STUDIA MATHEMATICA, T. LI. (1974)

On the semigroup of $\mathscr{D}^k$ mappings on Fréchet Montel space

by

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Abstract. It is known that the semigroup, under composition, of all differentiable maps from the reals into the reals is inner, [7]. We show that this result may be extended to the semigroup of k-times Fréchet differentiable selfmaps of an FM-space. Such semigroups will then characterize the FM-spaces.

Preliminaries. In 1937, J. Schreier showed that the semigroup, under composition, of all selfmaps of an arbitrary set has the property that every automorphism is inner [10]. Thirty years later K. D. Magill, Jr. [7] showed that this property also holds for the semigroup of all differentiable functions from the reals into the reals, while in [15] Yamamuro has generalized this result to Fréchet Montel (FM) spaces. We give here a further extension of this result.

Throughout, E and F will be Hausdorff locally convex spaces over the reals, $\boldsymbol{R}$. Roman letters will be used for elements of the former, and Greek for the latter. We shall denote the conjugate space of E by $\bar{E}$, while $\mathscr{L}(E,F)$ will be the space of all continuous linear maps from E into F with the topology of uniform convergence on bounded sets. Following [2], p. 90, a map $f\colon E\to F$ is said to be Fréchet differentiable at $a\in E$ if there exists a $u\in\mathscr{L}(E,F)$ such that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}r[f,a,\varepsilon x]=0,$$

uniformly for x in any bounded subset of E, where

$$r[f,a,y]=f(a+y)-f(a)-uy.$$

We call $u=f'(a)$. Higher derivatives are defined in the obvious way.

Let $f,g\in\mathscr{D}^k(E)$, the set of all k-times Fréchet differentiable selfmaps of E, and define the product fg by the composition,

$$(fg)(x)=f\big(g(x)\big),\quad \text{for every } x\in E.$$

Then $\mathscr{D}^k(E)$ is a semigroup ([1], p. 234). An automorphism of $\mathscr{D}^k(E)$, which we abbreviate now to $\mathscr{D}^k$, is a bijection φ of $\mathscr{D}^k$ such that

$$\varphi(fg)=\varphi(f)\varphi(g),\quad \text{for every } f,g\in\mathscr{D}^k.$$

An automorphism φ is inner if there exists a bijection h such that h, $h^{-1}\in\mathscr{D}^k$, and

$$\varphi(f) = hfh^{-1}, \quad \text{for every } f\in\mathscr{D}^k. \tag{1}$$

Yamamuro has noted that to show every automorphism of a semigroup of differentiable functions, $\mathscr{S}(E)$, is inner, is tantamount to showing that the semigroup characterizes the underlying locally convex space, E. Briefly, whenever an isomorphism exists between $\mathscr{S}(E)$ and $\mathscr{S}(F)$, with only a notational change in our proof we may find a bijection h from E onto F such that h and h^{-1} are differentiable. Since

$$(h^{-1})'\big(h(x)\big)h'(x) = h'(x)(h^{-1})'\big(h(x)\big) = 1,$$

for every $x\in E$, and $h'(x)\in\mathscr{L}(E,F)$, $h'(x)$ provides a linear homeomorphism between E and F. In view of this it is of interest to prove the following theorem, which is the purpose of this paper.

THEOREM. *Let E be an FM-space. Then every automorphism of the semigroup $\mathscr{D}^k$ is inner.*

Notation. Our notation and terminology follow that in [12], but certain frequently used items will now be given. By $\{\varepsilon_n\}\in(c_0)$ we mean $\{\varepsilon_n\}\subset \boldsymbol{R}$ and $\lim_{n\to\infty}\varepsilon_n = 0$. The constant map on E, whose single value is $a\in E$, is denoted by c_a. For $\bar{a}\in\bar{E}$, $x\in F$, the map $x\otimes\bar{a}$ from E into F is given by

$$(x\otimes\bar{a})(y) = \langle y,\bar{a}\rangle x, \quad \text{for } y\in E,$$

where $\langle y,\bar{a}\rangle$ denotes the value of $\bar{a}$ at y.

When $a\in E$, $a\otimes\bar{a}\in\mathscr{L}(E,E)$, abbreviated to $\mathscr{L}(E)$. Inductively we define $a\otimes^m\bar{a} = (a\otimes^{m-1}\bar{a})\otimes\bar{a}$, an element of $\mathscr{L}_m(E,E)$, also defined inductively by $\mathscr{L}_m(E,E) = \mathscr{L}\big(E,\mathscr{L}_{m-1}(E,E)\big)$. For $h\in\mathscr{D}^m(E)$, the mth Fréchet derivative of h at $x\in E$ is denoted by $h^{(m)}(x)$, an element of $\mathscr{L}_m(E,E)$. After m evaluations at $a\in E$, $h^{(m)}(x)$ is an element of E denoted by $h^{(m)}(x)(a)^m$.

In order to phrase our results in as general a form as possible, we introduce the following notion: a map $f\colon E\to E$ is said to be *weakly-$\mathscr{D}$* if the map $f\colon E\to E_w$ is Fréchet differentiable, where E_w denotes the space E endowed with the weak topology, $\sigma(E,\bar{E})$. Discussions of properties of locally convex spaces which are used without reference may be found in [6], or [8]. In the obvious way we also define *weakly-$\mathscr{D}^k$* maps. The bulk of the paper is devoted to the proof of the following lemma, from which the theorem readily follows.

LEMMA. *Let φ be an automorphism of $\mathscr{D}^k(E)$, E a Fréchet space. Then there exists a bijection h of E satisfying* (1), *such that both h and h^{-1} are weakly-$\mathscr{D}^k$.*

This is an extension of the result in [15], in which the case for $k = 1$ is given. That E be a Fréchet space is required for differentiability at a point to imply continuity in the proof of step 1, for the result of Banach in 2.1, and the separability assumption in 3.1.2. That E be Montel is used only in the deduction of the theorem.

PROOF OF LEMMA

The proof is in six stages.

1. *There exists a bijection h of E such that* (1) *holds. Further, for any $\bar{a}\in\bar{E}$ the function $\langle h(x),\bar{a}\rangle$ from E into $\boldsymbol{R}$ is continuous with respect to $x\in E$.*

The existence of h is essentially due to Schreier [10], but a demonstration in the convex space setting can be found in [15], step 1. Since differentiability at a point implies continuity at that point in a Fréchet space ([2], p. 105), the continuity of $\langle h(x),\bar{a}\rangle$ follows as in [15], step 2. See also [14], p. 505, where it is shown that we can assume $h(0) = 0$. In the same way that φ uniquely determines h, φ^{-1} determines h^{-1}, so that any property shown for h holds also for h^{-1}.

As was pointed out in [12], the elegant method of Magill which was used to show h once differentiable is no longer applicable when the space has dimension greater than one. A further difficulty is encountered in the present situation. Even in the case where $E = \boldsymbol{R}$, the derivative of the associated h is everywhere finite, and

$$(h^{-1})'\big(h(x)\big)h'(x) = 1,$$

for $x\in\boldsymbol{R}$, so that $h'(x)\neq 0$, for any x. Hence h' is certainly not a bijection, with the result that the method cannot be used in advancing to derivatives of higher order.

