Then we have

$$\lambda(\psi(X_i))+\varepsilon=\lambda(X_i)=\int\limits_{\psi(X_i)}\frac{\varphi(|a_{n_0}h(t)|)}{\varphi(a_{n_0})}\,d\lambda$$
$$\leqslant\frac{\varphi(a_{n_0}(i+2))}{\varphi(a_{n_0})}\lambda(\psi(X_i))\leqslant\lambda(\psi(X_i))+\varepsilon/2$$

which is a contradiction.

So altogether we have $\lambda(\psi(A))=\lambda(A)$ for all $A\in A_\lambda$ which implies (iii).

Let $M:=\{t\colon |h(t)|>1\}$. Without restriction we can suppose $\lambda(M)\leqslant 1$. We showed already that there is an $N\in A_\lambda$ with $\psi(N)=M$ and $\lambda(N)=\lambda(M)$. Then we have $\varphi(1)\cdot\lambda(N)=\|\chi_N\|=\|T(\chi_N)\|=\|h(t)\chi_M(t)\|=\int\limits_M\varphi(|h(t)|)\,d\lambda>\varphi(1)\cdot\lambda(M)$, which implies $\lambda(M)=0$. Similarly we get $\lambda\{t\colon |h(t)|<1\}=0$, which completes the proof.

The proof of the theorem is a trivial consequence of Lemma 2.

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Diagonal mappings between sequence spaces

by

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Abstract. Some general results are obtained about r-nuclear, r-integral and r-summing diagonal mappings from one sequence space to another. These are used to give a nearly complete characterization of such mappings from one l^p space to another, extending results of Schwartz and Tong.

1. Introduction. Schwartz ([4] Théorème (XXVI, 4; 1) and [5]) has given a complete account of 0-summing diagonal mappings from one l^p space to another. If a is a sequence, we denote by d_a the linear operator defined by coordinatewise multiplication by a. Recall that $a\in l^{p-}$ if $\sum_{n=1}^{\infty}|a_n|^p(1+\log|a_n^{-1}|)<\infty$. If $q<2<p$, we set $\varphi(p,q)=(p^{-1}+q^{-1}-\frac{1}{2})^{-1}$. Schwartz' result is then the following:

THEOREM 1. *d_a is 0-summing from $l^{p'}$ into l^q if and only if the following conditions are satisfied:*

(i) *if* $p=q<2$, $a\in l^{p-}$;

(ii) *if* $2\leqslant q<p$, $a\in l^p$;

(iii) *if* $q<2<p$, $a\in l^{\varphi(p,q)}$;

and

(iv) *otherwise*, $a\in l^{\min(p,q)}$.

The purpose of this paper is to extend this result to give a nearly complete account of r-summing, r-integral and r-nuclear diagonal nappings from $l^{p'}$ into l^q. For this, we shall need the three following theorems:

THEOREM 2. *Suppose that $1\leqslant p,\ q\leqslant 2$, that F is isomorphic to a closed subspace of $L^q(0,1)$ and that E' is isomorphic to a closed subspace of $L^p(0,1)$. Then the following are equivalent:*

(i) *u is 0-summing;*

(ii) *u is r-summing for some $r<p$;*

(iii) *u' is 0-summing;*

(iv) *u' is s-summing for some $s<q$.*

THEOREM 3. *Suppose that $1\leqslant p\leqslant 2$, and that E' is isomorphic to a closed subspace of $L^p(0,1)$ and that H is a Hilbert space. If $u\in L(E,H)$,* *the following are equivalent:*

(i) *u is 0-summing;*

(ii) *u is p-summing;*

(iii) *u' is 0-summing;*

(iv) *u' is s-summing for some s.*

THEOREM 4. *Suppose that $0 < r < \infty$, $p \geqslant 1$ and $q \geqslant 1$.*

(i) *If $r \leqslant q$ and $\alpha \in l^r$, then d_α is r-summing from $l^{p'}$ into l^q.*

(ii) *If $p \leqslant r$ and d_α is r-summing from $l^{p'}$ into l^q, then $\alpha \in l^r$.*

(iii) *If $p \leqslant r \leqslant q$, d_α is r-summing from $l^{p'}$ into l^q if and only if $\alpha \in l^r$.*

