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In non-locally bounded L_φ-spaces the norm is not almost transitive

by

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Abstract. In this note it is shown that the norm of those F-spaces $L_\varphi(X, \mathbf{A}, \mu)$, which are not locally bounded, is not almost transitive, in opposition to the norm of the locally bounded spaces $L_p[0,1]$. The proof follows from a representation of the linear isometries of these spaces.

Let $L_\varphi(X, \mathbf{A}, \mu)$ be the space of μ-measurable, φ-integrable, real-valued functions, defined on a σ-finite, non-atomic, separable measure space, where $\varphi\colon [0, \infty)\to[0, \infty)$ is a continuos, strictly increasing, subadditive function with $\varphi(0) = 0$. As usual functions are identified if they differ only on a set of measure zero.

$L_\varphi(X, \mathbf{A}, \mu)$ is an F-space with F-norm $\|f\|\colon = \int_X \varphi(|f|)\,d\mu$.

For an F-space X let $G(X)$ denote the group of all linear isometries which map X onto itself. The norm $\|x\|$ of an F-space X is called *transitive* resp. *almost transitive* if for all positive r and each $x \in X$ with $\|x\| = r$

$$\{A(x)\colon A \in G(X)\} = \{y \in X\colon \|y\| = r\}$$

resp.

$$\overline{\{A(x)\colon A \in G(X)\}} = \{y \in X\colon \|y\| = r\}, \quad \text{cf. [3]}.$$

Pełczyński and Rolewicz [3] proved that in the spaces $L_p[0,1]$, $0 < p < \infty$, the norm is almost transitive.

In this note we shall show

THEOREM. *In non-locally bounded L_φ-spaces the norm is not almost transitive.*

As $L_\varphi(X, \mathbf{A}, \mu)$ is isometrically isomorphic to $L_\varphi(\mathbf{R}, \mathbf{A}_\lambda, \lambda)$ resp. $L_\varphi([0,1], \mathbf{A}_\lambda, \lambda)$, if $\mu(X) = \infty$ resp. $\mu(X) = 1$, where λ is the Lebesgue measure, it clearly suffices to prove the theorem for these special cases.

The proof of the theorem is based on the following lemmas.

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LEMMA 1. *Let f and g be functions in $L_\varphi(X, \boldsymbol{A}, \mu)$. Then*

$$\|f+g\|+\|f-g\| = 2(\|f\|+\|g\|)$$

if and only if $fg = 0$ μ-a.e. on X (i.e. if and only if $\mu(\operatorname{supp} f \cap \operatorname{supp} g) = 0$).

Proof. Suppose that $\mu(\operatorname{supp} f \cap \operatorname{supp} g) = 0$. Then clearly

$$\|f+g\| = \int_X \varphi(|f+g|)\,d\mu = \|f\|+\|g\| \quad \text{and} \quad \|f-g\| = \|f\|+\|g\|.$$

As φ is subadditive, we have for real numbers a and b $\varphi(|a+b|)+\varphi(|a-b|) \leqslant 2(\varphi(|a|)+\varphi(|b|))$. Because φ is strictly increasing, equality holds if and only if either $a = 0$ or $b = 0$.

Suppose that

$$\int_X \{2(\varphi(|f|)+\varphi(|g|)) - \varphi(|f+g|) - \varphi(|f-g|)\}\,d\mu = 0.$$

Then $2(\varphi(|f|)+\varphi(|g|)) - \varphi(|f+g|) - \varphi(|f-g|) = 0$ μ-a.e. on X. This implies that for almost all $t \in X$ $f(t) = 0$ when $g(t) \neq 0$, and $g(t) = 0$ when $f(t) \neq 0$, i.e.

$$\mu(\operatorname{supp} f \cap \operatorname{supp} g) = 0.$$

LEMMA 2. *Let $L_\varphi(\boldsymbol{R}, \boldsymbol{A}_\lambda, \lambda)$ resp. $L_\varphi([0,1], \boldsymbol{A}_\lambda, \lambda)$ be non-locally bounded and $T \in G(L_\varphi)$.*

Then for every $f \in L_\varphi$

$$T(f)(t) = h(t) f(g(t)),$$

where g and h are λ-measurable functions such that

(i) $g(\boldsymbol{R}) = \boldsymbol{R}$ *resp.* $g([0,1]) = [0,1]$,

(ii) $|h(t)| = 1$,

(iii) $\lambda(g^{-1}(A)) = \lambda(A)$ *for all* $A \in \boldsymbol{A}_\lambda$.