In the remainder, $a\in E$ and $\bar{a}\in\bar{E}$ will be any pair of elements chosen such that $\langle a,\bar{a}\rangle = 1$. For brevity we let $h(a\otimes\bar{a}) = h_1$, $h^{-1}(a\otimes\bar{a}) = h_2$. The aim of steps 2, 3 and 4 will be to prove, using induction, that $h_1\in\mathscr{D}^k$, which will enable us to deduce, in steps 5 and 6, that h is weakly-$\mathscr{D}^k$. By noting that c_a and $a\otimes\bar{a}$ belong to $\mathscr{D}^k$, we may show $h_1\in\mathscr{D}$ as in [15], step 3 to step 7. Now we assume $h_1\in\mathscr{D}^m$, $1\leqslant m<k$, and show $h_1\in\mathscr{D}^{m+1}$. Step 2 will contain all preliminary results, while step 4 will be the straightforward completion of the proof that $h_1\in\mathscr{D}^{m+1}$. The bulk of the argument is contained in step 3.

This step reduces in essence to showing that the limit

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}[h_1^{(m)}(\varepsilon a)(a)^m - h_1^{(m)}(0)(a)^m] \tag{2}$$

exists. For this we are led to consider the differentiability of the real-valued functions of a real variable,

$$\lambda_{\bar{x}}(\xi) = \langle h_1^{(m)}(\xi a)(a)^m, \bar{x}\rangle, \quad \text{for each } \bar{x} \in \bar{E}.$$

When m is odd, we are able to show that for $\{\varepsilon_n\} \in (c_0)$ the sequence

$$\{\varepsilon_n^{-1}[h_1^{(m)}(\alpha a + \varepsilon_n a)(a)^m - h_1^{(m)}(\alpha a - \varepsilon_n a)(a)^m]\}$$

is bounded, for $\alpha \in \boldsymbol{R}$. Using a longstanding result of Khintchine [5] it is then shown relatively readily that the $\lambda_{\bar{x}}$ are differentiable almost everywhere. Yet in the even case pursuing a similar path with the sequence

$$\{\varepsilon_n^{-1}[h_1^{(m)}(\alpha a + \varepsilon_n a)(a)^m + h_1^{(m)}(\alpha a - \varepsilon_n a)(a)^m - 2h_1^{(m)}(\alpha a)(a)^m]\}$$

and using a result of Zygmund [16] yields only the finiteness of the Dini derivatives of each $\lambda_{\bar{x}}$ on a dense set in $\boldsymbol{R}$. With more effort differentiability almost everywhere does follow and the method of the odd case takes over. Regretfully the calculations are necessarily lengthy since we are constantly dealing with expansions of higher order derivatives of composition functions.

Specifically, if $f, g \in \mathscr{D}^m$, we have ([1], p. 234)

$$(fg)^{(m)}(x)(a)^m = \sum_{q=1}^{m} \sum \sigma_m f^{(q)}\big(g(x)\big)\,[g^{(i_1)}(x)(a)^{i_1}]\ldots[g^{(i_q)}(x)(a)^{i_q}]$$

where the second summation is over all q-tuples of positive integers $i_1, \ldots, i_q$ such that $i_1 + \ldots + i_q = m$, and σ_m is an integer coefficient. We shall frequently abbreviate the above expansion to

$$\sum_{q=1}^{m} \sum \sigma_m f^{(q)}\big(g(x)\big)[R(g, x, a; i_1, \ldots, i_q)].$$

By $\mathscr{L}\mathscr{J}(E^m, F)$ we shall mean all jointly continuous m-linear maps from E^m into F, while $\mathscr{L}\mathscr{S}(E^m, F)$ will refer to the corresponding family of separately continuous maps. It is readily shown that the inclusions

$$\mathscr{L}\mathscr{J}(E^m, F) \subseteq \mathscr{L}_m(E, F) \subseteq \mathscr{L}\mathscr{S}(E^m, F)$$

are valid at all times. For the lemma we require $\mathscr{L}_m$ to equal $\mathscr{L}\mathscr{J}$. When $m = 2$, Köthe has shown in [6], p. 172, (3), that $\mathscr{L}\mathscr{S} = \mathscr{L}\mathscr{J}$. This may be generalized to m-linear maps in a straightforward manner, and used to show, moreover, that the evaluation map from $\mathscr{L}_m(E, F) \times E^m$ into F is sequentially continuous.

Of the following preliminary results, 2.1, 2.4, and 2.5 will be used most frequently.

2. Preliminary results.

2.1. *$h_1^{(m)}(\xi a)(a)^m$ is continuous with respect to $\xi \in \boldsymbol{R}$.*

If $\{\varepsilon_n\} \in (c_0)$, $h_1^{(m)}(\xi a)(a)^m$ is the limit, as n tends to infinity, of

$$(3) \qquad \varepsilon_n^{-1}[h_1^{(m-1)}(\xi a + \varepsilon_n a)(a)^{m-1} - h_1^{(m-1)}(\xi a)(a)^{m-1}].$$

For fixed n, (3) is a continuous function of ξ, so by a result of Banach ([3], p. 397), the limit function, $h_1^{(m)}(\xi a)(a)^m$ is continuous on a dense set. Suppose that α is such a point of continuity and ξ an arbitrary real number. Then if $\{\varepsilon_n\} \in (c_0)$,

$$\begin{aligned} h_1^{(m)}(\xi a + \varepsilon_n a)(a)^m &= h_1^{(m)}[(1 + c_{\xi a - \alpha a})(\alpha a + \varepsilon_n a)](a)^m \\ &= [\varphi(1 + c_{\xi a - \alpha a})h_1]^{(m)}(\alpha a + \varepsilon_n a)(a)^m. \end{aligned}$$

That $\langle a, \bar{a}\rangle = 1$ is used here, to ensure the commutativity of the maps $a \otimes \bar{a}$ and $1 - c_{\xi a - \alpha a}$. Using the expansion for the higher order derivative of a composition function, it is evident that the last term converges to

$$[\varphi(1 + c_{\xi a - \alpha a})h_1]^{(m)}(\alpha a)(a)^m = h_1^{(m)}(\xi a)(a)^m,$$

as ε_n tends to zero.