The proofs of Theorems 2 and 3 will appear elsewhere [1]; we prove Theorem 4, which is much more elementary. As usual, let $e^{(j)}$ denote the sequence with 1 in the jth position, and with 0 elsewhere. Suppose that $r \leqslant q$ and that $\alpha \in l^r$. If $x^{(1)}, \ldots, x^{(n)} \in l^{p'}$,

$$\sum_{i=1}^{n} \|d_\alpha x^{(i)}\|^r = \sum_{i=1}^{n} \Big(\sum_{j=1}^{\infty} |\alpha_j x_j^{(i)}|^q\Big)^{r/q}$$

$$\leqslant \sum_{i=1}^{n} \sum_{j=1}^{\infty} |\alpha_j x_j^{(i)}|^r$$

$$= \sum_{j=1}^{\infty} |\alpha_j|^r \sum_{i=1}^{n} |\langle x^{(i)}, e^{(j)}\rangle|^r$$

$$\leqslant \Big(\sum_{j=1}^{\infty} |\alpha_j|^r\Big) \sup_{\|x'\| \leqslant 1} \sum_{i=1}^{n} |x'(x^{(i)})|^r,$$

so that d_α is r-summing. If $p \leqslant r$ and d_α is r-summing from $l^{p'}$ into l^q, and if $f \in l^p$,

$$\sum_{j=1}^{\infty} |\langle e^{(j)}, f\rangle|^r \leqslant \Big(\sum_{j=1}^{\infty} |\langle e^{(j)}, f\rangle|^p\Big)^{r/p} = \|f\|^r$$

so that $\sum_{j=1}^{\infty} \|d_\alpha e^{(j)}\|^r = \sum_{j=1}^{\infty} |\alpha_j|^r < \infty$. Finally (iii) is a consequence of (i) and (ii).

2. Spaces of diagonal mappings. In the next section we shall give a nearly complete account of the r-summing, r-integral and r-nuclear diagonal mappings from one l^p space to another. The procedure will be first to characterise the r-summing operators, and then to use duality theorems to characterize the r-integral and r-nuclear mappings (for $1 < r < \infty$). This technique has been used by Tong [6] to characterize the 1-nuclear diagonal mappings. In this section we shall establish the duality theorems required. In fact, it seems worth establishing these in a more general setting, as in [7].

Let ω denote the linear space of all sequences, and φ the space of all sequences with only finitely many non-zero terms. We recall that a BK-space E is a linear subspace E of ω, containing φ, and equipped with a Banach space norm under which all the coordinate functionals are continuous. E is *solid* if whenever $x \in E$ and $|y_i| \leqslant |x_i|$ for each i, then $y = (y_i) \in E$. If $(E, \|\ \|)$ is a solid BK-space, there is an equivalent norm $|||\ |||$ on E such that if $|y_i| \leqslant |x_i|$ for each i, then $|||y||| \leqslant |||x|||$. If E is solid, we shall always suppose that it is equipped with such a norm. We define the mapping P_n by $(P_n(x))_i = x_i$ if $i \leqslant n$, $(P_n(x))_i = 0$ if $i > n$. A BK-space E is an *AK-space* if $P_n(x) \to x$ for each x in E. If E is a sequence space, $E^x = \{y: \sum_{i=1}^{\infty} |x_i y_i| < \infty$, for each $x \in E\}$. E is a *Köthe space* if $E = E^{xx}$. If $(E, \|\ \|)$ is a solid BK-space, every element of E^x defines a continuous linear functional on E, with norm $\|y\| = \sup\{\sum_{i=1}^{\infty} |x_i y_i|: \|x\| \leqslant 1\}$. If in addition E is an AK-space, all continuous linear functionals are given by elements of E^x, so that we may identify E' and E^x. Thus a solid BK-space E is reflexive if and only if E is a Köthe space and E and E^x are both AK-spaces.

