Then we have
\[ \lambda \left( \psi (X_t) \right) + \varepsilon = \lambda (X_t) = \int_{X_t} \frac{\varphi \left( |a_n| h(t) \right)}{\varphi (a_n)} \, d\lambda \]
\[ \leq \frac{\varphi \left( |a_n| (i+2) \right)}{\varphi (a_n)} \lambda \left( \psi (X_t) \right) \leq \lambda (\psi (X_t)) + \varepsilon / 2 \]
which is a contradiction.

So altogether we have \( \lambda (\psi (A)) = \lambda (A) \) for all \( A \in \mathcal{A} \), which implies (iii).

Let \( \mathcal{M} := \{ t : |h(t)| > 1 \} \). Without restriction we can suppose \( \lambda (\mathcal{M}) < 1 \). We showed already that there is an \( N < \mathcal{A} \), with \( \varphi (X) = \mathcal{M} \) and \( \lambda (\mathcal{N}) = \lambda (\mathcal{M}) \). Then we have
\[ \varphi (1) \cdot \lambda (\mathcal{N}) = \| X_\mathcal{N} \| = \| X_\mathcal{M} \| = \| X_\mathcal{M} \| = \| \mathfrak{h} (1) X_\mathcal{M} (t) \| = \int \varphi (|h(t)|) \, d\lambda > \varphi (1) \cdot \lambda (\mathcal{M}) \]
which implies \( \lambda (\mathcal{N}) = 0 \). Similarly we get
\[ \lambda (\{ t : |h(t)| < 1 \}) = 0 \]
which completes the proof.

The proof of the theorem is a trivial consequence of Lemma 2.

References


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the following are equivalent:
(i) \( u \) is \( p \)-summing;
(ii) \( u \) is \( p^\prime \)-summing;
(iii) \( u' \) is \( 0 \)-summing;
(iv) \( u' \) is \( s \)-summing for some \( s \).

**Theorem 1.** Suppose that \( 0 < r < \infty, \ p \geq 1 \) and \( q \geq 1 \).
(i) If \( r < q \) and \( \alpha \in \Gamma \), then \( d_\alpha \) is \( r \)-summing from \( \ell^p \) into \( \ell^q \).
(ii) If \( p \leq r \) and \( \alpha \neq 0 \), then \( d_\alpha \) is \( r \)-summing from \( \ell^q \) into \( \ell^p \), then \( \alpha \in \Gamma \).
(iii) If \( p < r \leq q \), then \( d_\alpha \) is \( r \)-summing from \( \ell^q \) into \( \ell^p \) if and only if \( \alpha \in \Gamma \).

The proofs of Theorems 2 and 3 will appear elsewhere [1]; we prove Theorem 4, which is much more elementary. As usual, let \( d^{(0)} \) denote the sequence with 1 in the \( j \)-th position, and with 0 elsewhere. Suppose that \( r < q \) and that \( \alpha \in \Gamma \).

\[
\sum_{j=1}^{\infty} |d_\alpha^{(0)}(\alpha)|^r < \infty
\]

so that \( d_\alpha \) is \( r \)-summing. If \( p \leq r \) and \( \alpha \neq 0 \), then \( d_\alpha \) is \( r \)-summing from \( \ell^q \) into \( \ell^p \), and if \( f \in \ell^p \),

\[
\sum_{j=1}^{\infty} |d_\alpha^{(0)}(\alpha)|^r < \infty.
\]

Finally (iii) is a consequence of (i) and (ii).

2. Spaces of diagonal mappings. In the next section we shall give a nearly complete account of the \( r \)-summing, \( r \)-integral and \( r \)-nuclear diagonal mappings from one \( \ell^p \) space to another. The procedure will be first to characterise the \( r \)-summing operators, and then to use duality theorems to characterize the \( r \)-integral and \( r \)-nuclear mappings for \( 1 < r < \infty \).