2.2. *Given $\xi \in \boldsymbol{R}$ and $\{\varepsilon_n\} \in (c_0)$, the sequence $\{\varepsilon_n^{-1}[h(\xi a + \varepsilon_n a) - h(\xi a)]\}$ does not converge weakly to zero.*

We use [15], step 3, in which the case $\xi = 0$ is given. Suppose we can find an $\xi \in \boldsymbol{R}$ and sequence $\{\varepsilon_n\} \in (c_0)$ such that

$$\lim_{n\to\infty} \langle \varepsilon_n^{-1}[h(\xi a + \varepsilon_n a) - h(\xi a)], \bar{x}\rangle = 0,$$

for every $\bar{x} \in \bar{E}$. With some calculation, we may show

$$(4) \quad \begin{aligned} \varepsilon_n^{-1}h(\varepsilon_n a) &= \varphi(1 - c_{\xi a})'\big(h(\xi a)\big)\big(\varepsilon_n^{-1}[h(\xi a + \varepsilon_n a) - h(\xi a)]\big) + \\ &\quad + \varepsilon_n^{-1} r\Big[\varphi(1 - c_{\xi a}),\, h(\xi a),\, \varepsilon_n\Big(\varepsilon_n^{-1}\big(h(\xi a + \varepsilon_n a) - h(\xi a)\big)\Big)\Big]. \end{aligned}$$

Since $\varphi(1 - c_{\xi a})'\big(h(\xi a)\big) \in \mathscr{L}(E)$, we have

$$\lim_{n\to\infty} \langle \varphi(1 - c_{\xi a})'\big(h(\xi a)\big)\big(\varepsilon_n^{-1}[h(\xi a + \varepsilon_n a) - h(\xi a)]\big), \bar{x}\rangle = 0,$$

for every $\bar{x} \in \bar{E}$. Further, the set $\{\varepsilon_n^{-1}[h(\xi a + \varepsilon_n a) - h(\xi a)]\}$ is bounded, being weakly convergent to zero, so the limit of the second term in (4) is zero. Hence the sequence $\{\varepsilon_n^{-1}h(\varepsilon_n a)\}$ converges weakly to zero, contradicting [15], step 3.

2.3. *Given $\xi \in \boldsymbol{R}$, there exists an $\bar{x}_\xi \in \bar{E}$ such that*

$$\varphi[(a \otimes \bar{x}_\xi)(1 - c_{h^{-1}(\xi a)})]'(\xi a)(a) \neq 0.$$

The proof is almost the same as in [15], step 4, where the case $\xi = 0$ was given. For completeness we include it here. Suppose there exists an $\xi \in \boldsymbol{R}$ such that for all $\bar{x} \in \bar{E}$,

$$\varphi[(a \otimes \bar{x})(1 - c_{h^{-1}(\xi a)})]'(\xi a)(a) = 0.$$

Take a sequence $\{\delta_n\} \in (c_0)$ such that $\delta_n \neq 0$, any n, and let M be the set of all $\bar{x} \in \bar{E}$ such that the sequence $\{\langle h^{-1}(\xi a + \delta_n a) - h^{-1}(\xi a), \bar{x}\rangle\}$ contains infinitely many non-zero members. If $\bar{x} \notin M$ then the sequence

$\{\delta_n^{-1}\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a), \bar{x}\rangle\}$ converges to zero. If $\bar{x}\in M$ we have

$$(5)\qquad 0 = \lim_{n\to\infty}\delta_n^{-1}\varphi[(a\otimes\bar{x})(1-c_{h^{-1}(\xi a)})](\xi a+\delta_n a)$$
$$= \lim_{n\to\infty}\delta_n^{-1}\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a), \bar{x}\rangle\tau_n^{-1}h(\tau_n a)$$

where $\tau_n = \langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a), \bar{x}\rangle$. Suppose that the sequence $\{\delta_n^{-1}\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a), \bar{x}\rangle\}$ does not converge to zero. Then there is a subsequence $\{\delta_{n_k}\}$ and a $\gamma\in\boldsymbol{R}$, $\gamma>0$, such that

$$|\delta_{n_k}^{-1}\langle h^{-1}(\xi a+\delta_{n_k}a)-h^{-1}(\xi a), \bar{x}\rangle|\geqslant\gamma$$

for every k. Then by (5) the sequence $\{\tau_{n_k}^{-1}h(\tau_{n_k}a)\}$ converges to zero, which contradicts 2.2. So for any $\bar{x}\in\bar{E}$, the sequence

$$\{\delta_n^{-1}\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a), \bar{x}\rangle\}$$

converges to zero, again contradicting 2.2.

Note that if $\bar{x}$ satisfies $\varphi[(a\otimes\bar{x})(1-c_{h^{-1}(\xi a)})]'(\xi a)(a)\neq 0$, so too does $-\bar{x}$.

We may show

$$\varphi[(a\otimes-\bar{x})(1-c_{h^{-1}(\xi a)})]'(\xi a) = \varphi(-1)'(0)\varphi[(a\otimes\bar{x})(1-c_{h^{-1}(\xi a)})]'(\xi a).$$

But $\varphi(-1)'(0)$ is a linear bijection, because

$$\varphi(-1)'(0)\varphi(-1)'(0) = \varphi(1)'(0) = 1,$$

so the result follows.

2.4. *Given $\{\varepsilon_n\}\in(c_0)$ and $\xi\in\boldsymbol{R}$, there exists $\{\delta_k\}\in(c_0)$ and a subsequence $\{\varepsilon_{n_k}\}$ such that $\langle h^{-1}(\xi a+\delta_k a)-h^{-1}(\xi a), \bar{x}_\xi\rangle = \varepsilon_{n_k}$, for every k.*

It is evident from the equation

$$0\neq\varphi[(a\otimes\bar{x}_\xi)(1-c_{h^{-1}(\xi a)})]'(\xi a)(a)$$
$$= \lim_{\delta\to 0}\delta^{-1}h\big(\langle h^{-1}(\xi a+\delta a)-h^{-1}(\xi a), \bar{x}_\xi\rangle a\big),$$

that the function $\langle h^{-1}(\xi a+\delta a)-h^{-1}(\xi a), \bar{x}_\xi\rangle$ takes non-zero values in every zero-neighbourhood. Since $\langle h^{-1}(x), \bar{a}\rangle$ is continuous in x, there is a sequence $\{\delta_k\}\in(c_0)$ and a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that

$$\langle h^{-1}(\xi a+\delta_k a)-h^{-1}(\xi a), \bar{x}_\xi\rangle = \varepsilon_{n_k}\quad\text{or}\quad -\varepsilon_{n_k}.$$

So by taking a subsequence of $\{\varepsilon_{n_k}\}$ once more, and replacing $\bar{x}_\xi$ by $-\bar{x}_\xi$ if necessary, 2.4 follows.

Now, for fixed $\xi\in\boldsymbol{R}$, we let

$$S(\bar{x}_\xi) = \{\eta\in\boldsymbol{R}\colon \varphi[(a\otimes\bar{x}_\xi)(1-c_{h^{-1}(\eta a)})]'(\eta a)(a)\neq 0\}.$$

2.5. *$S(\bar{x}_\xi)$ is an open subset of $\boldsymbol{R}$.*

Suppose $\eta\in S(\bar{x}_\xi)$. Then

$$\langle h_2'(\eta a)(a), \bar{x}_\xi\rangle h_1'(0)(a) = h_1'(0)[a\otimes\bar{x}_\xi]h_2'(\eta a)(a)$$
$$= [h_1(a\otimes\bar{x}_\xi)(1-c_{h^{-1}(\eta a)})h_2]'(\eta a)(a)$$
$$= \varphi[(a\otimes\bar{x}_\xi)(1-c_{h^{-1}(\eta a)})]'(\eta a)(a),$$

since $(a\otimes\bar{a})(a\otimes\bar{x}_\xi) = (a\otimes\bar{x}_\xi)$.

So $\langle h_2'(\eta a)(a), \bar{x}_\xi\rangle\neq 0$. By 2.1 this is a continuous function of η, showing $S(\bar{x}_\xi)$ to be open.

We are now in a position to show that (2) exists but must deal with the cases where m is odd, and m even, separately.