If E and F are BK-spaces, and if $A = (a_{ij})$ is a matrix such that $Ax = \big(\sum_{j=1}^{\infty} a_{ij} x_j\big)_{i=1}^{\infty} \in F$ for each $x \in E$, then A defines a continuous linear mapping from E into F (which we shall again denote by A). If, further, E is an AK-space, then every continuous linear mapping is given in this way. If α is a sequence, we shall write d_α for the diagonal matrix diag$(\alpha_1, \alpha_2, \ldots)$. If A is a matrix, we shall write D_A for the associated diagonal matrix diag$(a_{11}, a_{22}, \ldots)$. If $1 \leqslant r < \infty$, we denote by $N_r(E, F)$, $\Pi_r(E, F)$ and $I_r(E, F)$ respectively the r-nuclear, r-summing and r-integral mappings from E into F, and denote the corresponding norms by $\nu_r(E, F)$, $\pi_r(E, F)$ and $i_r(E, F)$. If E is an AK-space, E trivially has the metric approximation property. Thus if E is reflexive or if F is an AK-space $N_r(E, F)$ is the closure of the operators of finite rank in $I_r(E, F)$, and $\nu_r(E, F) = i_r(E, F)|_{N_r(E,F)}$.

We denote by $\Delta N_r(E, F)$, $\Delta\Pi_r(E, F)$ and $\Delta I_r(E, F)$ the subspaces of $N_r(E, F)$, $\Pi_r(E, F)$ and $I_r(E, F)$ defined by diagonal matrices. Each is a closed subspace of its corresponding space, and is therefore a Banach space. We define $DN_r(E, F) = \{\alpha: d_\alpha \in \Delta N_r(E, F)\}$, and define $n_r(\alpha) = \nu_r(d_\alpha)$; then $(DN_r(E, F), n_r)$ is a BK-space. The BK-spaces $(D\Pi_r(E, F), p_r)$ and $(DI_r(E, F), i_r)$ are defined analogously. Note that if $\alpha \in DN_r(E, F)$ and if $|\beta_i| \leqslant |\alpha_i|$ for all i, then we can write $\beta = \gamma\alpha$, where $|\gamma_i| \leqslant 1$, for all i. Thus if F is a solid BK-space, $d_\beta = d_\gamma d_\alpha$, and

$\nu_r(d_\beta) \leqslant \|d_\gamma\| \nu_r(d_a) \leqslant \nu_r(d_a)$, so that $DN_r(E,F)$ is a solid BK-space. The same is clearly true if E is a solid BK-space, and corresponding results hold for $D\Pi_r(E,F)$ and $DI_r(E,F)$.

From now on we shall suppose that E and F are solid BK-spaces.

THEOREM 5. *Suppose that E is an AK-space, and that either E^x or F is also an AK-space. Then if $d_a \in \Delta I_r(E,F)$, $d_a \in \Delta N_r(E,F)$ if and only if $d_{P_n}(a) \to d_a$ with respect to the norm i_r.*

The condition is certainly sufficient, by the remarks above. Conversely, suppose that $d_a \in \Delta N_r(E,F)$. Then given $\varepsilon > 0$, there exists a mapping u of finite rank such that $i_r(d_a - u) < \varepsilon$. Let U be the corresponding matrix. By changing u a little, and using the conditions on E^x or F, we can suppose that U has either only finitely many non-zero rows or finitely many non-zero columns. Thus there exists n such that $u_{ij} = 0$ for $\min(i,j) > n$. Let $G_n = \{(g_i) : g_i = \pm 1 \text{ for } i \leqslant n,\ g_i = 1 \text{ for } i > n\}$. Then

$$i_r(d_a - d_g u d_g) = i_r\big(d_g(d_a - u)d_g\big) < \varepsilon \quad \text{for each } g \text{ in } G_n,$$

so that $i_r\big(d_a - 2^{-n}\sum_{g\in G_n} d_g u d_g\big) < \varepsilon$. But $2^{-n}\sum_{g\in G_n} d_g u d_g = \Delta_U$, so that if $m > n$

$$i_r(d_a - d_{P_m(a)}) \leqslant i_r(d_a - d_{P_n(a)}) \leqslant i_r(d_a - \Delta_u) < \varepsilon$$

(using the solid property of the norm i_r).

