$s$-Numbers of operators in Banach spaces

by

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Abstract. For each operator between Banach spaces one can define the sequence of approximation numbers, Kolmogorov numbers, Gelfand numbers, etc. As an unification we present an axiomatic theory of the so-called $s$-numbers, and we discuss related ideas of operators.

As has been shown in the famous book of I. Z. Gochberg and M. G. Krejn [1] the $s$-numbers are an important tool in the spectral theory of Hilbert space (cf. [18]). The $s$-number $s_n(S)$ of a compact operator $S$ from an infinite dimensional Hilbert space $H$ into itself is defined as the $n$th eigenvalue of the operator $|S| := (S^*S)^{1/2}$.

Particularly, the $s$-numbers can be used to describe the ideals in the ring $L(H, H)$ of operators. Let $S_c(H, H)$ be the closed ideal of compact operators. Then the most interesting ideals discovered by J. v. Neumann and R. Schatten are defined by

$$S_p(H, H) := \{S : S \in S_c(H, H) : \sum_{n=1}^{\infty} s_n(S)^p < \infty\}, \quad 0 < p < \infty.$$ 

For $p = 1$ we obtain the trace class of operators, and $S_1(H, H)$ is the ideal of Hilbert–Schmidt operators which are characterized by the inequality

$$\sum_{i, k} |(S_{ek}, f_k)|^2 < \infty$$

for arbitrary complete orthonormal systems $(e_k)$ and $(f_k)$.

The purpose of this paper is to present an axiomatic theory of $s$-numbers of operators in Banach spaces. Since we want to give a general survey, some known results for special $s$-numbers are reproduced.

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0. Prerequisites. Let $E$ be a real or complex Banach space with the closed unit ball $B_E$. The identity map of $E$ is denoted by $I_E$.

A subspace is a closed linear subset. The embedding map of a subspace $M$ into $E$ is denoted by $J_M$, and the canonical map of $E$ onto the quotient space $E/M$ is denoted by $Q_M$.

An operator is a bounded linear map. Let $L$ be the class of all operators between Banach spaces. The set of those operators which map $E$ into $F$ is denoted by $L(E, F)$.

Let $\dim(M)$ be the dimension of the subspace $M$, and let $\operatorname{codim}(M) = \dim(E/M)$ be the codimension. If the operator $S$ is of finite rank, then the dimension of the image is denoted by $\dim(S)$.

For $a \in E^\ast$ (dual Banach space) and $y \in F$, let $a \otimes y$ be the map: $x \mapsto (x, a(x))y$.

Next we state some important lemmas which are used in the following.

**Lemma 0.1.** (Principle of local reflexivity, cf. [15]). Let $M$ be a finite dimensional subspace of $E'$. If $s > 0$, then there exists $R \in L(M, E)$ such that

$$\|R\| \leq 1 + s\quad \text{and}\quad R(J_M x) = x \quad \text{for all} \quad x \in E \cap J_M^{-1}(M)$$

where $J_M$ denotes the canonical map of $E$ into $E'$.

**Lemma 0.2.** (Cf. [7], p. 199). Let $M$ and $N$ be finite dimensional subspaces of $E$ with $\dim(M) > \dim(N)$. Then there exists $x \in M$ such that

$$\|x\| \leq \|x\| = 1.$$

**Lemma 0.3.** (Cf. [9]). Let $M$ be a subspace of $E$ with $\dim(M) = m$. Then there exists a projection $P \in L(E, E)$ such that

$$M = P(E) \quad \text{and} \quad \|P\| \leq 2^m.$$

1. Axiomatic properties of $s$-numbers. For operators in Banach spaces there are several possibilities to define sequences of numbers which coincide with $s$-numbers in the case of Hilbert space. A report about such numbers was given in 1966 by B. S. Mitjagin and A. Pelczyński [11] at the Moscow Congress in 1966 (cf. [12]).

In the sequel we deal with an axiomatic theory of $s$-numbers. A map

$$s : S \to (s_n(S))$$

from $L$ into the set of sequences of non-negative numbers is called an $s$-number function if the following conditions are satisfied ($n = 1, 2, \ldots)$:

1. $|S| = s_1(S) \geq s_2(S) \geq \ldots \geq 0$ for $S \in L$.
2. $s_n(S + T) \leq s_n(S) + |T|$ for $S, T \in L(E, F)$.
3. $s_n(RST) \leq |S| s_n(T)$ for $T \in L(E, F), S \in L(E, F), R \in L(F, E)$.
4. If $\dim(S) < n$ then $s_n(S) = 0$.
5. If $\dim(E) \geq n$ then $s_n(I_E) = 1$.

The number $s_n(S)$ is said to be the $n$-th $s$-number of the operator $S$.

**Theorem 1.1.** The $s$-numbers are continuous functions since

$$|s_n(S) - s_n(T)| \leq |S - T|$$

for $S, T \in L(E, F)$.

**Proof.** By (2) we have

$$s_n(S) \leq s_n(T) + |S - T|.$$

Furthermore the inverse statement of condition (4) is valid.

**Theorem 1.2.** If $s_n(S) = 0$ then $\dim(S) < n$.

**Proof.** The assertion is an easy consequence of (3), (5) and

**Lemma 1.1.** Let $S \in L(E, F)$. If $\dim(S) > n$ then there exist a Banach space $G$ as well as operators $X \in L(G, E)$ and $B \in L(F, G)$ such that

$$I_G = BX \quad \text{and} \quad \dim(G) \geq n.$$

**Proof.** We choose $m_1, \ldots, m_n \in E$ such that $s_{m_1}, \ldots, s_{m_n}$ are linearly independent. Then by the Hahn--Banach theorem there are $b_1, \ldots, b_n \in F$ with $(s_{m_i}, b_i) = \delta_{m_i}$. Let $G := \sum_{i=1}^n m_i^\ast$,

$$X(m_i) := \sum_{i=1}^n \xi_i m_i \quad \text{for} \quad (\xi_i) \in \ell_2,$$

and

$$B(y) := (y, b_i) \quad \text{for} \quad y \in F.$$

2. $s$-Numbers of operators in Hilbert space. Now we show that $s$-numbers of operators in a Hilbert space are determined uniquely by their axiomatic properties.

**Theorem 2.1.** Let $S \in L(H, F)$, and let $P(\cdot)$ be the spectral measure of the positive operator $|S| := (S^*S)^{\frac{1}{2}}$. Then

$$s_n(S) = \inf \{\sigma \geq 0 : \dim(P(\sigma, \infty)) < n\}$$

for each $s$-number function.