3. *The limit, $\lim_{\varepsilon\to 0}\varepsilon^{-1}[h_1^{(m)}(\varepsilon a)(a)^m-h_1^{(m)}(0)(a)^m]$ exists.*

3.1. Case where m is odd. We show, for arbitrary $\bar{x}\in\bar{E}$, that the continuous map $\lambda_{\bar{x}}$, defined in 1, has the property that

$$\limsup_{\varepsilon\to 0}|\varepsilon^{-1}[\lambda_{\bar{x}}(\xi+\varepsilon)-\lambda_{\bar{x}}(\xi-\varepsilon)]|<\infty,$$

for every $\xi\in\boldsymbol{R}$. For this the essential step is 3.1.1. Using Khintchine's result, together with some further calculations in 3.1.4, we will be able to deduce (2) exists.

3.1.1. *For any $\{\varepsilon_n\}\in(c_0)$, the set*

$$\{\varepsilon_n^{-1}[h_1^{(m)}(\varepsilon_n a)(a)^m-h_1^{(m)}(-\varepsilon_n a)(a)^m]\}$$

is bounded.

With $\bar{x}_0$ the functional associated with $\xi=0$, as in 2.3, and arbitrary $\{\delta_n\}\in(c_0)$, we may expand the expression

$$[\varphi(a\otimes\bar{x}_0)a\otimes\bar{a}]^{(m+1)}(0)(a)^{m+1}+[\varphi(-a\otimes\bar{x}_0)a\otimes\bar{a}]^{(m+1)}(0)(a)^{m+1},$$

and noting that $(a\otimes\bar{a})(a\otimes\bar{x}_0) = a\otimes\bar{x}_0$, show it is the limit, as n tends to infinity, of

$$(6)\qquad \delta_n^{-1}\big[\big(h_1(a\otimes\bar{x}_0)\big)^{(m)}\big(h^{-1}(\delta_n a)\big)-\big(h_1(a\otimes\bar{x}_0)\big)^{(m)}\big(-h^{-1}(\delta_n a)\big)\big]\big(h_2'(\delta_n a)(a)\big)^m+$$
$$+\sum_{q=1}^{m-1}\sum\sigma_m\delta_n^{-1}\big[\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}\big(h_2(\delta_n a)\big)\big(R(h_2,\delta_n a,a;i_1,\ldots,i_q)\big)+$$
$$+(-1)^q\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}\big(-h_2(\delta_n a)\big)\big(R(h_2,\delta_n a,a;i_1,\ldots,i_q)\big)+$$
$$+\big(1+(-1)^q\big)\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}(0)\big(R(h_2,0,a;i_1,\ldots,i_q)\big)\big].$$

We wish to show that the sequences formed by the terms within the double summation are bounded. If q is odd the term becomes

$$\delta_n^{-1}\big[\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}\big(h_2(\delta_n a)\big)-\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}\big(-h_2(\delta_n a)\big)\big]\times \times\big(R(h_2,\,\delta_n a,\,a;\,i_1,\,\ldots,\,i_q)\big)$$

$$=\langle\delta_n^{-1}h^{-1}(\delta_n a),\,\bar{x}_0\rangle\langle h^{-1}(\delta_n a),\,\bar{x}_0\rangle^{-1}\times \times\big[[h_1^{(q)}\big(\langle h^{-1}(\delta_n a),\,\bar{x}_0\rangle a\big)-h_1^{(q)}(0)]-[h_1^{(q)}\big(-\langle h^{-1}(\delta_n a),\,\bar{x}_0\rangle a\big)-h_1^{(q)}(0)]\big]\times \times[\langle h_2^{(i_1)}(\delta_n a)(a)^{i_1},\,\bar{x}_0\rangle a]\ldots[\langle h_2^{(i_q)}(\delta_n a)(a)^{i_q},\,\bar{x}_0\rangle a],$$

since if $\langle h^{-1}(\delta_n a),\bar{x}_0\rangle=0$, the expression vanishes. By noting that $\{\delta_n^{-1}h^{-1}(\delta_n a)\}$ converges, $\{\langle h^{-1}(\delta_n a),\bar{x}_0\rangle\}\in(c_0)$, $h_1^{(q)}$ is Fréchet differentiable, and $h_1^{(q)}(\xi a)(a)^q$ is continuous in ξ, it is evident that the sequence converges.

If q is even, by adding and subtracting a suitable term we have,

$$\delta_n^{-1}\big[\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}\big(h_2(\delta_n a)\big)+\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}\big(-h_2(\delta_n a)\big)-2\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}(0)\big]\times \times\big(R(h_2,\,\delta_n a,\,a;\,i_1,\,\ldots,\,i_q)\big)+ +2\,\delta_n^{-1}\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}(0)\big(R(h_2,\,\delta_n a,\,a;\,i_1,\,\ldots,\,i_q)\big)- -2\,\delta_n^{-1}\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}(0)\big(R(h_2,\,0,\,a;\,i_1,\,\ldots,\,i_q)\big).$$

Convergence of the first term follows in a manner similar to the case where q is odd, while the second and third terms may be rearranged as,

$$2\big[\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}(0) \ldots\big(h_2^{(i_{q-1})}(\delta_n a)(a)^{i_{q-1}}\big)\Big(\delta_n^{-1}\big(h_2^{(i_q)}(\delta_n a)(a)^{i_q}-h_2^{(i_q)}(0)(a)^{i_q}\big)\Big) \;\vdots\; +\big(h_1(a\otimes\bar{x}_0)\big)^{(q)}(0)\Big(\delta_n^{-1}\big(h_2^{(i_1)}(\delta_n a)(a)^{i_1}-h_2^{(i_1)}(0)(a)^{i_1}\big)\Big) \ldots\big(h_2^{(i_q)}(0)(a)^{i_q}\big)\big].$$

Since $i_j<m$, $j=1,\ldots,q$, $h_2^{(i_j)}$ is Fréchet differentiable and so *a fortiori* Gâteaux differentiable. Moreover, by an earlier result, $[h_1(a\otimes\bar{x}_0)]^{(q)}(0)\colon E^q\to E$ is continuous, so convergence with n follows. These techniques for showing convergence will be used frequently, but elsewhere we shall refrain from presenting these detailed calculations.

The first term in (6) is

$$[\langle h_2'(\delta_n a)(a),\,\bar{x}_0\rangle]^m[\delta_n^{-1}\langle h_2(\delta_n a),\,\bar{x}_0\rangle]\langle h^{-1}(\delta_n a),\,\bar{x}_0\rangle^{-1}\times \times\big[h_1^{(m)}\big(\langle h^{-1}(\delta_n a),\,\bar{x}_0\rangle a\big)-h_1^{(m)}\big(-\langle h^{-1}(\delta_n a),\,\bar{x}_0\rangle a\big)\big](a)^m.$$

Recall that $\langle h_2'(0)(a),\bar{x}_0\rangle\neq 0$ and from 2.4 that given $\{\varepsilon_n\}\in(c_0)$ there is a sequence $\{\delta_k\}\in(c_0)$ and a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that

$$\langle h^{-1}(\delta_k a),\,\bar{x}_0\rangle=\varepsilon_{n_k},\quad\text{for every } k.$$

Hence we can conclude that given $\{\varepsilon_n\}\in(c_0)$ there is a subsequence $\{\varepsilon_{n_k}\}$ such that the set

$$\{\varepsilon_{n_k}^{-1}[h_1^{(m)}(\varepsilon_{n_k}a)(a)^m-h_1^{(m)}(-\varepsilon_{n_k}a)(a)^m]\}$$

is bounded. Immediately we have 3.1.1. This property is now transferred to an arbitrary $\xi\in\boldsymbol{R}$.