COROLLARY. *Under the hypotheses of the theorem, $DN_r(E,F)$ is an AK-space, and so $\big(DN_r(E,F)\big)' = \big(DN_r(E,F)\big)^x$.*

THEOREM 6. *Suppose that E is an AK-space, and that either E^x or F is an AK-space. Then if $A \in N_r(E,F)$, $\Delta_A \in N_r(E,F)$, and $\nu_r(\Delta_A) \leqslant \nu_r(A)$.*

Given $\varepsilon > 0$, there exists a u as in Theorem 5 such that $\nu_r(A-u) < \varepsilon/2$. Suppose that u has only finitely many non-zero rows, and that $u_{ij} = 0$ for $i > n$; then $P_m u = u$ for $m \geqslant n$. Thus $\nu_r(P_m A - u) = \nu_r\big(P_m(A-u)\big) \leqslant \|P_m\| \nu_r(A-u) < \varepsilon/2$, so that $\nu_r(A - P_m A) < \varepsilon$ for $m \geqslant n$. Let $a = (a_{11}, a_{22}, \ldots)$

$$\nu_r(d_{P(m)a}) = \nu_r\Big(2^{-m}\sum_{g\in G_m} gP_m Ag\Big) \leqslant \nu_r(P_m A) \leqslant \nu_r(A) + \varepsilon$$

and

$$\nu_r(d_{P(m)a} - d_{P(p)a}) = \nu_r\Big(2^{-m}\sum_{g\in G_m} g(P_m - P_p)Ag\Big) \leqslant \nu_r(P_m A - P_p A) \leqslant 2\varepsilon$$

for $m \geqslant p \geqslant n$, so that $(d_{P(m)a})$ is a Cauchy sequence in $N_r(E,F)$, which converges to d_a. Thus $d_a \in N_r(E,F)$, and $\nu_r(d_a) \leqslant \nu_r(A)$. A similar argument deals with the case where u has only finitely many non-zero columns.

THEOREM 7. *Suppose that E is an AK-space and a Köthe space, and that either E^x or F is an AK-space. Then, if $1 < r < \infty$, $D\Pi_r(F,E) = \big(DN_{r'}(E,F)\big)^x$.*

If $a \in D\Pi_r(F,E)$ and $\beta \in DN_{r'}(E,F)$, then $d_a d_\beta \in N_1(E,F)$ ([3] Satz 48), so that $\operatorname{tr}(d_a d_\beta) = \sum_{i=1}^{\infty} a_i\beta_i$ exists and is finite. Thus $D\Pi_r(F,E) \subseteq \big(DN_{r'}(E,F)\big)^x$. Conversely, suppose that $a \in \big(DN_{r'}(E,F)\big)^x$. It is easy to verify that d_a maps F into $E^{xx} = E$. In order to show that $d_a \in D\Pi_r(F,E)$ it is enough to show that there exists a constant K such that

$$|\operatorname{Tr}(d_a T)| \leqslant K\nu_{r'}(T),$$

for all continuous operators T of finite rank from E into F (cf. [3], Satz 52) and, as in Theorem 5, we can restrict attention to operators T given by a matrix with only finitely many non-zero rows or columns. If T is such a matrix, with diagonal $t = (t_{11}, t_{22}, \ldots)$,

$$|\operatorname{Tr}(d_a T)| = |\operatorname{Tr}(d_a \Delta_T)| = \Big|\sum_{i=1}^{\infty} a_i t_{ii}\Big| \leqslant K n_{r'}(t),$$

since a defines a continuous linear functional on $DN_{r'}(E,F)$. But

$$n_{r'}(t) = \nu_{r'}(\Delta_T) \leqslant \nu_{r'}(T), \qquad \text{by Theorem 6.}$$

Thus $DN_{r'}(E,F)^x \subseteq D\Pi_r(F,E)$.

THEOREM 8. *Suppose that E is an AK-space and a Köthe space, and that either E^x or F is an AK-space. Then if $1 < r < \infty$, $DI_r(F,E) = \big(D\Pi_{r'}(E,F)\big)^x$.*

If $a \in DI_r(F,E)$ and $\beta \in D\Pi_{r'}(E,F)$, then $d_a d_\beta \in I_1(E,F)$ ([3], Satz 48). Let $\gamma_i = |a_i\beta_i|$, and let $\gamma = (\gamma_i)$. Then

$$\sum_{i=1}^{n} \gamma_i = \operatorname{Tr} d_{P_n(\gamma)} \leqslant \nu_1(d_{P_n(\gamma)}) = i_1(d_{P_n(\gamma)}) \leqslant i_1(d_a d_\beta).$$

Thus $\sum_{i=1}^{\infty} |a_i\beta_i| < \infty$, and $DI_r(F,E) \subseteq \big(D\Pi_{r'}(E,F)\big)^x$. In order to obtain the converse inclusion, we argue as in Theorem 7, using [3], Satz 53, and the observation that E is complemented in E'', so that a linear mapping T of a Banach space into E is r-integral if and only if $i_E T$ is r-integral, where i_E is the inclusion of E into E''. (E is complemented in E'' because $E = G'$, where G is the closure of φ in E^x.)