**Proof.** By the theorem of polar representation (cf. [11], p. 21; [17], p. 284) there is a partially isometric operator $U$ such that

$$S = U|S| \quad \text{and} \quad S^* = U^* S.$$

Hence

$$s_n(S) = s_n(U|S|).$$

We set $a_{\alpha} := \inf \{x \geq 0 : \dim(P(\sigma, \infty)) < n\}$. If $\varepsilon > 0$ then from

$$|S| = \int \sigma P(d\sigma) = \int \sigma P(d\sigma) + \int \sigma P(d\sigma)$$

$$= \int \sigma P(d\sigma) = \int \sigma P(d\sigma) + \int \sigma P(d\sigma)$$

$$= \int \sigma P(d\sigma) = \int \sigma P(d\sigma) + \int \sigma P(d\sigma).$$
and
\[
\dim \left( \int_{a_n^{+}}^{a_n^{-}} \sigma_n(\sigma) \, d\sigma \right) < n
\]
we obtain
\[
s_n(|S|) \leq \int_{a_n^{+}}^{a_n^{-}} \sigma_n(\sigma) \, d\sigma + s_n \left( \int_{a_n^{+}}^{a_n^{-}} \sigma_n(\sigma) \, d\sigma \right) \leq s_n + \varepsilon.
\]
On the other hand, let \( s_n > \varepsilon > 0 \). Then
\[
P(s_n - \varepsilon, \infty) = \int_{s_n - \varepsilon}^{\infty} P(\sigma) \, d\sigma \leq s_n \int_{s_n - \varepsilon}^{\infty} \sigma^{-1} P(\sigma) \, d\sigma
\]
and
\[
\dim \left( \int_{s_n - \varepsilon}^{\infty} P(\sigma) \, d\sigma \right) \geq n
\]
imply
\[1 = s_n \left( \int_{s_n - \varepsilon}^{\infty} P(\sigma) \, d\sigma \right) \leq s_n \left( |S| \right) \int_{s_n - \varepsilon}^{\infty} \sigma^{-1} P(\sigma) \, d\sigma \leq s_n (|S|) (s_n - \varepsilon)^{-1}.
\]
So we have
\[s_n - \varepsilon \leq s_n (|S|) \leq s_n + \varepsilon \quad \text{for all } \varepsilon > 0.
\]
**Corollary.** Let \( S \in S_n(H, H) \). Then \( s_n(S) \) is the \( n \)-th eigenvalue of the positive operator \(|S|\).

3. **Approximation numbers and isomorphism numbers.** Now we present two examples of s-number functions.

For every operator \( S \in L(E, F) \) the approximation numbers are defined by
\[a_n(\varepsilon) := \inf \left\{ \|S - A\| : A \in L(E, F), \dim(A) < n \right\}.
\]

**Theorem 3.1.** The map
\[\text{app} : S \rightarrow a_n(S)\]
is an s-number function.

**Proof.** Since the other properties are trivial we prove the condition (5). Let us assume \( a_n(I_E) < 1 \). Then there exists \( A \in L(E, E) \) such that \( \|I_E - A\| < 1 \) and \( \dim(A) < n \). Consequently, \( A = I_E - (I_E - A) \) is invertible by the Neumann series, and we have \( \dim(A) > n \). Contradiction.

**Theorem 3.2.** The approximation numbers are the largest s-numbers.

**Proof.** Let \( S \in L(E, F) \). Then for each s-number function and \( A \in L(E, F) \) with \( \dim(A) < n \) we have
\[s_n(S) \leq s_n(A) + \|S - A\| = \|S - A\|.
\]
Hence
\[s_n(S) \leq a_n(S) \quad \text{for all } S \in L.
\]

For every operator \( S \in L(E, F) \) the isomorphism numbers are defined as follows. If \( \dim(S) < n \) we set \( i_n(S) := 0 \). If \( \dim(S) \geq n \) then by Lemma 1.1 there exist a Banach space \( G \) as well as operators \( X \in L(G, E) \) and \( B \in L(F, G) \) such that
\[\mathcal{I}_G = BSX \quad \text{and} \quad \dim(G) \geq n.
\]
In this case let
\[i_n(S) := \sup\{\|B^{-1} X^{-1}\|^{-1}\},
\]
where the supremum is taken over all possibilities.

**Theorem 3.3.** The map
\[\text{iso} : S \rightarrow i_n(S)\]
is an s-number function.

**Proof.** Since the other properties are trivial, we prove
\[i_n(S + T) \leq i_n(S) + \|T\| \quad \text{for } S, T \in L(E, F).
\]
We may assume \( i_n(S + T) > \|T\| \). If \( 0 < \varepsilon < i_n(S + T) - \|T\| \) then there exist a Banach space \( G \) as well as operators \( X \in L(G, E) \) and \( B \in L(F, G) \) such that
\[\mathcal{I}_G = B(S + T) X, \quad \dim(G) \geq n, \quad \|B^{-1} X^{-1}\|^{-1} \geq i_n(S + T) - \varepsilon > \|T\|.
\]
Since \( \|B TX\| < 1 \), the operator
\[BSX = B(S + T) X - BTX = \mathcal{I}_G - B TX\]
is invertible. From
\[\mathcal{I}_G = (I_G - B TX)^{-1} BSX \quad \text{and} \quad \|I_G - B TX\|^{-1} \leq (1 - \|B TX\|)^{-1}
\]
it follows that
\[i_n(S) \geq \|I_G - B TX\|^{-1} \|B^{-1} X^{-1}\|^{-1}
\]
\[\geq (1 - \|B TX\|)\|B^{-1} X^{-1}\|^{-1}
\]
\[\geq \|B^{-1} X^{-1}\|^{-1} - \|T\|
\]
\[\geq i_n(S + T) - \varepsilon - \|T\|.
\]
Consequently,
\[i_n(S + T) \leq i_n(S) + \|T\| + \varepsilon.
\]

**Theorem 3.4.** The isomorphism numbers are the smallest s-numbers.

**Proof.** Let \( S \in L(E, F) \), \( X \in L(G, E) \) and \( B \in L(F, G) \) such that
\[\mathcal{I}_G = BSX \quad \text{and} \quad \dim(G) \geq n.
\]
Then for each s-number function we have
\[1 = i_n(I_G) \leq \|B\| i_n(S) \|X\|.
\]
Hence
\[i_n(S) \leq i_n(S) \quad \text{for all } S \in L.
\]
4. Injective s-numbers. An s-number function \( s \) is called injective if the following property is satisfied:

Let \( M \) be a subspace of \( F \); then

\[
\sigma_n(J^F_S) = \sigma_n(S) \quad \text{for all } S \in L(E, M).
\]

In other words, injectivity means that the s-numbers \( \sigma_n(S) \) do not depend on the codomain of \( S \).

