 i_n(S)$$

where ϱ is a positive constant.

9. Ideals of operators. For each subclass $\boldsymbol{A}$ of $\boldsymbol{L}$ we set

$$\boldsymbol{A}(E, F) := \boldsymbol{A} \cap \boldsymbol{L}(E, F).$$

$\boldsymbol{A}$ is called an *ideal of operators* if the following conditions are satisfied (cf. [15]):

(1) *If* $a_0 \in E'$ *and* $y_0 \in F$ *then* $a_0 \otimes y_0 \in \boldsymbol{A}(E, F)$.

(2) *If* $T \in \boldsymbol{L}(E_0, E)$, $S \in \boldsymbol{A}(E, F)$ *and* $R \in \boldsymbol{L}(F, F_0)$ *then* $RST \in \boldsymbol{A}(E_0, F_0)$.

(3) *If* $S_1, S_2 \in \boldsymbol{A}(E, F)$ *then* $S_1 + S_2 \in \boldsymbol{A}(E, F)$.

A subclass $\boldsymbol{A}$ of $\boldsymbol{L}$ with properties (1) and (2) is said to be an *idol of operators*.

Let $\boldsymbol{s}$ be an s-number function. Then we define

$$\boldsymbol{S}_p^s := \Big\{S \in \boldsymbol{L}:\ \sum_1^\infty s_n(S)^p < \infty\Big\} \quad \text{for } 0 < p < \infty,$$

and

$$\boldsymbol{S}_\infty^s := \{S \in \boldsymbol{L}:\ \lim_n s_n(S) = 0\}.$$

We have the trivial

THEOREM 9.1. *The class* $\boldsymbol{S}_p^s$ *is an idol,* $0 < p < \infty$.

THEOREM 9.2. *The class* $\boldsymbol{S}_\infty^s$ *is a closed idol.*

Proof. Let $S \in \boldsymbol{L}(E, F)$. We suppose that, for every positive $\varepsilon > 0$, there is $S_0 \in \boldsymbol{S}_\infty^s(E, F)$ with $\|S - S_0\| \leqslant \varepsilon$. Then we find a natural number n_0 such that

$$s_n(S_0) \leqslant \varepsilon \quad \text{for } n \geqslant n_0.$$

Consequently,

$$s_n(S) \leqslant \|S - S_0\| + s_n(S_0) \leqslant 2\varepsilon \quad \text{for } n \geqslant n_0,$$

and therefore $S \in \boldsymbol{S}_\infty^s(E, F)$. This proves the closedness of $\boldsymbol{S}_\infty^s(E, F)$.

Let $\boldsymbol{K}$ be the class of compact operators. Then we state the known (cf. [14], p. 146)

THEOREM 9.3. $\boldsymbol{S}_\infty^{\mathrm{gel}} = \boldsymbol{S}_\infty^{\mathrm{kol}} = \boldsymbol{K}$.

Proof. Let $S \in \boldsymbol{K}(E, F)$. If $\varepsilon > 0$, we choose $y_1, \ldots, y_m \in F$ such that

$$S(U_E) \subset \bigcup_1^m \{y_i + \varepsilon U_F\}.$$

Let N be a finite dimensional subspace of F with $y_1, \ldots, y_m \in N$. Then $\|Q_N^F S\| \leqslant \varepsilon$. Consequently,

$$d_n(S) \leqslant \varepsilon \quad \text{for all } n \geqslant n_0 := \dim(N).$$

This proves $\boldsymbol{K} \subset \boldsymbol{S}_\infty^{\mathrm{kol}}$.

Now the inverse statement will be established. Let $S \in \boldsymbol{S}_\infty^{\mathrm{kol}}(E, F)$. If $\varepsilon > 0$, we choose a natural number n with $d_n(S) < \varepsilon$. Hence there is a subspace N of F such that

$$\|Q_N^F S\| < \varepsilon \quad \text{and} \quad \dim(N) < n.$$

Since U_N is compact, we find $y_1, \ldots, y_m \in F$ such that

$$(\|S\| + \varepsilon)\, U_N \subset \bigcup_1^m \{y_i + \varepsilon U_F\}.$$

Let $x \in U_E$. Then $\|Q_N^F Sx\| < \varepsilon$ and, therefore, $\|Sx - y\| < \varepsilon$ for some $y \in N$. Since $\|y\| \leqslant \|S\| + \varepsilon$, we have

$$y \in \bigcup_1^m \{y_i + \varepsilon U_F\}.$$

Consequently,

$$Sx \in \bigcup_1^m \{y_i + 2\varepsilon U_F\} \quad \text{for all } x \in U_E.$$

This proves $\boldsymbol{S}_\infty^{\mathrm{kol}} \subset \boldsymbol{K}$.

Finally, $\boldsymbol{S}_\infty^{\mathrm{gel}} = \boldsymbol{S}_\infty^{\mathrm{kol}}$ follows from $c_n(S) = d_n(S')$ and Schauder's theorem (cf. [21], p. 275).

An s-number function $\boldsymbol{s}$ is called *additive* if the following improvement of condition (2) of § 2 is satisfied:

(2*) $s_{m+n-1}(S + T) \leqslant s_m(S) + s_n(T)$ *for* $S, T \in \boldsymbol{L}(E, F)$ *and* $m, n = 1, 2, \ldots$

THEOREM 9.4. *Let* $\boldsymbol{s}$ *be an additive s-number function. Then* $\boldsymbol{S}_p^s$ *is an ideal of operators,* $0 < p \leqslant \infty$.

Proof. Let $S_1, S_2 \in \boldsymbol{S}_p^s(E, F)$. Since

$$(\sigma_1 + \sigma_2)^p \leqslant \varrho_p(\sigma_1^p + \sigma_2^p) \quad \text{for } \sigma_1, \sigma_2 \geqslant 0$$

with $\varrho_p := \max(2^{p-1}, 1)$, we have

$$\begin{aligned}\sum_1^\infty s_n(S_1 + S_2)^p &\leqslant 2 \sum_1^\infty s_{2n-1}(S_1 + S_2)^p \\ &\leqslant 2 \sum_1^\infty \big(s_n(S_1) + s_n(S_2)\big)^p \\ &\leqslant 2\varrho_p \Big(\sum_1^\infty s_n(S_1)^p + \sum_1^\infty s_n(S_2)^p\Big).\end{aligned}$$

If $p = \infty$ then

$$\lim_n s_n(S_1 + S_2) = \lim_n s_{2n-1}(S_1 + S_2) \leqslant \lim_n s_n(S_1) + \lim_n s_n(S_2) = 0.$$

Remark. By the definition

$$\Sigma_p^s(S) := \left\{\sum_{1}^{\infty} s_n(S)^p\right\}^{1/p} \quad \text{for } S \in \mathbf{S}_p^s$$

we obtain a quasinorm Σ_p^s which is in general not a norm even in the case $1 \leqslant p < \infty$.

The following statement is proved in [14].

THEOREM 9.5. *The approximation numbers, Gelfand numbers and Kolmogorov numbers are additive.*

Remark. It seems to be unknown whether the isomorphism numbers, Bernstein numbers and Mitiagin numbers are additive.

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Received February 14, 1973 (648)