COROLLARY. *Suppose that E and F are AK-spaces and Köthe spaces. Then if $1 < r < \infty$, every r-integral mapping from E into F is r-nuclear if and only if $DN_r(E,F)$ is a Köthe space.*

3. Diagonal mappings between l^p-spaces. We now consider diagonal mappings between l^p-spaces.

THEOREM 9. *The mapping d_a is r-summing $(0 \leqslant r < \infty)$ from $l^{p'}$ into l^q if and only if the following conditions are satisfied:*

(i) *if* $1 \leqslant p \leqslant 2$ *and* $p < 2$,

$$\alpha \in l^p \quad \text{for } 0 \leqslant r \leqslant p,$$
$$\alpha \in l^r \quad \text{for } p \leqslant r \leqslant q,$$
$$\alpha \in l^q \quad \text{for } q \leqslant r;$$

(ii) *if* $1 \leqslant p = q < 2$,

$$\alpha \in l^{p-} \quad \text{for } 0 \leqslant r < p,$$
$$\alpha \in l^p \quad \text{for } p \leqslant r;$$

(iii) *if* $p = q = 2$, $\alpha \in l^2$ *for all values of* r;

(iv) *if* $1 \leqslant q < p \leqslant 2$, $\alpha \in l^q$ *for all values of* r;

(v) *if* $1 \leqslant q \leqslant 2$ *and* $2 < p \leqslant \infty$,

$$\alpha \in l^{\varphi(p,q)} \quad \text{for all values of } r;$$

(vi) *if* $2 < q \leqslant p < \infty$,

$$\alpha \in l^p \quad \text{for } 0 \leqslant r \leqslant p;$$

(vii) *if* $2 < p < q \leqslant \infty$,

$$\alpha \in l^p \quad \text{for } 0 \leqslant r \leqslant p,$$
$$\alpha \in l^r \quad \text{for } p \leqslant r \leqslant q;$$

and (viii) *if* $2 \leqslant q \leqslant p = \infty$, $\alpha \in l^\infty$ *for all values of* r.

The results of this theorem can be expressed diagrammatically; I am grateful to Professor Pietsch for suggesting this. In the diagrams, p is plotted horizontally, q vertically on an inverse scale (so that the bottom left-hand corner of the square corresponds to $p = q = 1$, the top right-hand corner to $p = q = \infty$ and the centre of the square to $p = q = 2$). In the diagrams on the left, the space of r-summing mappings is indicated; on the right, points (p, q) with the same space of r-summing mappings are joined by contour lines.

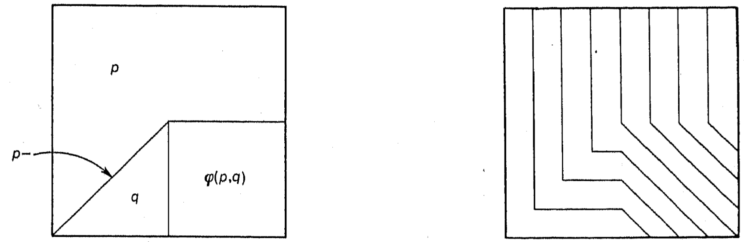
We begin with case (iii) (which is of course well-known). d_α is 2-summing if and only if $\alpha \in l^2$, by Theorem 4. Then d_α is r-summing for all r if and only if $\alpha \in l^2$, by Theorem 3.

Next we consider case (iv). First suppose that $p = 2$. Then if $\alpha \in l^q$, d_α is q-summing, by Theorem 4. If d_α is q-summing, d'_α is q-summing from $l^{q'}$ into l^2 by Theorem 3, and so $\alpha \in l^q$, by Theorem 4. Further d_α is r-summing, for $0 \leqslant r < \infty$, if and only if d_α is q-summing, again by Theorem 3. This deals with the case $p = 2$. If $p < 2$, and if $\alpha \in l^q$, then d_α is 0-summing (by Theorem 1) and r-summing for all r. If d_α is r-summing, then *a fortiori* it is r-summing from l^2 into l^q, and so $\alpha \in l^q$.

