STUDIA MATHEMATICA, T. LI. (1974)

Riesz products on non-commutative groups

by

JACEK CYGAN (Wrocław)

Abstract. A condition of lacunarity on a subset of a group and a construction of a set satisfying it are given. The following result is then obtained. Suppose that G is a discrete amenable group, X — a subset of G whose elements form a sequence which satisfies this lacunarity condition and, moreover, there is a constant c such that $|Ox| < c$ for all $x \in X$. Then by replacing every $x \in X$ by a suitable $y \in Ox$ we obtain a Sidon set Y in G. Thus examples of Sidon sets in FC groups are obtained.

The aim of this note is to investigate an intriguing notion of a Sidon subset of a non-abelian discrete group introduced by A. Figà-Talamanca.

(1) *If G is an abelian discrete group, then a subset S is called a Sidon set if every bounded function on S is a restriction of a Fourier–Stieltjes transform of a bounded measure on $\hat{G}$.*

Even though for non-abelian group there is no natural dual group $\hat{G}$ the algebra of Fourier–Stieltjes transforms on G was defined by P. Eymard [1] and following the idea of A. Figà-Talamanca we use this notion to define Sidon sets for non-abelian groups in a complete analogy to (1). The definition and the elementary properties of Sidon sets, most of them noticed by A. Figà-Talamanca, are given in Section 1; see also [2].

One of the main tools in constructing Sidon sets in abelian groups are Riesz products. Our main objective here is to test this method for non-commutative groups. It has turned out, however, that it is much better adapted to the commutative case. However, under some restrictive assumptions on a group we are able to use it to produce non-trivial infinite Sidon subsets.

1. Sidon sets. Let G be a discrete group. Let $B(G)$ be the Fourier–Stieltjes algebra of G as defined by P. Eymard in [1]. Let ϱ denote the left regular representation of $L^1(G)$ on $L^2(G)$, i.e.

$$\varrho(f)g = f*g, \quad f \in L^1(G), \quad g \in L^2(G).$$

We write $\|f\|_\varrho$ for the norm of the operator $\varrho(f)$.

We shall use the following theorem of A. Hulanicki (cf. [4]):

A group G is amenable if and only if $\|f\|_{\varrho} = \sup_{T}\|T_f\| = \|f\|_{\Sigma}$ for all hermitian functions f in $L^1(G)$, where the least upper bound is taken over all $*$-representations T of $L^1(G)$.

Definition 1. We say that $E \subset G$ is a *Sidon set* if every bounded function on E is the restriction of a function in $B(G)$.

Theorem 1. *Suppose that E is a subset of an amenable discrete group G; then the following three conditions are equivalent:*

(a) *for every complex valued function c on E with $\|c\|_{\infty} \leqslant 1$ there exists a function $u \in B(G)$ such that $u(g) = c(g)$ for $g \in E$ and $\|u\|_{B(G)} \leqslant M$, where the constant M depends only on E;*

(b) *for every function $d\colon E \to \{\pm 1\}$ there exists a function $u \in B(G)$ such that $\|u\|_{B(G)} \leqslant K$ and*

$$\sup_{g \in E} |u(g) - d(g)| \leqslant 1 - \delta \qquad (\delta > 0), \tag{2}$$

where K and δ are positive constants depending only on E;

(c) *there exists a constant $c > 0$ such that for every function $f \in L^1(G)$ with the support in E we have:*

$$\|f\|_1 \leqslant c\|f\|_{\varrho}.$$

Remark. Condition (a) is only formally stronger than the one in Definition 1. In fact, if E is a Sidon set, then the mapping

$$T\colon B(G) \ni u \mapsto u|_E \in L^{\infty}(E)$$

is a surjective bounded linear operator, so we may apply the open mapping theorem to get the constant M.

Proof of the theorem. (a)$\Rightarrow$(b) is obvious. (b)$\Rightarrow$(c): Let $f \in L^1(G)$ be real with the support in E. Define d on E so that $d = \pm 1$ and $d \cdot f = |f|$. By the hypothesis, there is a function $u \in B(G)$ which satisfies (2).

If $\operatorname{Re} u = \frac{1}{2}(u + \bar{u})$, then $\operatorname{Re} u \in B(G)$, $\|\operatorname{Re} u\|_{B(G)} \leqslant K$ and it also satisfies (2).