For every operator \( S \in L(E, F) \) the Gelfand numbers are defined by

\[
\sigma_n(S) := \inf \{ \| SJ^F_M \| : \dim(M) < n \}.
\]

**Theorem 4.1.** The map

\[
gel: S \rightarrow \{ \sigma_n(S) \}
\]

is an injective s-number function.

The proof is left to the reader.

A Banach space \( F \) is said to have the extension property if for every operator \( S \) mapping a subspace \( M \) of an arbitrary Banach space \( E \) into \( F \) there is an extension \( S \) from \( E \) into \( F \) such that \( \| S \| = \| S \| \).

**Theorem 4.2.** If \( F \) has the extension property then

\[
\sigma_n(S) = \sigma_n(S) \quad \text{for all } S \in L(E, F).
\]

**Proof.** Let \( S \in L(E, F) \). Since \( \sigma_n(S) \) are the largest s-numbers, it is enough to show \( \sigma_n(S) \leq \sigma_n(S) \).

If \( \varepsilon > 0 \) we choose a subspace \( M \) of \( E \) such that

\[
\| SJ^F_M \| \leq \sigma_n(S) + \varepsilon \quad \text{and} \quad \dim(M) < n.
\]

Then there exists an extension \( T \) of \( S \) with \( \| T \| = \| SJ^F_M \| \). We set \( A := S - T \). Since \( A \) is an operator, we have \( \dim(A) < n \). Hence

\[
\sigma_n(S) \leq \| S - A \| = \| T \| = \| SJ^F_M \| \leq \sigma_n(S) + \varepsilon.
\]

Every Banach space \( F \) is a subspace of a Banach space \( F^\infty \) which has the extension property. The embedding map of \( F \) into \( F^\infty \) is denoted by \( J^F \).

**Theorem 4.3.** Let \( S \in L(E, F) \); then

\[
\sigma_n(S) = \sigma_n(J^F_S).
\]

**Proof.** From the injectivity of the Gelfand numbers and Theorem 4.2 it follows

\[
\sigma_n(S) = \sigma_n(J^F_S) = \sigma_n(J^F_S).
\]

**Theorem 4.4.** The Gelfand numbers are the largest injective s-numbers.

**Proof.** Let \( S \in L(E, F) \). Then for each injective s-number function \( \sigma_n(S) = \sigma_n(J^F_S) \leq \sigma_n(J^F_S) = \sigma_n(S) \).

Let \( S \in L(E, F) \). Then the modulus of injectivity is defined by

\[
j(S) := \sup \{ \varepsilon > 0 : \| S \| \geq \varepsilon \| S \| \}.
\]

Without proof we state the following lemmas.

**Lemma 4.1.** Let \( S, T \in L(E, F) \); then

\[
j(S + T) \leq j(S) + j(T).
\]

**Lemma 4.2.** Let \( T \in L(E, F) \) and \( S \in L(F, F) \); then

\[
j(ST) \leq \| S \| j(T).
\]

Moreover, if \( T \) is onto then

\[
j(ST) \leq j(S) \| T \|.
\]

For every operator \( S \in L(E, F) \) the Bernstein numbers are defined by

\[
u_n(S) := \sup \{ j(SF^M) : \dim(M) < n \}.
\]

**Remark.** It is enough to take the supremum over all subspaces \( M \) with \( \dim(M) = n \).

**Theorem 4.5.** The map

\[
bern: S \rightarrow \{ \nu_n(S) \}
\]

is an injective s-number function.

**Proof.** We only show

\[
u_n(RST) \leq \| R \| \nu_n(S) \| T \| \quad \text{for} \quad T \in L(E, E), \quad S \in L(E, F), \quad E \in L(F, F).
\]

Let \( 0 < \varepsilon < \nu_n(RST) \). Then there is a subspace \( M \) of \( E \) such that

\[
u_n(RST) - \varepsilon \leq j(RSF^M) \quad \text{and} \quad \dim(M) = n.
\]

Let \( M := T(M) \), and let \( T \) be the restriction of \( T \) to \( M \) considered as a map into \( M \). Then

\[
RSTF^M = RSTF^M T \quad \text{and} \quad \| T \| \leq \| T \|.
\]
Since by Lemma 4.1

\[
0 < u_n(RST) - \varepsilon \leq j(RST)^{n}_{\varepsilon} |T_{\varepsilon}| \leq \|RST\| \|T_{\varepsilon}\|
\]

we have \(j(T_{\varepsilon}) > 0\). Hence \(T_{\varepsilon}\) is one-to-one, and we obtain \(\dim(M) \geq n\). Consequently, since \(T_{\varepsilon}\) is onto, Lemma 4.1 implies

\[
u_n(RST) - \varepsilon \leq j(RST)^{n}_{\varepsilon} |T_{\varepsilon}| \leq \|RST\| \|T_{\varepsilon}\| \leq \|RST\| u_n(S) |T_{\varepsilon}|
\]

**Theorem 4.5.** The Bernstein numbers are the smallest injective \(n\)-numbers.

**Proof.** Let \(S \in L(E, F)\). For each injective \(n\)-number function we show that \(\dim(M) \geq n\) implies \(j(S)^{n}_{\varepsilon} \leq \varepsilon_n(S)\). This proves

\[
u_n(S) \leq \varepsilon_n(S) \quad \text{for all} \quad S \in L.
\]

We may assume \(j(S)^{n}_{\varepsilon} > 0\). Let \(M_{\varepsilon} := S(M)\). Then the restriction \(S_{\varepsilon}\) of \(S\) to \(M_{\varepsilon}\) considered as a map into \(M_{\varepsilon}\) is invertible, and we have

\[
\|S_{\varepsilon}^{-1}\| = j(S)^{n}_{\varepsilon}^{-1}.
\]

Now the conclusion follows from

\[
1 = \varepsilon_n(M_{\varepsilon}) \leq \varepsilon_n(S_{\varepsilon}) \|S_{\varepsilon}^{-1}\| = j(S)^{n}_{\varepsilon} \|S_{\varepsilon}^{-1}\| \leq \varepsilon_n(S) j(S)^{n}_{\varepsilon}^{-1}.
\]

**3. Surjective \(n\)-numbers.** An \(n\)-number function \(a\) is called surjective if the following property is satisfied:

Let \(E/\mathcal{N}\) be a quotient space of \(E\); then

\[
s_n(SQ_E^{\mathcal{N}}) = \varepsilon_n(S) \quad \text{for all} \quad S \in L(E/\mathcal{N}, F).
\]

In other words, surjectivity means that the \(n\)-numbers \(s_n(S)\) do not depend on the domain of \(S\).