3.1.2. *For any* $\{\varepsilon_n\}\in(c_0)$, *and* $\xi\in\boldsymbol{R}$, *the set*

$$\{\varepsilon_n^{-1}[h_1^{(m)}(\xi a+\varepsilon_n a)(a)^m-h_1^{(m)}(\xi a-\varepsilon_n a)(a)^m]\}$$

is bounded.

Using the translation map $1+c_{\xi a}$, and combining the technique of 2.1 with the technique and result of 3.1.1, we may obtain the above.

As was pointed out in [15], step 6, no loss of generality is suffered if at this stage we assume E to be separable. A result in [6], p. 259, then gives that $\bar{E}$ is weakly sequentially separable, which means that every element of $\bar{E}$ is the weak limit of a subsequence of a fixed sequence, $\{\bar{a}_i\}$ of elements of $\bar{E}$. Notice that such a set is also total. We now show

3.1.3. *For some* $\alpha\in\boldsymbol{R}$ *the limit*

$$\lim_{\varepsilon\to 0}\langle\varepsilon^{-1}[h_1^{(m)}(\alpha a+\varepsilon a)(a)^m-h_1^{(m)}(\alpha a)(a)^m],\,\bar{a}_i\rangle$$

exists, for every $i=1,2,\ldots$

Recall that each $\lambda_i=\lambda_{\bar{a}_i}$ is continuous, while from 3.1.2 it follows that

$$\limsup_{\varepsilon\to 0}|\varepsilon^{-1}[\lambda_i(\xi+\varepsilon)-\lambda_i(\xi-\varepsilon)]|<\infty$$

for every $\xi\in\boldsymbol{R}$. An early result of Khintchine ([5], p. 217) shows that this is sufficient for each λ_i to be differentiable almost everywhere. We deduce the existence of an $\alpha\in\boldsymbol{R}$ at which each of the functions λ_i is differentiable. Coupled with the following, this enables us to show that the limit

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}[h_1^{(m)}(\varepsilon a)(a)^m-h_1^{(m)}(0)(a)^m]$$

exists.

3.1.4. *Given that* $\{\varepsilon_n\}\in(c_0)$ *there is a subsequence* $\{\varepsilon_{n_k}\}$ *such that the limit*

$$\lim_{k\to\infty}\varepsilon_{n_k}^{-1}[h_1^{(m)}(\varepsilon_{n_k}a)(a)^m-h_1^{(m)}(0)(a)^m],$$

exists.

Although the inductive assumption was that $h_1\in\mathscr{D}^m(E)$ we may also assume that $h_2\in\mathscr{D}^m$, since any property true of h can also be shown for h^{-1}. In fact we have $h_2^{(m)}(\xi a)(a)^m$ continuous in ξ. With $\bar{x}_0$ as before, and $\xi \in \boldsymbol{R}$, we examine the expression

$$\big[\varphi\big((a\otimes\bar{a})(a\otimes\bar{x}_0)(1-c_{h^{-1}(\xi a)})\big)(a\otimes\bar{a})\big]^{(m+1)}(\xi a)(a)^{m+1}.$$

For $\{\delta_n\}\in(c_0)$, this is the limit of the sequence with nth term

$$\delta_n^{-1}\big[\big(h_1(a\otimes\bar{x}_0)(1-c_{h^{-1}(\xi a)})h_2\big)^{(m)}(\xi a+\delta_n a)(a)^m - \\ -\big(h_1(a\otimes\bar{x}_0)(1-c_{h^{-1}(\xi a)})h_2\big)^{(m)}(\xi a)(a)^m\big],$$

which with some computation may be shown to equal,

$$\delta_n^{-1}\big[h_1^{(m)}\big(\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\bar{x}_0\rangle a\big)-h_1^{(m)}(0)\big]\big(\langle h_2'(\xi a)(a),\bar{x}_0\rangle a\big)^m+ \\ +\delta_n^{-1}\big[h_1^{(m)}\big(\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\bar{x}_0\rangle a\big)\big(\langle h_2'(\xi a+\delta_n a)(a),\bar{x}_0\rangle a\big)^m- \\ -h_1^{(m)}\big(\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\bar{x}_0\rangle a\big)\big(\langle h_2'(\xi a)(a),\bar{x}_0\rangle a\big)^m\big]+ \\ +\delta_n^{-1}\Big[\sum_{1<q<m}\sum\sigma_m\big\{\big(h_1(a\otimes\bar{x}_0)(1-c_{h^{-1}(\xi a)})\big)^{(q)}\big(h^{-1}(\xi a+\delta_n a)\big)\times \\ \times\big(R(h_2,\xi a+\delta_n a,a;i_1,\ldots,i_q)\big)- \\ -\big(h_1(a\otimes\bar{x}_0)(1-c_{h^{-1}(\xi a)})\big)^{(q)}\big(h^{-1}(\xi a)\big)\big(R(h_2,\xi a,a;i_1,\ldots,i_q)\big)\big\}\Big]+ \\ +\delta_n^{-1}\big[h_1'\big(\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\bar{x}_0\rangle a\big)-h_1'(0)\big]\big(\langle h_2^{(m)}(\xi a)(a)^m,\bar{x}_0\rangle a\big)+ \\ +\langle\delta_n^{-1}\big(h_2^{(m)}(\xi a+\delta_n a)(a)^m-h_2^{(m)}(\xi a)(a)^m\big),\bar{x}_0\rangle\times \\ \times h_1'\big(\langle h^{-1}(\xi a+\delta_n a)-h^{-1}(\xi a),\bar{x}_0\rangle a\big)(a).$$

Firstly, assume $m>1$. When suitable terms are added to and subtracted from the general term of the second sequence, the scalar coefficients taken out, and the continuity result of 2.1 applied to the remainder, the second sequence may be shown to converge. By fixing q in the third sequence, again adding and subtracting suitable terms, and observing that $i_j<m$, $j=1,\ldots,q$ so that $h_1^{(i_j)}$ is Fréchet differentiable, we are able to use the fact that differentiability implies continuity when the first space is sequential to show that the third sequence converges. Thus the three central terms form convergent sequences, as does the final term for all ξ in a set of full measure A, by the results of 3.1.3. Choosing $\xi\in S(\bar{x}_0)\cap A$ we are then able to find a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ for which the limit

$$\lim_{k\to\infty}\varepsilon_{n_k}^{-1}[h_1^{(m)}(\varepsilon_{n_k}a)-h_1^{(m)}(0)](a)^m$$

exists, since $\langle h_2'(\xi a)(a),\bar{x}_0\rangle\neq 0$. Note that when $m=1$ only the final pair of sequences remain and the proof goes through as before.