Proof. Using Lemma 1, we easily get a generalization of [1], Theorem 3.8.5, to all L_φ-spaces. Hence, for each $f \in L_\varphi$, T has the form

$$T(f)(t) = h(t) f(g(t)), \tag{1}$$

where g and h are λ-measurable functions, g has property (i), and $g^{-1}(A) = \operatorname{supp} T\chi_A$ for each $A \in \boldsymbol{A}_\lambda$. So we only have to verify properties (ii) and (iii).

Define

$$\psi\colon \boldsymbol{A}_\lambda \to \boldsymbol{A}_\lambda$$

by

$$\psi(A) := \operatorname{supp} T\chi_A.$$

Then $\lambda(\psi(A) \cap \psi(B)) = 0$ if $A \cap B = \emptyset$ (cf. [1]).

As T is onto, we can choose $h(t) \neq 0$ for all t. Further on, for each $A \in \boldsymbol{A}_\lambda$ there is an $f \in L_\varphi$ with

$$T(f) = \chi_A.$$

Now (1) implies $\chi_A(t) = h(t) f(g(t))$ for all t. Let $D := \operatorname{supp} f$.

Then we have

$$\begin{aligned} t \in A &\Leftrightarrow f(g(t)) \neq 0 \\ &\Leftrightarrow g(t) \in \operatorname{supp} f \\ &\Leftrightarrow h(t) \cdot \chi_D(g(t)) \neq 0 \\ &\Leftrightarrow T(\chi_D)(t) \neq 0. \end{aligned}$$

So for each $A \in \boldsymbol{A}_\lambda$ there is a $D \in \boldsymbol{A}_\lambda$ with

$$\psi(D) = A. \tag{2}$$

Suppose now that property (iii) is not true. Then there is a $B \in \boldsymbol{A}_\lambda$ with $\lambda(\psi(B)) \neq \lambda(B)$. Obviously we can assume $\lambda(\psi(B)) \leqslant 1$ and $\lambda(B) \leqslant 1$. Now (1) implies that, for all real $a > 0$, $\|a \cdot \chi_B\| = \|T(a\chi_B)\| = \|h(t) a \chi_{\psi(B)}\|$ or

$$\lambda(B) = \int_{\psi(B)} \frac{\varphi(|ah(t)|)}{\varphi(a)}\,d\lambda. \tag{3}$$

Let $E := \psi(B)$ and for each $i \in \boldsymbol{N}$

$$E_i := \begin{cases} \{t \in E\colon i \leqslant |h(t)| < i+2\} & i \text{ odd}, \\ \left\{t \in E\colon \dfrac{1}{i+1} \leqslant |h(t)| < \dfrac{1}{i-1}\right\} & i \text{ even}. \end{cases}$$

Clearly, $E = \bigcup_{i \in \boldsymbol{N}} E_i$ and $E_i \cap E_j = \emptyset$, $i \neq j$.

(2) implies that for each $i \in \boldsymbol{N}$ there is an $X_i \in \boldsymbol{A}_\lambda$ with $\psi(X_i) = E_i$. By definition of ψ we have $\bigcup_{i \in \boldsymbol{N}} X_i = B$ and $\lambda(X_i \cap X_j) = 0$, $i \neq j$.

As we only consider classes of measurable sets, we can choose X_i in such a way that $X_i \cap X_j = \emptyset$, $i \neq j$. Now $\lambda(\psi(B)) \neq \lambda(B)$ implies that there is an $i \in \boldsymbol{N}$ with $\lambda(\psi(X_i)) \neq \lambda(X_i)$. This leads to a contradiction.

We only consider the case: i odd, as the proof is nearly the same for even i.

(3) implies $\lambda(X_i) \geqslant \lambda(\psi(X_i))$. Suppose there is an $\varepsilon > 0$ such that $\lambda(X_i) = \lambda(\psi(X_i)) + \varepsilon$.