This process has been used by Tong [6] to characterize the 1-nuclear diagonal mappings. In this section we shall establish the duality theorems required. In fact, it seems worth establishing these in a more general setting, as in [7].

Let \( \mathcal{E} \) denote the linear space of all sequences, and \( \varphi \) the space of all sequences with only finitely many non-zero terms. We recall that a \( BK \)-space \( E \) is a linear subspace of \( \mathcal{E} \) containing \( \varphi \), and equipped with a Banach space norm under which all the coordinate functionals are continuous. \( E \) is said to be \( \varphi \)-space if whenever \( x \in E \) and \( |y| < |x| \) for each \( i \), then \( y = (y_i) \in E \). If \( (E, \| \cdot \|) \) is a solid \( BK \)-space, there is an equivalent norm \( \| \cdot \| \) on \( E \) such that \( |y| < |x| \) for each \( i \), then \( \|y\| < \|x\| \). If \( E \) is solid, we shall always suppose that it is equipped with such a norm. We define the mapping \( F_n \) by \( F_n(x) = x_i \) if \( i \leq n \), \( F_n(x) = 0 \) if \( i > n \). 

**ABK-space** \( \mathcal{E} \) is an \( \mathcal{A} \)-space if \( \mathcal{F}_n(x) \to x \) for each \( x \in \mathcal{E} \). If \( \mathcal{E} \) is a sequence space, \( \mathcal{E}^{\infty} = \{ \sum_{i=1}^{\infty} |y_i| < \infty, \, \text{for each} \, x \in \mathcal{E} \} \). \( \mathcal{E} \) is a \( \mathcal{K} \)-space if \( \mathcal{E} = \mathcal{E}^{\infty} \).

If \( (E, \| \cdot \|) \) is a solid \( BK \)-space, every element of \( E \) defines a continuous linear functional on \( E \), with norm \( \|y\| = \sup \left\{ \sum_{i=1}^{\infty} |y_i| : \|x\| \leq 1 \right\} \). In addition \( E \) is an \( \mathcal{A} \)-space, all continuous linear functionals are given by elements of \( E^* \), so that we may identify \( E^* \) and \( E^* \). Thus a solid \( BK \)-space \( E \) is reflexive if and only if \( E \) is a \( \mathcal{K} \)-space and \( E \) and \( E^* \) are both \( \mathcal{A} \)-spaces.

If \( E \) and \( F \) are \( BK \)-spaces, and if \( A = (a_{ij}) \) is a matrix such that \( A = (\sum_{i=1}^{\infty} a_{ij} e_i) \cdot e_j \) for each \( x \in E \), then \( A \) defines a continuous linear mapping from \( E \) into \( F \) (which we shall again denote by \( A \)). If further, \( E \) is an \( \mathcal{A} \)-space, then every continuous linear mapping is given in this way. If \( A \) is a sequence, we shall write \( d_\alpha \) for the associated diagonal matrix \( \sigma(n_1, n_2, \ldots) \). If \( A \) is a matrix, we shall write \( D_\alpha \) for the associated diagonal matrix \( \sigma(n_1, n_2, \ldots) \).