In view of the conclusion of 3.1.4, in order to show (2) exists we

must show that if $\{\delta_n\}$, $\{\varepsilon_n\}\in(c_0)$ and

$$\lim_{n\to\infty}\varepsilon_n^{-1}[h_1^{(m)}(\varepsilon_n a)(a)^m-h_1^{(m)}(0)(a)^m]=a_1,$$
$$\lim_{n\to\infty}\delta_n^{-1}[h_1^{(m)}(\delta_n a)(a)^m-h_1^{(m)}(0)(a)^m]=a_2,$$

then $a_1=a_2$. Now with a as in 3.1.3,

$$\varepsilon_n^{-1}[h_1^{(m)}(\alpha a+\varepsilon_n a)(a)^m-h_1^{(m)}(\alpha a)(a)^m] \\ =\varphi(1+c_{\alpha a})'(0)\big(\varepsilon_n^{-1}[h_1^{(m)}(\varepsilon_n a)(a)^m-h_1^{(m)}(0)(a)^m]\big)+ \\ +\varepsilon_n^{-1}\big[\varphi(1+c_{\alpha a})'\big(h_1(\varepsilon_n a)\big)-\varphi(1+c_{\alpha a})'(0)\big]\big(h_1^{(m)}(\varepsilon_n a)(a)^m\big)+ \\ +\sum_{2\leqslant q\leqslant m}\sum\sigma_m\varepsilon_n^{-1}\big[\varphi(1+c_{\alpha a})^{(q)}\big(h_1(\varepsilon_n a)\big)\big(R(h_1,\varepsilon_n a,a;i_1,\ldots,i_q)\big)- \\ -\varphi(1+c_{\alpha a})^{(q)}(0)\big(R(h_1,0,a;i_1,\ldots,i_q)\big)\big].$$

All but the first term on the right hand side converge to a value independent of the sequence $\{\varepsilon_n\}\in(c_0)$. Hence, by 3.1.3,

$$\langle\varphi(1+c_{\alpha a})'(0)(a_1),\bar{a}_i\rangle=\langle\varphi(1+c_{\alpha a})'(0)(a_2),\bar{a}_i\rangle,$$

for every i. Since $\{\bar{a}_i\}$ is total and $\varphi(1+c_{\alpha a})'(0)$ is one-to-one, we have $a_1=a_2$.

3.2. Case where m is even. Due to the fact that

$$h_1^{(m)}(-\varepsilon a)(a)^m=\begin{cases}(h_1-1)^{(m)}(\varepsilon a)(a)^m, & \text{for } m \text{ even},\\ -(h_1-1)^{(m)}(\varepsilon a)(a)^m, & \text{for } m \text{ odd},\end{cases}$$

we are led in the even case to examine a second order difference quotient. Using steps analogous to those in 3.1.1, 3.1.2, and 3.1.3 we show λ_i, $i\in\boldsymbol{N}$, the natural numbers, have finite Dini derivatives on an everywhere dense set in $\boldsymbol{R}$. Employing the calculation of 3.1.4 we are able to show the set $\{\varepsilon_n^{-1}[h_1^{(m)}(\varepsilon_n a)(a)^m-h_1^{(m)}(0)(a)^m]\}$ is bounded, and with the aid of a translation argument and a theorem concerning the Dini derivatives, that the λ_i are differentiable almost everywhere. The argument is completed as in 3.1.4 and 3.1.5.

3.2.1. *For any* $\{\varepsilon_n\}\in(c_0)$, *the set*

$$\{\varepsilon_n^{-1}[h_1^{(m)}(\varepsilon_n a)(a)^m+h_1^{(m)}(-\varepsilon_n a)(a)^m-2h_1^{(m)}(0)(a)^m]\}$$

is bounded.

As in the odd case we calculate

$$[\varphi(a\otimes\bar{x}_0)a\otimes\bar{a}]^{(m+1)}(0)(a)^{m+1}+[\varphi(-a\otimes\bar{x}_0)a\otimes\bar{a}]^{(m+1)}(0)(a)^{m+1}$$

which for $\{\delta_n\} \in (c_0)$ is the limit, as n tends to infinity, of

$$\langle h_2'(\delta_n a)(a), \bar{x}_0\rangle^m \big(\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{x}_0\rangle\big) \langle h^{-1}(\delta_n a), \bar{x}_0\rangle^{-1} \times$$
$$\times \big[h_1^{(m)}\big(\langle h^{-1}(\delta_n a), \bar{x}_0\rangle a\big) + h_1^{(m)}\big(-\langle h^{-1}(\delta_n a), \bar{x}_0\rangle a\big) - 2 h_1^{(m)}(0)\big](a)^m +$$
$$+ 2\,\delta_n^{-1}\big[\big(h_1(a\otimes\bar{x}_0)\big)^{(m)}(0)\big(h_2'(\delta_n a)(a)\big)^m - \big(h_1(a\otimes\bar{x}_0)\big)^{(m)}(0)\big(h_2'(0)(a)\big)^m\big]$$

+ (term identical to that involving double summation in (6)).

Since m is even, h_2' is Fréchet differentiable so the second sequence converges. Again, $\langle h_2'(0)(a), \bar{x}_0\rangle \neq 0$ allowing us to conclude that given $\{\varepsilon_n\} \in (c_0)$, there is a subsequence $\{\varepsilon_{n_k}\}$ such that the set

$$\{\varepsilon_{n_k}^{-1}[h_1^{(m)}(\varepsilon_{n_k} a)(a)^m + h_1^{(m)}(-\varepsilon_{n_k} a)(a)^m - 2 h_1^{(m)}(0)(a)^m]\}$$

is bounded. Then 3.2.1 follows.

3.2.2. *For* $\{\varepsilon_n\} \in (c_0)$, *and* $\xi \in \boldsymbol{R}$, *the set*

$$\{\varepsilon_n^{-1}[h_1^{(m)}(\xi a + \varepsilon_n a)(a)^m + h_1^{(m)}(\xi a - \varepsilon_n a)(a)^m - 2 h_1^{(m)}(\xi a)(a)^m]\}$$

is bounded.

This follows using the translation map $1 + c_{\xi a}$, 3.2.1, and standard techniques.

In the terminology of Zygmund, [16], the continuous functions λ_i, $i \in \boldsymbol{N}$, have the property Λ on $\boldsymbol{R}$. That is,

$$\lambda_i(\xi + \varepsilon) + \lambda_i(\xi - \varepsilon) - 2\lambda_i(\xi) = O(\varepsilon), \qquad \xi \in \boldsymbol{R}.$$

As indicated in [16], p. 55, this is insufficient to ensure the differentiability of λ_i at even a single point. However, it does mean that the set of points at which all four Dini derivatives of λ_i are finite is everywhere dense ([16], p. 55).

3.2.3. *Given* $\{\varepsilon_n\} \in (c_0)$, *the set*

$$\{\varepsilon_n^{-1}[h_1^{(m)}(\varepsilon_n a)(a)^m - h_1^{(m)}(0)(a)^m]\}$$

is bounded.