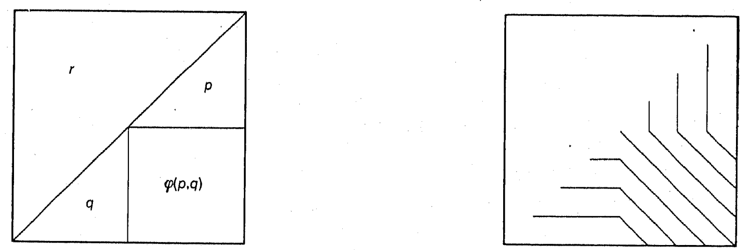
Case (v). If $\alpha \in l^{\varphi(p,q)}$, then d_α is 0-summing (Theorem 1), and therefore r-summing for all values of r. If d_α is r-summing, and if d_β maps l^2 into $l^{p'}$, then $d_{\alpha\beta}$ is r-summing from l^2 into l^q, so that $\alpha\beta \in l^q$, by case (iv). Since this holds for all such β, $\alpha \in l^{\varphi(p,q)}$.

Case (vi) is obtained by combining Theorem 1 with Theorem 4; this deals with case (vii) when $0 \leqslant r \leqslant p$, and the result for $p \leqslant r \leqslant q$ follows from Theorem 4.

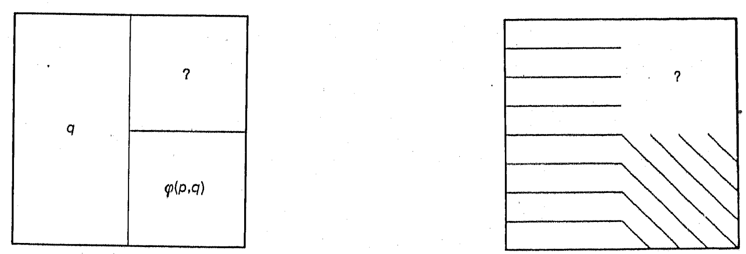
Case 1: $r < \min(p, q)$



Case 2: $\min(p, q) \leqslant r \leqslant \max(p, q)$



Case 3: $\max(p, q) < r$



Case (viii) follows directly from the well-known fact that the inclusion mapping from l^1 to l^2 is 0-summing (which is included in Theorem 1) and the fact that if d_α is continuous, then $\alpha \in l^\infty$.

Case (i). If $\alpha \in l^p$, d_α is 0-summing by Theorem 1, and therefore r-summing, for $0 \leqslant r \leqslant p$. Also d_α is r-summing if and only if $\alpha \in l^r$, for $p \leqslant r \leqslant q$, by Theorem 4. This deals with $0 \leqslant r \leqslant q$. If $\alpha \in l^q$, d_α is r-summing for $r \geqslant q$, and if d_α is r-summing, d_α is *a fortiori* r-absolutely summing from l^2 into l^q, and so $\alpha \in l^q$, by (iv).

Finally we consider case (ii). d_α is 0-summing if and only if $\alpha \in l^{p-}$, by Theorem 1, and this happens if and only if d_α is r-summing for $0 < r < p$, by Theorem 2. d_α is p-summing if and only if $\alpha \in l^p$, by Theorem 4, and this implies that d_α is r-summing, for $p \leqslant r$. Finally if d_α is r-summing, for some $r > p$, $\alpha \in l^p$, as in case (i).

Tong [17] has shown that d_α is 1-nuclear from $l^{p'}$ into l^q if and only if

(i) $\alpha \in l^1$ if $1 \leqslant q \leqslant p' \leqslant \infty$;

(ii) $\alpha \in l^{j(p,q)}$, where $j(p,q) = (1/p+1/q)^{-1}$, if $1 \leqslant p' < q < \infty$;

(iii) $\alpha \in l^p$ if $1 \leqslant p' < q = \infty$;

and (iv) $\alpha \in c_0$ if $p = q = \infty$.

In the next theorem, we obtain corresponding results for r-nuclear and r-integral diagonal mappings, when $1 < r < \infty$.