For every operator \(S \in L(E, F)\) the Kolmogorov numbers are defined by

\[
d_n(S) := \inf \{\|Q_E^{\mathcal{N}} S\| : \dim(N) < n\}.
\]

**Theorem 5.1.** The map

\[
\text{kol} : S \rightarrow \{d_n(S)\}
\]

is a surjective \(n\)-number function.

The proof is left to the reader (cf. [13]).

A Banach space \(E\) is said to have the lifting property if for every operator \(S\), mapping \(E\) into a quotient space \(E/\mathcal{N}\) of an arbitrary Banach space \(F\), and for \(\varepsilon > 0\), there is a lifting \(S\) from \(E\) into \(F\) such that

\[
\|S\| < \varepsilon + \|S\|,
\]

**Theorem 5.2.** If \(E\) has the lifting property then

\[
d_n(S) = \varepsilon_n(S) \quad \text{for all} \quad S \in L(E, F).
\]

**Proof.** Let \(S \in L(E, F)\). Since \(\varepsilon_n(S)\) are the largest \(n\)-numbers, it is enough to show \(\varepsilon_n(S) \leq d_n(S)\).

If \(\varepsilon > 0\) we choose a subspace \(N\) of \(F\) such that

\[
\|Q_E^{\mathcal{N}} S\| = d_n(S) + \varepsilon \quad \text{and} \quad \dim(N) < n.
\]

Then there exists a lifting \(T \in L(E, F)\) of \(Q_E^{\mathcal{N}} S\) with \(\|T\| \leq (1 + \varepsilon) \|Q_E^{\mathcal{N}} S\|\).

We set \(A := S - T\). Since \(A\) is a \(N\)-dimensional operator, we have \(\dim(A) < n\).

Hence \(a_n(A) = \|S - A\| = (1 + \varepsilon) \|d_n(S + \varepsilon)\|\).

Every Banach space \(E\) is a quotient space of a Banach space \(E^\mathcal{N}\) which has the lifting property. The canonical map of \(E^\mathcal{N}\) onto \(E\) is denoted by \(Q\).

**Theorem 5.3.** Let \(S \in L(E, F)\); then

\[
d_n(S) = \varepsilon_n(SQ_E^{\mathcal{N}}).
\]

**Proof.** From the surjectivity of the Kolmogorov numbers and Theorem 5.2 it follows

\[
d_n(S) = \varepsilon_n(SQ_E^{\mathcal{N}}) = \varepsilon_n(SQ_E).
\]

**Theorem 5.4.** The Kolmogorov numbers are the largest surjective \(n\)-numbers.

**Proof.** Let \(S \in L(E, F)\). Then for each surjective \(n\)-number function we have

\[
s_n(S) = \varepsilon_n(SQ_E^{\mathcal{N}}) \leq \varepsilon_n(SQ_E) = d_n(S).
\]

Let \(S \in L(E, F)\). Then the modulus of surjectivity is defined by

\[
q(S) := \sup \{q \geq 0 : \{S(U) \neq q U\} \}.
\]

**Lemma 5.1.** Let \(S, T \in L(E, F)\); then

\[
q(S + T) \leq q(S) + q(T) + \|T\|.
\]
Proof. We may assume \(q(S+T) > \|T\|\). If \(0 < \varepsilon < q(S+T)-\|T\|\) we set \(q := q(S+T)-\varepsilon\). Let \(y \in U_F\). We choose inductively a sequence of elements \(x_n \in E\) such that

\[
Sx_1 + Tx_1 = (q-\|T\|)y \quad \text{and} \quad \|x_1\| \leq \frac{q-\|T\|}{\varepsilon},
\]

\[
Sx_2 + Tx_2 = Tx_1 \quad \text{and} \quad \|x_2\| \leq \frac{\|Tx_1\|}{\varepsilon},
\]

\[
Sx_{n+1} + Tx_{n+1} = Tx_n \quad \text{and} \quad \|x_{n+1}\| \leq \frac{\|Tx_n\|}{\varepsilon},
\]

Then

\[
\|x_n\| \leq \left(\frac{\|T\|}{\varepsilon}\right)^{n-1} \frac{q-\|T\|}{\varepsilon} \quad \text{for} \quad n = 1, 2, \ldots
\]

Thus, it is possible to define

\[a := \sum_{n=1}^{\infty} a_n,
\]

and we have

\[Sx = (q-\|T\|)y \quad \text{and} \quad \|x\| \leq 1.
\]

This proves \(S(U_F) \supset (q-\|T\|)U_F\). Consequently,

\[q(S) \geq q - \|T\| = q(S+T)-\|T\|-\varepsilon.
\]

Without proof we state

**Lemma 5.2.** Let \(T \in L(E,F)\) and \(S \in L(F,G)\); then

\[g(ST) \leq g(S)\|T\|.
\]

Moreover, if \(S\) is one-to-one then

\[g(ST) \leq \|S\|g(T).
\]

For every operator \(S \in L(E,F)\) the Mitiagin numbers are defined by

\[v_n(S) := \sup\{g(Q_{F_n}^S) : \text{codim}(N) \geq n\}.
\]

Remark. It is enough to take the supremum over all subspaces \(N\) with \(\text{codim}(N) = n\).

**Theorem 5.5.** The map

\[\text{mit: } S \mapsto (v_n(S))\]

is a surjective \(s\)-number function.
THEOREM 6.3 Let \( S \in L \) such that \( S \) is compact; then
\[
\|s_n(S)\| = \|s_n(S')\|.
\]

Proof. Using similar arguments as in the preceding proof we obtain
\[
\|s_n(S)\| \geq \|s_n(S')\|.
\]
To show the inverse inequality we need the compactness of \( S \).