**1**. If \( 1 < r < \infty \), we denote by \( D_{\mathcal{N}}(E, F) \), \( \mathcal{A}_{\mathcal{N}}(E, F) \) and \( D_{\mathcal{A}}(E, F) \) the subspaces of \( \mathcal{N}(E, F) \), \( \mathcal{A}(E, F) \) and \( \mathcal{A}(E, F) \) respectively by diagonal matrices. Each is a closed subspace of its corresponding space, and is therefore a Banach space. We define \( D_{\mathcal{N}}(E, F) = \{ \alpha : d_\alpha \in D_{\mathcal{N}}(E, F) \} \) and \( \mathcal{N}(E, F) = \{ \alpha : d_\alpha \in \mathcal{N}(E, F) \} \). We denote by \( D_{\mathcal{A}}(E, F) \), \( \mathcal{A}(E, F) \) and \( D_{\mathcal{A}}(E, F) \) the subspaces of \( \mathcal{A}(E, F) \), \( \mathcal{A}(E, F) \) and \( \mathcal{A}(E, F) \) respectively by diagonal matrices. Each is a closed subspace of its corresponding space, and is therefore a Banach space. We define \( D_{\mathcal{A}}(E, F) = \{ \alpha : d_\alpha \in D_{\mathcal{A}}(E, F) \} \) and \( \mathcal{A}(E, F) = \{ \alpha : d_\alpha \in \mathcal{A}(E, F) \} \). Note that if \( \alpha \in D_{\mathcal{N}}(E, F) \) and if \( |b| \leq |a| \) for all \( a \), then we can write \( b = \gamma a \), where \( |\gamma| \leq 1 \), for all \( a \). Thus if \( E \) is a solid \( BK \)-space, \( d_\alpha = d_\beta d_\alpha \), and
If $a \in DI_{r}(F, E)$ and $b \in DN_{r}(E, F)$, then $d_{a} d_{b} \in N_{r}(E, F)$ ([3] Satz 18), so that $\text{Tr}(d_{a} d_{b}) = \sum_{i=1}^{n} a_{i} b_{i}$ exists and is finite. Thus $DI_{r}(F, E) \subseteq (DN_{r}(E, F))^{r}$. Conversely, suppose that $a \in (DN_{r}(E, F))^{r}$. It is easy to verify that $d_{a}$ maps $F$ into $E^{r} = E$. In order to show that $d_{a} \in DI_{r}(F, E)$ it is enough to show that there exists a constant $K$ such that

$$|\text{Tr}(d_{a} T)| \leq K r_{r}(T),$$

for all continuous operators $T$ of finite rank from $E$ into $F$ (cf. [3], Satz 52) and, as in Theorem 5, we can restrict attention to operators $T$ given by a matrix with only finitely many non-zero rows or columns. If $T$ is such a matrix, with diagonal $t = (t_{1}, t_{2}, \ldots)$,

$$|\text{Tr}(d_{a} T)| = |\text{Tr}(d_{a} A_{T})| = \sum_{i=1}^{n} a_{i} t_{i}^{r} \leq K r_{r}(t),$$

since $a$ defines a continuous linear functional on $DN_{r}(E, F)$. But

$$r_{r}(t) = r_{r}(d_{A_{T}}) \leq r_{r}(T),$$

by Theorem 6. Thus

$$DN_{r}(E, F^{r}) \subseteq DI_{r}(F, E).$$

Theorem 8. Suppose that $E$ is an AK-space and a Köthe space, and that either $E^{r}$ or $F$ is an AK-space. Then if $1 < r < \infty$, $DI_{r}(F, E) = (DN_{r}(E, F))^{r}$.

If $a \in DI_{r}(F, E)$ and $b \in DI_{r}(E, F)$, then $d_{a} d_{b} \in DI_{r}(E, F)$ ([3], Satz 48). Let $\gamma^{r} = (\gamma_{1}, \gamma_{2}, \ldots)$ and $\gamma = (\gamma_{1})$. Then

$$\sum_{i=1}^{n} \gamma_{i} = \text{Tr} d_{\gamma^{r}}(T) \leq \text{Tr} d_{\gamma_{1}^{r}}(A_{T}) \leq \text{Tr} d_{\gamma_{1}^{r}}(A_{T}) \leq r_{r}(T),$$

for $t > t_{0}$, and $DI_{r}(F, E) \subseteq (DI_{r}(E, F))^{r}$. In order to obtain the converse inclusion, we argue as in Theorem 5, using ([3], Satz 53), and the observation that $E$ is complemented in $E^{r}$, so that a linear mapping $T$ of a Banach space into $E$ is $r$-integral if and only if $t_{p} T$ is $r$-integral, where $t_{p}$ is the inclusion of $E$ into $E^{r}$. ($F$ is complemented in $E^{r}$ because $G = G^{r}$, where $G$ is the closure of $F$ in $E^{r}$).