The space l^{r-} is an Orlicz sequence space. It is not difficult to verify (cf. [2], § 7) that when $1 < r < \infty$, $(l^{r-})^x = l^{s+}$, where $1/r+1/s = 1$, and

$$l^{s+} = \Big\{\alpha\colon \sum_{n=1}^{\infty} |\alpha_n|^s\big(1+\big|\log|\alpha_n|\big|\big)^{1-s} < \infty\Big\}.$$

Further, l^{r-} is reflexive if $1 < r < \infty$. Combining Theorem 7 and Theorem 9, we obtain

THEOREM 10. (i) *The mapping d_α is r-integral $(1 < r < \infty)$ from $l^{p'}$ into l^q if and only if it is r-summing, except perhaps when $2 < \min(p,q) \leqslant \max(p,q) < r$ and when $1 < r < \min(p,q) \leqslant \max(p,q) < 2$.*

(ii) *If $2 < p < q < r$, d_α is r-integral if and only if $\alpha \in l^q$.*

(iii) *If $2 < p = q < r$, d_α is r-integral if and only if $\alpha \in l^{p+}$.*

(iv) *If $2 < q < p < r$, d_α is r-integral if and only if $\alpha \in l^p$.*

(v) *d_α is r-nuclear $(1 < r < \infty)$ if and only if it is r-integral, except when $2 \leqslant q \leqslant p = \infty$, when the condition is that $\alpha \in c_0$, and perhaps when $1 < r < \min(p,q) \leqslant \max(p,q) < 2$.*

It is natural to conjecture that a diagonal mapping d_α from $l^{p'}$ into l^q is r-integral if and only if it is r-summing, for $1 < r < \infty$. If this were so, we would have a complete account of the diagonal r-summing mappings.

Note also that the inclusion mapping of l^1 into l^2 is r-integral, for $1 < r \leqslant \infty$. Is every continuous linear mapping from l^1 into l^2 r-integral, for $1 < r < \infty$? Is every compact linear mapping from l^1 into l^2 r-nuclear, for $1 < r < \infty$? If this were so, every r-summing mapping from l^2 into l^1 would be r-integral.

It is natural to ask what the corresponding results for L^p spaces are. If μ is a measure on Ω, we can write $\mu = \mu_1+\mu_2$, where μ_1 is purely atomic and μ_2 is continuous; then $L^p(\Omega,\mu) \cong L^p(\Omega,\mu_1) \oplus L^p(\Omega,\mu_2)$. L^p is isometrically isomorphic to l^p or l^p_n, and we can use Theorem 9 to deal with this. Thus it is sufficient to consider the case where μ is a continuous measure. If $g \in L^0(\Omega,\mu)$, M_g denotes the operation of multiplication by g.

THEOREM 11. *Suppose that μ is a continuous probability measure on Ω, that $g \in L^q(\Omega,\mu)$ (where $g \geqslant 1$) and that $g \neq 0$. Then*

(i) *M_g is q-summing from $L^\infty(\Omega,\mu)$ into $L^q(\Omega,\mu)$;*

(ii) *M_g is not r-summing from $L^\infty(\Omega,\mu)$ into $L^q(\Omega,\mu)$ for any $r < q$, and*

(iii) *M_g is not r-summing from $L^s(\Omega,\mu)$ into $L^q(\Omega,\mu)$ for any finite r and s.*

The proof of Theorem 4(i) carries over, with obvious modifications, to prove part (i). Replacing g by $-g$ if necessary, we can find $\varepsilon > 0$ and a subset E of positive measure m such that $g(\omega) \geqslant \varepsilon$ on E. Let $a_1, a_2, \ldots$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i = m$, but otherwise to be determined, and let $E_1, E_2, \ldots$ be a sequence of disjoint subsets of E such that $\mu(E_i) = a_i$. Let $S_s\colon l^s \to L^s(\Omega,\mu)$ be defined by

$$S_s(x) = \sum_{i=1}^{\infty} a_i^{-1/s} x_i x_{E_i} \quad \text{for } 1 \leqslant s < \infty,$$

$$S_\infty(x) = \sum_{i=1}^{\infty} x_i X_{E_i}.$$

S_s is of course an isometric embedding. Let $T_q\colon L^q(\Omega,\mu) \to l^q$ be defined by

$$\big(T_q(f)\big)_n = a^{-1/q'} \int_{E_i} f(\omega)\,d\mu(\omega) \quad \text{for } 1 \leqslant q < \infty.$$

$\|T_q\| = 1$, and $T_s S_s$ is the identity mapping on l^s.