The calculations of 3.1.4 suffice to show that if $\{\delta_n\} \in (c_0)$, and the set $\{\langle \delta_n^{-1}[h_2^{(m)}(\xi a + \delta_n a)(a)^m - h_2^{(m)}(\xi a)(a)^m], \bar{x}_0\rangle\}$ is bounded in $\boldsymbol{R}$, then so too is the set

$$\{\langle h^{-1}(\xi a + \delta_n a) - h^{-1}(\xi a), \bar{x}_0\rangle^{-1} \times$$
$$\times \big[h_1^{(m)}\big(\langle h^{-1}(\xi a + \delta_n a) - h^{-1}(\xi a), \bar{x}_0\rangle a\big) - h_1^{(m)}(0)\big](a)^m\}$$

in E. Choosing ξ to be in the dense set in which all four Dini derivatives of $\lambda_{\bar{x}_0}$ are finite, as well as in the open set $S(\bar{x}_0)$, we deduce the existence of a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ for which the set

$$\{\varepsilon_{n_k}^{-1}[h_1^{(m)}(\varepsilon_{n_k} a)(a)^m - h_1^{(m)}(0)(a)^m]\}$$

is bounded. Immediately we have 3.2.3.

3.2.4. *Given* $\{\varepsilon_n\} \in (c_0)$, $\xi \in \boldsymbol{R}$, *the set*

$$\{\varepsilon_n^{-1}[h_1^{(m)}(\xi a + \varepsilon_n a)(a)^m - h_1^{(m)}(\xi a)(a)^m]\}$$

is bounded.

This follows in the usual way from 3.2.3.

3.2.5. *For any* $i \in \boldsymbol{N}$, λ_i *is differentiable almost everywhere.*

If $\{\varepsilon_n\} \in (c_0)$ the set

$$\{\langle \varepsilon_n^{-1}[h_1^{(m)}(\xi a + \varepsilon_n a)(a)^m - h_1^{(m)}(\xi a)(a)^m], \bar{a}_i\rangle\}$$

is bounded, any $\xi \in \boldsymbol{R}$, any $i \in \boldsymbol{N}$. Thus all four Dini derivatives of λ_i are finite at every point in $\boldsymbol{R}$, so by [9], p. 270, λ_i is differentiable almost everywhere. Following the argument of the odd case from here leads to the existence, in the even case also, of the limit

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[h_1^{(m)}(\varepsilon a)(a)^m - h_1^{(m)}(0)(a)^m].$$

We call this $[h_1^{(m)}]^*(0)(a)^{m+1}$.

4. $h_1^{(m)}$ *is Fréchet differentiable.*

We begin by showing

$$h_1^{(m+1)}(0) = [h_1^{(m)}]^*(0)(a)^{m+1}\otimes^{m+1}\bar{a}$$

which is certainly an element of $\mathscr{L}_{m+1}(E, E)$. Thus $h_1^{(m+1)}(0)(x)$ will equal $\langle x, \bar{a}\rangle [h_1^{(m)}]^*(0)(a)^{m+1}\otimes^m \bar{a}$.

We have to show that for each bounded set B in E

$$\varepsilon^{-1}[h_1^{(m)}(\varepsilon x) - h_1^{(m)}(0)] - \langle x, \bar{a}\rangle [h_1^{(m)}]^*(0)(a)^{m+1}\otimes^m \bar{a}$$

is uniformly convergent to zero for $x \in B$. Since

$$h_1^{(m)}(x) = h_1^{(m)}(\langle x, \bar{a}\rangle a)(a)^m\otimes^m \bar{a}$$

the expression is zero if $\langle x, \bar{a}\rangle = 0$. So we need consider only those x for which $\langle x, \bar{a}\rangle \neq 0$. Suppose the result is false. Then

$$\langle x, \bar{a}\rangle \big[(\varepsilon\langle x, \bar{a}\rangle)^{-1}[h_1^{(m)}(\varepsilon\langle x, \bar{a}\rangle a)(a)^m\otimes^m \bar{a} - h_1^{(m)}(0)(a)^m\otimes^m \bar{a}] -$$
$$- [h_1^{(m)}]^*(0)(a)^{m+1}\otimes^m \bar{a}\big]$$

does not converge to zero uniformly for $x \in B$. Hence we can find a zero-neighbourhood U in E, $\{\varepsilon_n\} \in (c_0)$, $\{x_n\} \subset B$, and bounded sequences $\{x_n^1\}, \ldots, \{x_n^m\}$, such that

$$\langle x_n^1, \bar{a}\rangle \ldots \langle x_n^m, \bar{a}\rangle \langle x_n, \bar{a}\rangle \big[(\varepsilon_n\langle x_n, \bar{a}\rangle)^{-1}[h_1^{(m)}(\varepsilon_n\langle x_n, \bar{a}\rangle a)(a)^m -$$
$$- h_1^{(m)}(0)(a)^m] - [h_1^{(m)}]^*(0)(a)^{m+1}\big] \notin U,$$

for every n. But the sets $\{\langle x_n^i, \bar{a}\rangle\}$ are bounded, $i = 1, \ldots, m$, and $\{\varepsilon_n \langle x_n, \bar{a}\rangle\} \in (c_0)$ so from the definition of $[h_1^{(m)}]^*(0)(a)^{m+1}$ we have a contradiction. We now show

$$h_1^{(m+1)}(x) = [\varphi(1+\langle x, \bar{a}\rangle c_a) h_1]^{(m+1)}(0),$$

which certainly exists in $\mathscr{L}_{m+1}(E, E)$, since we have shown $h_1^{(m)}$ is Fréchet differentiable at zero. Given a bounded set B in E we must show

$$\begin{aligned}\varepsilon^{-1}\big[h_1^{(m)}(x+\varepsilon y) - h_1^{(m)}(x) - [\varphi(1+\langle x, \bar{a}\rangle c_a)h_1]^{(m+1)}(0)(\varepsilon y)\big] \\ = \varepsilon^{-1}\big[h_1^{(m)}(\langle x, \bar{a}\rangle a + \varepsilon\langle y, \bar{a}\rangle a) - h_1^{(m)}(\langle x, \bar{a}\rangle a) - \\ - [\varphi(1+\langle x, \bar{a}\rangle c_a)h_1]^{(m+1)}(0)(\varepsilon y)\big]\end{aligned}$$

converges to zero uniformly for y in B. As before it is evident we need consider only those y in B for which $\langle y, \bar{a}\rangle \neq 0$. But the above expression is

$$\begin{aligned}\langle y, \bar{a}\rangle\Big[(\varepsilon\langle y, \bar{a}\rangle)^{-1}\Big[\big(\varphi(1+\langle x, \bar{a}\rangle c_a)h_1\big)^{(m)}(\varepsilon\langle y, \bar{a}\rangle a) - \\ - \big(\varphi(1+\langle x, \bar{a}\rangle c_a)h_1\big)^{(m)}(0)\Big] - \big(\varphi(1+\langle x, \bar{a}\rangle c_a)h_1\big)^{(m+1)}(0)(a)\Big]\end{aligned}$$

which converges uniformly to zero for y in B. Hence $h_1 \in \mathscr{D}^{m+1}(E)$, so by induction, $h_1 \in \mathscr{D}^k(E)$.

5. $(a\otimes\bar{a})h \in \mathscr{D}^k(E)$.