As L_φ is not locally bounded, for every positive real r there is a sequence $\{a_i\}_{i \in \boldsymbol{N}}$ of positive reals such that $\lim a_i = \infty$ or $\lim a_i = 0$, and $\lim \dfrac{\varphi(a_i r)}{\varphi(a_i)} = 1$ (cf. [2], [3]). So there exists an $n_0 \in \boldsymbol{N}$ with $\dfrac{\varphi(a_{n_0}(i+2))}{\varphi(a_{n_0})} \leqslant 1 + \varepsilon/2$.

Then we have

$$\lambda(\psi(X_i))+\varepsilon = \lambda(X_i) = \int_{\psi(X_i)} \frac{\varphi(|a_{n_0}h(t)|)}{\varphi(a_{n_0})}\,d\lambda$$
$$\leqslant \frac{\varphi(a_{n_0}(i+2))}{\varphi(a_{n_0})}\lambda(\psi(X_i)) \leqslant \lambda(\psi(X_i))+\varepsilon/2$$

which is a contradiction.

So altogether we have $\lambda(\psi(A)) = \lambda(A)$ for all $A \in \boldsymbol{A}_\lambda$ which implies (iii).

Let $M := \{t\colon |h(t)| > 1\}$. Without restriction we can suppose $\lambda(M) \leqslant 1$. We showed already that there is an $N \in \boldsymbol{A}_\lambda$ with $\psi(N) = M$ and $\lambda(N) = \lambda(M)$. Then we have $\varphi(1)\cdot\lambda(N) = \|\chi_N\| = \|T(\chi_N)\| = \|h(t)\chi_M(t)\| = \int_M \varphi(|h(t)|)\,d\lambda > \varphi(1)\cdot\lambda(M)$, which implies $\lambda(M) = 0$. Similarly we get $\lambda\{t\colon |h(t)| < 1\} = 0$, which completes the proof.

The proof of the theorem is a trivial consequence of Lemma 2.

References

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Diagonal mappings between sequence spaces

by

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Abstract. Some general results are obtained about r-nuclear, r-integral and r-summing diagonal mappings from one sequence space to another. These are used to give a nearly complete characterization of such mappings from one l^p space to another, extending results of Schwartz and Tong.

1. Introduction. Schwartz ([4] Théorème (XXVI, 4; 1) and [5]) has given a complete account of 0-summing diagonal mappings from one l^p space to another. If a is a sequence, we denote by d_a the linear operator defined by coordinatewise multiplication by a. Recall that $a \in l^{p-}$ if $\sum_{n=1}^{\infty} |a_n|^p(1+\log|a_n^{-1}|) < \infty$. If $q < 2 < p$, we set $\varphi(p,q) = (p^{-1}+q^{-1}-\frac{1}{2})^{-1}$. Schwartz' result is then the following:

THEOREM 1. *d_a is 0-summing from $l^{p'}$ into l^q if and only if the following conditions are satisfied:*

(i) *if* $p = q < 2$, $a \in l^{p-}$;

(ii) *if* $2 \leqslant q < p$, $a \in l^p$;

(iii) *if* $q < 2 < p$, $a \in l^{\varphi(p,q)}$;

and

(iv) *otherwise*, $a \in l^{\min(p,q)}$.

The purpose of this paper is to extend this result to give a nearly complete account of r-summing, r-integral and r-nuclear diagonal nappings from $l^{p'}$ into l^q. For this, we shall need the three following theorems:

THEOREM 2. *Suppose that $1 \leqslant p, q \leqslant 2$, that F is isomorphic to a closed subspace of $L^q(0,1)$ and that E' is isomorphic to a closed subspace of $L^p(0,1)$. Then the following are equivalent:*

(i) *u is 0-summing;*

(ii) *u is r-summing for some $r < p$;*

(iii) *u' is 0-summing;*

(iv) *u' is s-summing for some $s < q$.*

THEOREM 3. *Suppose that $1 \leqslant p \leqslant 2$, and that E' is isomorphic to a closed subspace of $L^p(0,1)$ and that H is a Hilbert space. If $u \in L(E,H)$,*