If \( \varepsilon > 0 \) then we find \( x_1, \ldots, x_n \in U \) with
\[
S(U) \subseteq \bigcup_{i=1}^{n} (S_{x_i} + \varepsilon U_p)
\]
as well as a subspace \( N \) of \( F'' \) such that
\[
\|Q_{F''} s''\| \leq \|s_n(S)\| + \varepsilon
\]
and \( \dim(N) < n \).

Then there is a finite dimensional subspace \( M \) of \( F'' \) with \( N \subseteq M \) and \( J_P S_{x_i} M \) for \( i = 1, \ldots, n \).

By Lemma 6.1 there exists \( E \in L(M, F) \) such that
\[
\|E\| \leq 1 + \varepsilon
\]
and \( RD_P S_{x_i} = S_{x_i} \) for \( i = 1, \ldots, n \).

We set \( N_2 = R(N) \). Using the definition of the quotient norm on \( F''/N \), we choose \( x'_i \in N \) with
\[
\|s'' J_P x'_i - S_x x_i\| \leq \|Q_{F''} s''\| \leq \|s_n(S)\| + \varepsilon.
\]

Let \( x_i := R x'_i \). Then \( x_i \in N_2 \), and therefore
\[
\|Q_{F''} S_{x_i}\| = \|Q_{F''} J_P S_{x_i}\| = \|Q_{F''} J_P S_{x_i}\| \leq \|s'' J_P x'_i - S_x x_i\| \leq (1 + \varepsilon) \|s_n(S)\| + 2 \varepsilon.
\]

For each \( x \in U \) with some index \( i_x \) there holds
\[
\|s_n(S) - S_{x_x}\| \leq 2 \varepsilon.
\]

Consequently,
\[
\|Q_{F''} S_x x_i\| = \|Q_{F''} S_{x_x}\| \leq 1 + \varepsilon \|s_n(S')\| + 2 \varepsilon.
\]

This proves \( s_n(S) \leq s_n(S') \).

The proof of the following lemma is implicitly contained in [2], p. 62, or [31], p. 234.

**LEMMA 6.1.** Let \( S \in L \); then
\[
j(s) = q(S') \quad \text{and} \quad q(s) = j(S).
\]

**THEOREM 6.4.** Let \( S \in L \); then
\[
v_n(S) = u_n(S').
\]

---

**Proof.** Let \( N \) be a subspace of \( F \) with \( \text{codim}(N) = n \), and let \( M := \{b \in F' \mid \langle y, b \rangle = 0 \quad \text{for all} \quad y \in N \} \)

be the corresponding subspace in \( F' \) with \( \dim(M) = n \). Then by Lemma 6.1 we have
\[
q(Q_{F''} S) = J(S')^n S.
\]

Now the assertion follows using the same duality arguments as in the proof of Theorem 6.2.

**Remark.** It is unknown whether
\[
u_n(S) = v_n(S)
\]
holds for all \( S \in L \).

Finally we state the trivial

**THEOREM 6.5.** Let \( S \in L \); then
\[
u_n(S) = \varepsilon_n(S').
\]

---

**7. \( s \)-Numbers of diagonal operators.** In this section we compute the \( s \)-numbers of diagonal operators \( S \),
\[
S(x_1, \ldots, x_m) := (c_1 x_1, \ldots, c_m x_m)
\]
with \( c_1 \geq \ldots \geq c_m > 0 \),

mapping \( \ell^p \) onto \( \ell^q \). Since \( s_n(S) = 0 \) for \( n > m \), the following let \( n = 1, \ldots, m \).

**THEOREM 7.1.** If \( 1 \leq p = q \leq \infty \) then
\[
s_n(S) = c_n
\]
for each \( s \)-number function.

**Proof.** Since \( \text{dim}(A) < n \) for
\[
A(x_1, \ldots, x_m) := (c_1 x_1, \ldots, c_m x_m) \leq (1 + \varepsilon) \|s_n(S')\| + 2 \varepsilon.
\]

We have \( s_n(S) \leq s_n(S) \leq \|s - A\| = c_n \).

One the other hand, let
\[
J(x_1, \ldots, x_m) := (x_1, \ldots, x_m, 0, \ldots, 0),
\]
and
\[
S(x_1, \ldots, x_m) := (c_1 x_1, \ldots, c_m x_m),
\]

Then \( S = QJS \), and therefore
\[
1 = s_n(J) \leq s_n(S) \|S_1\| \leq \|Q||s_n(S)\| \|S^{-1}\| \leq s_n(S) s_n^{-1}.
\]

Consequently,
\[
s_n(S) \geq c_n.
\]
To prove the following theorem we need two lemmas (cf. [9]).

**Lemma 7.1.** Let \( M \) be a subspace of \( \ell^p_n \) with \( \text{codim}(M) < n \); then there exists \( e = (e_1, \ldots, e_n) \in M \) with \( \|e\|_m = 1 \) such that the set
\[
K := \{ k : |e_k| < 1 \}
\]
has less than \( n \) elements.

Proof. We consider an extremum point \( e \in U_M \). Let us assume that \( K \) has at least \( n \) elements. If
\[
N := \{ x \in \ell^p_n : e_k = 0 \text{ for } k \notin K \}
\]
then \( \dim(N) \geq n \). Hence we find \( y \in M \cap N \) with \( \|y\|_m = 1 \). Since
\[
\varnothing := \max\{ |e_k| : k \in K \} < 1,
\]
we have
\[
e_k \neq 0 \quad \text{for } 0 < \delta < 1 - \varnothing.
\]
So \( e \) cannot be an extremum point of \( U_M \). Contradiction.

**Lemma 7.2.** If \( 0 < q < p < \infty \), \( \mu_1, \ldots, \mu_{n+1} > 0 \), and \( |\xi_{n+1}| \leq |\xi_k| \) for \( k = 1, \ldots, n \), then
\[
\frac{\sum_{k=1}^{n+1} |\xi_k|^p \mu_k}{\sum_{k=1}^{n+1} |\xi_k|^q \mu_k}^{1/q} \geq \frac{\sum_{k=1}^{n} |\xi_k|^p \mu_k}{\sum_{k=1}^{n} |\xi_k|^q \mu_k}^{1/q}.
\]

Proof. We set
\[
a := \left( \frac{\sum_{k=1}^{n} |\xi_k|^p \mu_k}{\sum_{k=1}^{n+1} |\xi_k|^p \mu_k} \right)^{1/p}
\]
and \( \beta := \left( \frac{\sum_{k=1}^{n} |\xi_k|^q \mu_k}{\sum_{k=1}^{n+1} |\xi_k|^q \mu_k} \right)^{1/q} \).