Since $h(a\otimes\bar{a}) \in \mathscr{D}^k(E)$, it follows that

$$\varphi^{-1}[h(a\otimes\bar{a})] = h^{-1}[h(a\otimes\bar{a})]h = (a\otimes\bar{a})h \in \mathscr{D}^k(E).$$

6. *h is weakly-$\mathscr{D}^k(E)$.*

The proof is by induction. The case $k = 1$ was treated in [15], steps 8 and 9. Now assume h is weakly-$\mathscr{D}^m(E)$, some m, $1 \leqslant m < k$. Unless otherwise stated $[(a\otimes\bar{a})h]^{(m)}$ and $[h(a\otimes\bar{a})]^{(m)}$ will refer in this section to strong mth Fréchet derivatives, while $h^{(m)}$ will denote the weak mth Fréchet derivative of h. Note that since $\mathscr{L}(E)$ and $\mathscr{L}(E, E_w)$ are equal as sets, strong differentiability implies weak differentiability and the derivatives coincide.

6.1. *$h^{(m)}$ is Gâteaux differentiable at zero.*

With a, $\bar{a}$ as before, $h_1^{(m+1)}(0)(a)$ exists and equals $\lim_{\varepsilon\to 0}\varepsilon^{-1}[h_1^{(m)}(\varepsilon a) - h_1^{(m)}(0)]$, an element of $\mathscr{L}_m(E, E)$. But this is $\lim_{\varepsilon\to 0}\varepsilon^{-1}[h^{(m)}(\varepsilon a) - h^{(m)}(0)]$, an element of $\mathscr{L}_m(E, E_w)$, since the topology on $\mathscr{L}(E, E_w)$ is weaker than the topology on $\mathscr{L}(E)$. We denote this limit by $[h^{(m)}]^*(0)(a)$.

6.2. $[h^{(m)}]^*(0) \in \mathscr{L}\big(E, \mathscr{L}_m(E, E_w)\big)$.

It is readily shown that if $\{S_a\}$ is a net in $\mathscr{L}_m(E, E_w)$ then S_a converges to zero in $\mathscr{L}_m(E, E_w)$ if and only if $(a\otimes\bar{a})S_a$ converges to zero in $\mathscr{L}_m(E, E_w)$,

for every a, $\bar{a}$ such that $\langle a, \bar{a}\rangle = 1$. Now

$$\begin{aligned}[(a\otimes\bar{a})h]^{(m+1)}(0)(y) &= \lim_{\varepsilon\to 0}\varepsilon^{-1}\Big[\big((a\otimes\bar{a})h\big)^{(m)}(\varepsilon y) - \big((a\otimes\bar{a})h\big)^{(m)}(0)\Big] \\ &\qquad \text{in } \mathscr{L}_m(E, E) \\ &= \lim_{\varepsilon\to 0}(a\otimes\bar{a})\varepsilon^{-1}[h^{(m)}(\varepsilon y) - h^{(m)}(0)] \\ &\qquad \text{in } \mathscr{L}_m(E, E_w) \\ &= (a\otimes\bar{a})[h^{(m)}]^*(0)(y),\end{aligned}$$

using the 'only if' of the above result. Since for non-zero $\bar{a}$ we can find non-zero a such that $\langle a, \bar{a}\rangle = 1$, it follows that $[h^{(m)}]^*(0)$ is linear. Any net convergent to zero is mapped by $[(a\otimes\bar{a})h]^{(m+1)}(0)$ into a net convergent to zero, so using the 'if' direction of the result gives $[h^{(m)}]^*(0) \in \mathscr{L}_{m+1}(E, E_w)$.

6.3. *$h^{(m)}$ is weakly Fréchet differentiable.*

We show this property at zero. Let

$$r[h^{(m)}, 0, y] = h^{(m)}(y) - h^{(m)}(0) - [h^{(m)}]^*(0)(y).$$

We require that $\varepsilon^{-1}r[h^{(m)}, 0, \varepsilon y]$ should converge to zero in $\mathscr{L}_m(E, E_w)$ uniformly for y in any bounded subset of E. Suppose this is false. Then there exists a sequence $\{\varepsilon_n\} \in (c_0)$, bounded sequences $\{y_n\}$, $\{y_n^i\}$, $i = 1, \ldots, m$, and $\bar{a} \in \bar{E}$, such that

$$\langle \varepsilon_n^{-1} r[h^{(m)}, 0, \varepsilon_n y_n](y_n^1)\ldots(y_n^m), \bar{a}\rangle$$

does not converge to zero with n. But $[(a\otimes\bar{a})h]^{(m)}$ is Fréchet differentiable at zero, so for any bounded sets B, B_i, $i = 1, \ldots, m$ in E,

$$\varepsilon^{-1} r\Big[\big((a\otimes\bar{a})h\big)^{(m)}, 0, \varepsilon y\Big](y^1)\ldots(y^m)$$

converges to zero in E, uniformly for $y \in B$, $y^i \in B_i$, $i = 1, \ldots, m$. That is,

$$\langle \varepsilon^{-1} r[h^{(m)}, 0, \varepsilon y](y^1)\ldots(y^m), \bar{a}\rangle$$

converges to zero uniformly for $y \in B$, $y^i \in B_i$, $i = 1, \ldots, m$, a contradiction. We may use a method similar to that in [4] to move this point of weak differentiability to any other point, so completing the proof of the lemma.

PROOF OF THEOREM

By the lemma we have a weakly-$\mathscr{D}^k$ bijection h of E, associated with an automorphism φ of $\mathscr{D}^k$, such that (1) holds. We use induction to show $h \in \mathscr{D}^k$. The case $k = 1$ follows as in [15]. Assume $h \in \mathscr{D}^m$, some m, $1 \leqslant m < k$, and suppose $h^{(m)}$, the strong mth derivative, does not have Fréchet derivative at zero given by $h^{(m+1)}(0)$. This is the weak Fréchet derivative at zero of the strong mth derivative of h. Then there exists a neighbourhood U of zero, bounded sets $B, B_1, \ldots, B_m$, a sequence $\{\varepsilon_n\} \in (c_0)$ and sequences $\{y_n\} \subset B$, $\{y_n^i\} \subset B_i$, $i = 1, \ldots, m$ such that

$$\varepsilon_n^{-1} r[h^{(m)}, 0, \varepsilon_n y_n](y_n^1) \ldots (y_n^m) \notin U, \quad \text{for every } n.$$

Using the fact that every weakly convergent sequence in a Montel space is strongly convergent to the same limit, we contradict the lemma.

Remarks

1. With a little additional effort parallel results may be found for the semigroup $\mathscr{C}^k(E)$, the k-times continuously Fréchet differentiable selfmaps of an FM-space, E. In the finite dimensional case a far quicker proof of this result is available in [11] using a theorem of Bochner and Montgomery [4].

2. A similar treatment of the semigroups $\mathscr{D}^\infty$ and $\mathscr{C}^\infty$ of indefinitely Fréchet differentiable and indefinitely continuously Fréchet differentiable selfmaps of FM-space respectively, shows that each of their automorphisms is inner.

3. For the semigroup $\mathscr{D}(E)$, E Banach, certain partial results of the above type have been found by placing restrictions on the automorphism, φ. See, for example, [14] and [13], Theorem 4.

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Received October 2, 1972
Changed version June 4, 1973 (699)