Corollary. Suppose that $E$ and $F$ are AK-spaces and Köthe spaces. Then if $1 < r < \infty$, every $r$-integral mapping from $E$ into $F$ is $r$-nuclear if and only if $DN_{r}(E, F)$ is a Köthe space.

3. Diagonal mappings between $\ell^{p}$ spaces. We now consider diagonal mappings between $\ell^{p}$ spaces.

Theorem 9. The mapping $d_{a}$ is $r$-summing ($0 < r < \infty$) from $\ell^{p}$ into $\ell^{p}$ if and only if the following conditions are satisfied:
(i) if $1 \leq p \leq 2$ and $p < 2$,
\[ a \in \ell^p \quad \text{for} \quad 0 \leq r < p, \]
\[ a \in \ell^r \quad \text{for} \quad p \leq r < q, \]
\[ a \in \ell^q \quad \text{for} \quad q \leq r. \]

(ii) if $1 < p = q < 2$,
\[ a \in \ell^p \quad \text{for} \quad 0 \leq r < p, \]
\[ a \in \ell^r \quad \text{for} \quad p \leq r. \]

(iii) if $p = q = 2$, $a \in \ell^2$ for all values of $r$;

(iv) if $1 \leq q < p \leq 2$, $a \in \ell^p$ for all values of $r$;

(v) if $1 \leq q < 2$ and $2 < p \leq \infty$,
\[ a \in \ell^{pq0-q} \quad \text{for} \quad all \quad values \quad of \quad r; \]

(vi) if $2 < q \leq p < \infty$,
\[ a \in \ell^p \quad \text{for} \quad 0 \leq r < p, \]
\[ a \in \ell^r \quad \text{for} \quad p \leq r < q; \]

(vii) if $2 < p < q < \infty$,
\[ a \in \ell^p \quad \text{for} \quad 0 \leq r < p, \]
\[ a \in \ell^r \quad \text{for} \quad p \leq r < q; \]

and (viii) if $2 < q \leq p < \infty$, $a \in \ell^p$ for all values of $r$.

The results of this theorem can be expressed diagrammatically:

I am grateful to Professor Pietsch for suggesting this. In the diagrams, $p$ is plotted horizontally, $q$ vertically on an inverse scale (so that the bottom left-hand corner of the square corresponds to $p = q = 1$, the top right-hand corner to $p = q = \infty$ and the centre of the square to $p = q = 2$). In the diagrams on the left, the space of $r$-summing mappings is indicated; on the right, points $(p, q)$ with the same space of $r$-summing mappings are joined by contour lines.

We begin with case (iii) (which is of course well-known). $d_r$ is 2-summing if and only if $a \in p$, by Theorem 4. Then $d_r$ is $r$-summing for all $r$ if and only if $a \in \ell^p$, by Theorem 3.

Next we consider case (iv). First suppose that $p = 2$. Then if $a \in \ell^p$, $d_r$ is $q$-summing, by Theorem 4. If $d_r$ is $q$-summing, $d_r'$ is $q$-summing from $\ell^p$ into $\ell^q$ by Theorem 3, and so $a \in \ell^p$, by Theorem 4. Further $d_r$ is $r$-summing, for $0 \leq r < \infty$, if and only if $d_r$ is $q$-summing, again by Theorem 3. This deals with the case $p = 2$. If $p < 2$, and if $a \in \ell^p$, then $d_r$ is 0-summing (by Theorem 1) and $r$-summing for all $r$. If $d_r$ is $r$-summing, then a fortiori it is $r$-summing from $\ell^p$ into $\ell^q$, and so $a \in \ell^p$.