From
\[
\frac{|\xi_k|}{|\xi_{n+1}|} \leq a \quad \text{for } k = 1, \ldots, n
\]
it follows
\[
\frac{|\xi_{n+1}|}{a} \leq \frac{|\xi_{n+1}|}{\beta}.
\]

Consequently,
\[
\frac{\sum_{k=1}^{n+1} |\xi_k|^p \mu_k}{\sum_{k=1}^{n+1} |\xi_k|^q \mu_k}^{1/q} \geq \frac{\sum_{k=1}^{n} |\xi_k|^p \mu_k}{\sum_{k=1}^{n} |\xi_k|^q \mu_k}^{1/q}.
\]

**Theorem 7.2.** If \( 1 < q < p < \infty \) then
\[
a_p(\delta) = a_q(\delta) = d_p(\delta) = \left( \sum_{k=1}^{n} |\xi_k|^{1/p} \right)^{1/p}
\]
where \( 1/p := 1/q + 1/p \).

Proof. Since \( \dim(M) < n \) for
\[
A(\xi_1, \ldots, \xi_n) := (\xi_1, \xi_2, \ldots, \xi_n, 0, \ldots, 0),
\]
we have
\[(a) \quad a_p(\delta) \leq \|S - \delta\| = \left( \sum_{k=1}^{n} |\xi_k|^{1/p} \right)^{1/p}.
\]

On the other hand, let \( M \) be an arbitrary subspace of \( \ell^p_n \) with \( \text{codim}(M) < n \). If
\[
D(\xi_1, \ldots, \xi_n) := (\xi_1, \xi_2, \ldots, \xi_n, 0, \ldots, 0)
\]
is considered as a map from \( \ell^p_n \) onto \( \ell^q_n \) by Lemma 7.1 there exists \( e = (e_1, \ldots, e_n) \in D(M) \) with \( \|e\|_m = 1 \) such that
\[
K := \{ k : |e_k| < 1 \}
\]
has less than \( n \) elements. We set \( x := D^{-1} e \). Then from Lemma 7.2 it follows
\[
\frac{\|Sx\|_q}{\|x\|_q} = \frac{\sum_{k=1}^{n} |\xi_k|^p \mu_k}{\sum_{k=1}^{n+1} |\xi_k|^p \mu_k}^{1/q} = \frac{\sum_{k=1}^{n} |\xi_k|^q \mu_k}{\sum_{k=1}^{n+1} |\xi_k|^q \mu_k}^{1/p} \geq \left( \sum_{k=1}^{n} |\xi_k|^{1/p} \right)^{1/p} \geq \left( \sum_{k=1}^{n} |\xi_k|^{1/q} \right)^{1/q}.
\]

Consequently,
\[\text{(c)} \quad a_q(\delta) \geq \left( \sum_{k=1}^{n} |\xi_k|^{1/q} \right)^{1/q}.
\]

By Theorem 6.2 we have
\[\text{(d)} \quad d_p(\delta) = a_q(\delta) \geq \left( \sum_{k=1}^{n} |\xi_k|^{1/q} \right)^{1/q}.
\]

Finally, the assertion follows from \((a), (c), \) and \((d), \).

If \( p = \infty \) the proof must be changed in an obvious way and we do not need Lemma 7.2.

In the case \( 1 < p < \infty \) the \( s \)-numbers \( a_p(\delta), a_q(\delta) \) and \( d_p(\delta) \) seem to be unknown. A special result was proved by S. A. Smoljak (cf. [19]).
Theorem 7.3. If \( p = 1 \) and \( q = 2 \) then 
\[
\alpha_n(S) = d_n(S) = \max_{\sigma = \text{diag}} \left\{ \frac{n - m + 1}{2} \right\}.
\]

Remark. Let \( I_m \) be the identity map of \( \ell_m^q \) onto \( \ell_v^q \). If 
\[
M := \left\{ x \in \ell_v^q : \sum_{k=1}^{\infty} x_k = 0 \right\}
\]
then \( \text{codim}(M) < 2 \). Since 
\[
\|X^r_{\ell_v^q}\|_2 = \frac{1}{\sqrt{2}},
\]
it follows 
\[
\alpha_v(I) \leq \frac{1}{\sqrt{2}}.
\]
On the other hand, from Theorem 7.3 we obtain 
\[
\alpha_v(I) = d_v(I) = \sqrt{(m-1)/m} > 1/\sqrt{2} \quad \text{for} \ m = 3, 4, \ldots
\]
This proves that the \( \varepsilon \)-numbers \( \alpha_v \) and \( d_v \) are different in general.

Remark. In the next step one should try to compute the value of \( \alpha_v(I_m) \) for the identity map \( I_m \) of \( \ell_m^q \) onto \( \ell_v^q \). Using the operators 
\[
A_v := \frac{1}{2} \ e \otimes e \quad \text{and} \quad A_v^* := I_m - \frac{1}{m} \ e \otimes e \quad \text{with} \ e = (1, \ldots, 1)
\]
it can be proved that 
\[
\alpha_v(I) = 1/2 \quad \text{and} \quad \alpha_v(I) = 1/m.
\]
From B. S. Ismagilov the author was informed about the following result: 
\[
\alpha_v(I_m) = O \left( \frac{m^{1+\varepsilon}}{n} \right) \quad \text{for each} \ \varepsilon > 0.
\]

There is some kind of duality between \( \varepsilon \)-numbers.

Lemma 7.3. Let \( \dim(E) = \dim(F) = m \), and let \( S \in L(E, F) \). If \( S \) is invertible then 
\[
\alpha_v(S) d_{m-n+1}(S^{-1}) = 1
\]
and 
\[
\alpha_v(S) d_{m-n+1}(S^{-1}) = 1.
\]

Proof. Let \( M \) be a subspace of \( E \) with \( \dim(M) \geq n \). If \( N := S\{M\} \) then \( \dim(N) < m - n + 1 \), and from 
\[
\|S\{x\}\|_{\ell_v^q} = 1
\]
we obtain 
\[
\alpha_v(S) d_{m-n+1}(S^{-1}) = 1.
\]
The proof of the other equality is analogous.