Suppose first that M_g were r-summing from $L^\infty(\Omega,\mu)$ into $L^q(\Omega,\mu)$ for some $r < q$. Then $T_q M_g S_\infty$ would be r-summing from l^∞ into l^q. But $T_q M_g S_\infty$ is a diagonal mapping, d_α say, with $\alpha_n \geqslant \varepsilon a_n^{1/q}$. Now if $q > 1$ we can choose the sequence (a_n) in such a way that $\alpha \notin l^{\max(r,1)}$ and if $q = 1$ in such a way that $\alpha \notin l^{1-}$. This contradicts Theorem 9(i) and (ii).

Next suppose that M_g were r-summing from $L^s(\Omega,\mu)$ into $L^q(\Omega,\mu)$, for some finite s and r. Then $T_q M_g S_s$ would be r-summing from l^s into l^q. Again $T_q M_g S_s$ is a diagonal mapping, d_β say, where $\beta_n \geqslant \varepsilon a_n^{1/q-1/p}$. We can therefore choose the sequence (a_n) in such a way that $\beta \notin l^q$. Note also that since M_g is continuous, $s \geqslant q$. Inspection of Theorem 9 (i)–(iv) shows that this provides a contradiction when $s \geqslant 2$ (so that $s' \leqslant 2$). But if M_g were r-summing from $L^s(\Omega, \mu)$ into $L^q(\Omega, \mu)$ for some $s < 2$, M_g would be r-summing from $L^2(\Omega, \mu)$ into $L^q(\Omega, \mu)$, since $L^2(\Omega, \mu) \subseteq L^s(\Omega, \mu)$, and we again obtain a contradiction.

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Sur l'analyse harmonique du groupe affine de la droite *

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Abstract. Nous définissons et étudions la transformation de Plancherel pour le groupe G des transformations affines de la droite. Nous en déduisons une caractérisation des fonctions f de $L^1(G)$ telles que les opérateurs $\pi(f)$, où $\pi \in \hat{G}$, sont compacts. De plus, nous étudions l'algèbre de Fourier $A(G)$ et l'algèbre de Fourier–Stieltjes $B(G)$ de ce groupe, établissant notamment une „décomposition de Lebesgue" et prouvant que $A(G) = B(G) \cap \mathscr{C}_0(G)$.

Introduction. Soit G le groupe affine de la droite, c'est-à-dire des transformations $x \to ax+b$, de $\boldsymbol{R}$ dans $\boldsymbol{R}$, où $a > 0$ et b sont réels. On connaît par Gelfand et Naimark la description complète de l'ensemble $\hat{G}$ des (classes de) représentations unitaires irréductibles de G. Cet ensemble contient, à équivalence près, une famille indéxée par $\boldsymbol{R}$ de représentations de dimension 1, et deux représentations, π_+ et π_-, de dimension infinie, opérant dans le même espace hilbertien $L^2(\boldsymbol{R}_+^*)$. En analysant de près $\hat{G}$, on voit que seules les deux représentations π_+ et π_- jouent un rôle essentiel, en tout cas dans les questions que nous avons abordées. Cela tient au fait bien connu que l'ensemble des deux points π_+ et π_- est dense, au sens de la topologie de J. M. G. Fell, dans $\hat{G}$.

Le troisième paragraphe de ce travail est consacré à établir une formule de Plancherel explicite sur ce groupe G. Elle précise notablement dans ce cas particulier, et par des méthodes différentes, le résultat de Kleppner et Lipsman [19]. Nous avons ici à vaincre le fait que G n'est pas unimodulaire, et aussi que, même pour des fonctions f suffisamment régulières, il peut arriver que les opérateurs $\pi_+(f)$ et $\pi_-(f)$ ne soient pas des opérateurs compacts, encore moins des opérateurs de Hilbert–Schmidt sur $L^2(\boldsymbol{R}_+^*)$. Cependant, en composant ces opérateurs par un opérateur convenablement choisi δ non borné de $L^2(\boldsymbol{R}_+^*)$ de domaine dense, on aboutit aux résultats que $\mathscr{P}_\pm(f) = \delta\pi_\pm(\Delta^{-1/2}f)$, où Δ est la fonction

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