Case (v). If $a \in \ell^{pq0-q}$, then $d_r$ is 0-summing (Theorem 1), and therefore $r$-summing for all values of $r$. If $d_r$ is $r$-summing, and if $d_r$ maps $\ell^p$ into $\ell^q$, then $d_{\lambda r}$ is $r$-summing from $\ell^p$ into $\ell^q$, so that $a(f, r)$, by case (iv). Since this holds for all such $\beta$, $a \in \ell^{pq0-q}$.

Case (vi) is obtained by combining Theorem 1 with Theorem 4; this deals with case (vii) when $0 \leq r \leq p$, and the result for $p \leq r \leq q$ follows from Theorem 4.

Case 1: $r < \min(p, q)$

Case 2: $\min(p, q) \leq r \leq \max(p, q)$

Case 3: $\max(p, q) < r$
It is natural to ask what the corresponding results for $L^p$ spaces are. If $\mu$ is a measure on $\Omega$, we can write $\mu = \mu_1 + \mu_2$, where $\mu_1$ is purely atomic and $\mu_2$ is continuous; then $L^p(\Omega, \mu) \cong L^p(\Omega, \mu_1) \oplus L^p(\Omega, \mu_2)$. $L^p$ is isometrically isomorphic to $P$ or $P^\ast$, and we can use Theorem 9 to deal with this. Thus it is sufficient to consider the case where $\mu$ is a continuous measure. If $g \in L^p(\Omega, \mu)$, $M_g$ denotes the operation of multiplication by $g$.

**Theorem 11.** Suppose that $\mu$ is a continuous probability measure on $\Omega$, that $g \in L^p(\Omega, \mu)$ (where $p \geq 1$) and that $g \neq 0$. Then:

(i) $M_g$ is $g$-summing from $L^2(\Omega, \mu)$ into $L^p(\Omega, \mu)$;

(ii) $M_g$ is not $r$-summing from $L^2(\Omega, \mu)$ into $P(\Omega, \mu)$ for any $r < q$,

and

(iii) $M_g$ is not $r$-summing from $L^p(\Omega, \mu)$ into $L^2(\Omega, \mu)$ for any finite $r$ and $\mu$.

The proof of Theorem 4(i) carries over, with obvious modifications, to prove part (i). Replacing $g$ by $-g$ if necessary, we can find $s > 0$ and a subset $E$ of positive measure such that $g(x) \geq s$ on $E$. Let $\lambda_1, \lambda_2, \ldots$, be a sequence of positive numbers such that $\sum \lambda_i = m$, but otherwise to be determined, and let $E_1, E_2, \ldots$ be a sequence of disjoint subsets of $E$ such that $\mu(E_i) = \lambda_i$. Let $S_i : P \to P^\ast$ be defined by

$$S_i(a) = \sum_{k \in \mathbb{N}} a_k^2 \chi_{E_k}$$

Further, $P^\ast$ is reflexive if $1 < r < \infty$. Combining Theorem 7 and Theorem 9, we obtain

**Theorem 10.** (i) The mapping $d_n$ is $r$-integral $(1 < r < \infty)$ from $P^\ast$ into $P$ if and only if it is $r$-summing, except perhaps when $2 < \min(p, q) < \max(p, q) < r$ and when $1 < r < \min(p, q) < \max(p, q) < 2$.

(ii) If $2 < p < q < r$, $d_n$ is $r$-integral if and only if $\alpha \in P^\ast$.

(iii) If $2 < q < r = p$, $d_n$ is $r$-integral if and only if $\alpha \in P^\ast$.

(iv) If $2 < q < r < p$, $d_n$ is $r$-integral if and only if $\alpha \in P^\ast$.

(v) $d_n$ is $r$-nuclear $(1 < r < \infty)$ if and only if it is $r$-integral, except when $2 < q < r = \infty$, when the condition is $\alpha \in P$, and perhaps when $1 < r < \min(p, q) < \max(p, q) < 2$.