Remark. It is unknown whether, with the same assumption as in Lemma 7.3, there holds 
\[
\alpha_v(S) d_{m-n+1}(S^{-1}) = 1
\]
Up to this time we can only prove an inequality. For this purpose, if \( \varepsilon > 0 \), we choose a Banach space \( G \) as well as operators \( X \in L(G, E) \) and \( B \in L(F, G) \) such that 
\[
\alpha_v(S) - \varepsilon \leq \|B\|^{-1} \quad \text{and} \quad \dim(G) \geq n.
\]
Let \( A := S^{-1} - XB \). Then \( \dim(X) \geq n \) and \( A(F) \cap X(G) = 0 \) it follows \( \dim(A) < m - n + 1 \). Consequently, 
\[
\alpha_v(S) - \varepsilon \leq \|B\|^{-1} \quad \text{for} \ \varepsilon = (1, \ldots, 1)
\]
As an immediate consequence of Theorem 7.2 and Lemma 7.3 we obtain 

Theorem 7.4. If \( 1 \leq p < q \leq \infty \) then 
\[
\alpha_v(S) - \varepsilon \leq \frac{1}{\sqrt{2}} \quad \text{for} \ \varepsilon = (1, \ldots, 1)
\]
where \( 1/p = 1/2 - 1/q \).

8. Relations between some \( \varepsilon \)-numbers. As a consequence of the preceding results (Theorems 3.2, 3.4, 4.4, 4.6, 5.4, and 5.6) we have 

Theorem 8.1. Let \( S \in L(E) \); then 
\[
\alpha_v(S) \geq \alpha_v(S) \geq \alpha_v(S) \geq \alpha_v(S)
\]
and 
\[
\alpha_v(S) \geq \alpha_v(S) \geq \alpha_v(S) \geq \alpha_v(S).
\]
The following statement is well-known (cf. [8]).

Theorem 8.2. Let \( S \in L(E) \); then 
\[
\alpha_v(S) \geq \alpha_v(S).
\]
Proof. Let \( S \in L(E) \). Since 
\[
\alpha_v(S) := \inf \|Q_{\varepsilon} S\| \quad \text{dim}(N) < n
\]
and 
\[
\alpha_v(S) := \sup \|Q_{\varepsilon} S\| \quad \text{dim}(M) \geq n,
\]
it is enough to show 
\[
\|Q_{\varepsilon} S\| \geq \|Q_{\varepsilon} S\|.
\]
We may assume $j(S/J)^{\parallel}_M > 0$. If $M_{a} := S(M)$ then $\dim(M_{a}) \geq n$. Consequently, by Lemma 0.2 there exists $a_{n} \in N$ such that

$$\|Q_{a_{n}} S\| = \|S\| = 1.$$  

Now the inequality which we want to prove follows from

$$1 = \|S\| \geq j(S/J)^{\parallel}_M \|x\| \quad \text{and} \quad 1 = \|Q_{a_{n}} S\| \leq \|Q_{a_{n}} S\| \|x\|.$$  

**Theorem 8.3.** Let $S \in L_{1}$ then

$$c_{a_{n}}(S) \geq u_{a_{n}}(S).$$

**Proof.** Using Theorems 6.3 and 6.4 we have

$$c_{a_{n}}(S) = d_{a_{n}}(S') \geq u_{a_{n}}(S') = u_{a_{n}}(S).$$

The results are represented in the following diagram where the arrows point from the larger $a_{n}$-numbers to the smaller ones:

$$c_{a_{n}}(S) \quad u_{a_{n}}(S) \quad a_{n}(S) \quad d_{a_{n}}(S) \quad i_{a_{n}}(S).$$

**Theorem 8.4.** Let $S \in L_{1}$ then

$$a_{n}(S) \leq g^{n^{1/2}} d_{a_{n}}(S) \quad \text{and} \quad a_{n}(S) \leq g^{n^{1/3}} c_{a_{n}}(S)$$

where $g$ is a positive constant.

**Proof.** Let $S \in L(F, E)$. For $\varepsilon > 0$ we choose a subspace $N$ of $F$ such that

$$\|Q_{a_{n}} S\| \leq d_{a_{n}}(S) + \varepsilon \quad \text{and} \quad \dim(N) < n.$$  

Then by Lemma 0.3 there exists a projection $P : L(F, F)$ with $N = P(F)$ and $\|P\| \leq (n-1)^{1/2}$. Next, by

$$J(x + N) := y - Py$$

we define an operator $J : L(F/N, F)$. Then

$$\|J\| \leq \|P\| \leq 1 + (n-1)^{1/2} \leq g^{n^{1/3}}$$

where $g = \sqrt{2}$. From

$$S - P = (I_{P} - P)S = JQ_{a_{n}} S$$

we obtain

$$a_{n}(S) \leq \|S - P\| \leq \|J\| \|Q_{a_{n}} S\| \leq g^{n^{1/3}} (d_{a_{n}}(S) + \varepsilon).$$

The proof of the other inequality is similar and will be omitted.

**Remark.** It is unknown whether

$$a_{n}(S) \leq g^{n^{1/2}} d_{a_{n}}(S)$$

holds for an exponent $\alpha < 1/2$.

**Theorem 8.5.** Let $S \in L_{1}$ then

$$u_{n}(S) \leq n^{1/2} u_{n}(S) \quad \text{and} \quad v_{n}(S) \leq n^{1/2} u_{n}(S).$$

**Proof.** Let $S \in L(E, F)$. If $0 < \varepsilon < u_{n}(S)$ we choose a subspace $M$ of $E$ such that

$$u_{n}(S) \leq j(S/M \|x\|) \quad \text{and} \quad \dim(M) = n.$$  

Let $N = S(M)$. Since $j(S/M \|x\|) > 0$, the restriction $S_{a}$ of $S$ to $M$ considered as a map onto $N$ is invertible, and we have

$$j(S/M \|x\|) = \|S_{a}^{-1}\|^{-1}.$$  

Let $P : L(F, N)$ such that $P_{E} = I_{N}$ and $\|P\| \leq n^{1/2}$. Then

$$I_{M} = S_{a}^{-1} P S_{a} M.$$  

Consequently,

$$u_{n}(S) \leq \|S_{a}^{-1}\|^{-1} \|J_{a}\|^{-1} \leq n^{1/2} \|S_{a}^{-1}\|^{-1} = n^{1/2} (u_{a_{n}}(S) - \varepsilon).$$

The proof of the other inequality is similar and will be omitted.

The next statement was proved by B. S. Mitiagin and G. M. Henkin [10].