It is natural to conjecture that a diagonal mapping $d_n$ from $P^\ast$ into $P$ is $r$-integral if and only if it is $r$-summing, for $1 < r < \infty$. If this were so, we would have a complete account of the diagonal-summability properties.

Note also that the inclusion mapping of $P$ into $P$ is $r$-integral, for $1 < r < \infty$ is every continuous linear mapping from $P$ into $P$ $r$-integral, for $1 < r < \infty$. Is every compact linear mapping from $P$ into $P$ $r$-integral, for $1 < r < \infty$? If this were so, every $r$-summing mapping from $P$ into $P$ would be $r$-integral.
that since $M_\rho$ is continuous, $s \geq q$. Inspection of Theorem 9(i)-(iv) shows that this provides a contradiction when $s \geq 2$ (so that $s' \leq 2$). But if $M_\rho$ were $r$-summing from $L^s(\Omega, \mu)$ into $L^{s'}(\Omega, \mu)$ for some $s < 2$, $M_\rho$ would be $r$-summing from $L^s(\Omega, \mu)$ into $L^{s'}(\Omega, \mu)$, since $L^s(\Omega, \mu) \subseteq L^{s'}(\Omega, \mu)$, and we again obtain a contradiction.

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**STUDIA MATHEMATICA, T. LI (1974)**

**Sur l'analyse harmonique du groupe affine de la droite**

par

IDRISS KHALIL (Nancy et Rabat)

**Abstract.** Nous définissons et étudions la transformation de Plancherel pour le groupe $G$ des transformations affines de la droite. Nous en déduisons une caractérisation des fonctions $f$ de $L^2(G)$ telles que les opérateurs $\pi(f)$, où $\pi \in \hat{G}$, sont compacts. De plus, nous étudions l'algèbre de Fourier $A(G)$ et l'algèbre de Fourier-Stieljes $L^2(G)$ de ce groupe, établissant notamment une "décomposition de Lebesgue" et montrant que $A(G) = L^2(G) \cap \nu(G)$.

**Introduction.** Soit $G$ le groupe affine de la droite, c'est-à-dire des transformations $a \rightarrow ax + b$, de $R$ dans $R$, où $a > 0$ et $b$ sont réels. On connaît par Gelfand et Naimark la description complète de l'ensemble $\hat{G}$ des (classe de) représentations unitaires irréductibles de $G$. Cet ensemble contient, à équivalence pres, une famille indexée par $R$ de représentations de dimension 1, et deux représentations, $\pi_+$ et $\pi_-$, de dimension infinie, opérant dans le même espace hilbertien $L^2(R)$. En analysant de près $\hat{G}$, on voit que seules les deux représentations $\pi_+$ et $\pi_-$ jouent un rôle essentiel, en tout cas dans les questions que nous avons abordées. Cela tient au fait bien connu que l'ensemble des deux points $\pi_+$ et $\pi_-$ est dense, au sens de la topologie de J. M. G. Fell, dans $\hat{G}$.

Le troisième paragraphe de ce travail est consacré à établir une formule de Plancherel explicite sur ce groupe $G$. Elle précise notablement dans ce cas particulier, et par des méthodes différentes, le résultat de Kleppner et Lipsman [19]. Nous avons ici à vaincre le fait que $G$ n'est pas unimodulaire, et aussi que, même pour des fonctions $f$ suffisamment régulières, il peut arriver que les opérateurs $\pi_+(f)$ et $\pi_-(f)$ ne soient pas des opérateurs compacts, encore moins des opérateurs de Hilbert–Schmidt sur $L^2(R)$. Cependant, en composant ces opérateurs par un opérateur convenablement choisi $\delta$ non borné de $L^2(R)$ de domaine dense, on aboutit aux résultats que $\mathcal{F}_\mu(f) = \delta_\mu(A^{-1}f)$, où $A$ est la fonction

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