**Theorem 8.6.** Let $S \in L_{1}$ then

$$c_{a_{n}}(S) \leq n^{1/2} u_{n}(S) \quad \text{and} \quad d_{a_{n}}(S) \leq n^{2} u_{n}(S).$$

**Remark.** Probably there holds

$$c_{a_{n}}(S) \leq n^{1/2} u_{n}(S) \quad \text{and} \quad d_{a_{n}}(S) \leq n^{2} u_{n}(S).$$

A smaller exponent of $n$ as $\alpha = 1$ is impossible since for the identity map $I$ of $I_{a_{n}}$ into $I_{a_{n}}$ we have

$$u_{n}(I) = v_{a_{n}}(I) = 1/n \quad \text{and} \quad c_{a_{n}}(I) = d_{a_{n}}(I) = 1/2, \quad \text{cf. [3].}$$

As an immediate consequence of the preceding results we obtain

**Theorem 8.7.** Let $S \in L_{1}$ then

$$a_{n}(S) \leq g^{n^{1/2}} d_{a_{n}}(S)$$

where $g$ is a positive constant.
9. Ideals of operators. For each subclass $A$ of $L$ we set

$$A(E, F) := A \cap L(E, F).$$

$A$ is called an ideal of operators if the following conditions are satisfied (cf. [15]):

1. If $a_n E$ and $y_n E$ then $a_n y_n x E, F$.
2. If $T \in L(E, E)$, $S \in A(E, F)$ and $R \in L(F, F_0)$ then $RST \in A(E, F)$.
3. If $T_1, T_1 E, A(E, F)$ then $T_1 + T_1 E, A(E, F)$.

A subclass $A$ of $L$ with properties (1) and (2) is said to be an ideal of operators.

Let $s$ be an $\varepsilon$-number function. Then we define

$$S_0^\varepsilon := \{a \in L : \sum_1^n s_n(a) < \varepsilon \}$$

and

$$S_0 := \{a \in L : \lim_n s_n(a) = 0\}.$$

We have the trivial

Theorem 9.1. The class $S_0^\varepsilon$ is an ideal, $0 < \varepsilon < \infty$.

Theorem 9.2. The class $S_0$ is a closed ideal.

Proof. Let $S \in L(E, F)$. We suppose that, for every positive $\varepsilon > 0$, there is $T \in S \in L(E, F)$ with $\|T - S\| < \varepsilon$. Then we find a natural number $n_0$ such that

$$s_n(S) < \varepsilon \quad \text{for} \quad n \geq n_0.$$

Consequently,

$$s_n(S) < \|S - S_n\| + s_n(S_n) < 2\varepsilon \quad \text{for} \quad n \geq n_0,$$

and therefore $S \in S_0^\varepsilon \subseteq L(E, F)$. This proves the closedness of $S_0^\varepsilon \subseteq L(E, F)$.

Let $K$ be the class of compact operators. Then we state the known

(cf. [14], p. 146)

Theorem 9.3. $S_0^\varepsilon = S_0^{\text{com}} = K$.

Proof. Let $S \in K(E, E)$. If $\varepsilon > 0$, we choose $y_1, \ldots, y_m E$ such that

$$S(U_m) = \sum_1^m (y_i + \varepsilon U_m).$$

Let $N$ be a finite dimensional subspace of $E$ with $y_1, \ldots, y_m E$. Then $\|Q_0^N S\| \leq \varepsilon$. Consequently,

$$s_n(S) < \varepsilon \quad \text{for all} \quad n \geq n_{1} := \dim(N).$$

This proves $K \subseteq S_0^{\text{com}}$.

Now the inverse statement will be established. Let $S \in S_0^{\text{com}} \subseteq L(E, F)$. If $\varepsilon > 0$, we choose a natural number $n$ with $d_n(S) < \varepsilon$. Hence there is a subspace $N$ of $F$ such that

$$\|Q_0^N S\| < \varepsilon \quad \text{and} \quad \dim(N) < n.$$

Since $U_m$ is compact, we find $y_1, \ldots, y_m E$ such that

$$\|y_i + \varepsilon U_m\| = \|y_i + \varepsilon U_m\| < \varepsilon.$$

Let $x \in U_m$. Then $\|Q_0^N S x\| < \varepsilon$ and therefore $\|S x - y_i\| < \varepsilon$ for some $y_i E$. Since $|y_i| \leq \|y_i\| + \varepsilon$, we have

$$y_i \in \bigcup_1^m (y_i + \varepsilon U_m).$$

Consequently,

$$S x \in \bigcup_1^m (y_i + \varepsilon U_m)$$

for all $x \in U_m$.

This proves $S_0^{\text{com}} \subseteq K$.

Finally, $S_0 \subseteq S_0^{\text{com}}$ follows from $s_n(S) = d_n(S)$ and Schauder's theorem (cf. [21], p. 275).

An $\varepsilon$-number function $s$ is called additive if the following improvement of condition (2) of § 2 is satisfied:

1. $s_{n+1}(S + T) \leq s_n(S) + s_n(T)$ for $S, T \in L(E, F)$ and $m, n = 1, 2, \ldots$

Theorem 9.4. Let $s$ be an additive $\varepsilon$-number function. Then $S_0^\varepsilon$ is an ideal of operators, $0 < \varepsilon < \infty$.

Proof. Let $S, T \in S_0^\varepsilon \subseteq L(E, F)$. Since

$$s_n(S + T) \leq s_n(S) + s_n(T)$$

for $s_1, s_2, \ldots \geq 0$ with $\varepsilon := \max(2^{p-1}, 1)$, we have

$$s_n(S + T) \leq s_n(S) + s_n(T).$$

If $p = \infty$ then

$$\lim_n s_n(S + T) = \lim_n s_n(S) + s_n(T) < \lim_n s_n(S) + \lim_n s_n(T) = 0.$$
Remark. By the definition
\[ \Sigma_p^2(S) = \left( \sum_{k=1}^{\infty} S_k^2(S') \right)^{1/2} \text{ for } S \in S_p \]
we obtain a quasinorm \( \Sigma_p^2 \) which is in general not a norm even in the case \( 1 \leq p < \infty \).

The following statement is proved in [14].

Theorem 9.3. The approximation numbers, Gelfand numbers and Kolmogorov numbers are additive.

Remark. It seems to be unknown whether the isomorphism numbers, Bernstein numbers and Mitiagin numbers are additive.

References


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