We have

$$|f \operatorname{Re} u - |f|| = |f|\,|\operatorname{Re} u - d| \leqslant (1-\delta)|f|$$

hence we have $f \operatorname{Re} u \geqslant |f|\,\delta$ and

$$\delta\|f\|_1 = \sum_{g \in E} |f(g)|\,\delta \leqslant \sum_{g \in E} f(g)(\operatorname{Re} u)(g) \leqslant \|\operatorname{Re} u\|_{B(G)}\|f\|_{\Sigma} \leqslant 2K\|f\|_{\varrho}.$$

If f is not real, put $f_1 = \dfrac{1}{2}(f + \bar{f})$, $f_2 = \dfrac{1}{2i}(f - \bar{f})$; then f_1 and f_2 are real with supports in E, so:

$$\tfrac{1}{2}\|f + \bar{f}\|_1 \leqslant K\delta^{-1}\|f + \bar{f}\|_{\varrho} \leqslant 2K\delta^{-1}\|f\|_{\varrho},$$

similarly

$$\tfrac{1}{2}\|f - \bar{f}\|_1 \leqslant 2K\delta^{-1}\|f\|_{\varrho}$$

and finally

$$\|f\|_1 \leqslant 4K\delta^{-1}\|f\|_{\varrho}.$$

(c)$\Rightarrow$(a): Let $c \in L^{\infty}(E)$ and $\|c\|_{\infty} \leqslant 1$. The set

$$\mathscr{L}_E = \{\varrho(f) \in \mathscr{L}(L^2(G))\colon f \in L^1(G) \text{ and } \operatorname{supp} f \subset E\}$$

is a closed subspace of $\mathscr{L}(L^2(G))$ (norms $\|\cdot\|_{\varrho}$ and $\|\cdot\|_1$ are equivalent because of (c)). The function Φ_c defined on $\mathscr{L}_E$ by the formula

$$\Phi_c(\varrho(f)) = \sum_{g \in E} f(g)c(g)$$

is a linear functional on $\mathscr{L}_E$ and it is bounded:

$$|\Phi_c(\varrho(f))| \leqslant \|f\|_1\|c\|_{\infty} \leqslant c\|f\|_{\varrho}.$$

By the Hahn–Banach theorem it can be extended to a functional Φ on $\mathscr{L}(L^2(G))$ with $\|\Phi\| \leqslant c$. Now we define the function u on G by the formula $u(g) = \Phi(\varrho(\delta_g))$, where $\delta_g(h)$ is equal to 1 for $h = g$ and 0 for $h \neq g$. For $f \in L^1(G)$, $\|f\|_{\Sigma} \leqslant 1$ we have:

$$\Big|\sum_{g \in G} f(g)u(g)\Big| = \Big|\sum_{g \in G} f(g)\Phi(\varrho(\delta_g))\Big| = |\Phi(\varrho(f))| \leqslant \|\Phi\|\,\|f\|_{\varrho} \leqslant c$$

hence:

$$|u(g)| = \Big|\sum_{g \in G} \delta_g(h)u(h)\Big| \leqslant c$$

and

$$\sup\Big\{\Big|\sum_{g \in G} f(g)u(g)\Big|\colon f \in L^1(G) \text{ and } \|f\|_{\Sigma} \leqslant 1\Big\} \leqslant c$$

so $u \in B(G)$ and $\|u\|_{B(G)} \leqslant c$ (see [1], Proposition (2.1), p. 191).

Finally we check that for $g \in E$

$$u(g) = \Phi_c(\varrho(\delta_g)) = \sum_{h \in G} \delta_g(h)c(h) = c(g).$$

2. Riesz products. First we introduce some notations. Let $\boldsymbol{N}$ be the set of natural numbers ordered in the reverse order, i.e. $\boldsymbol{N} = \{\ldots, 2, 1\}$ and let A be a finite subset of $\boldsymbol{N}$. We denote by I_A the set of all sequences $\langle e_l\rangle_{l \in \boldsymbol{N}}$ such that $e_l = 0$ for $l \notin A$ and $e_l \in \{0, 1\}$ for $l \in A$. Let $\underline{i} = \langle e_l\rangle_{l \in \boldsymbol{N}}$ be such a sequence. By $\underline{i}'$ we denote a sequence $\langle e_l'\rangle_{l \in \boldsymbol{N}}$, where $e_l' = 0$ for $l \notin A$ and $e_l' = 1 - e_l$ for $l \in A$. Now let $a = \langle a_l\rangle_{l \in \boldsymbol{N}}$ be a sequence of real numbers.

We write $a^{\underline{i}} = \ldots \cdot a_n^{e_n} \cdot \ldots \cdot a_1^{e_1}$. It is a real number because A is finite. Similarly, if $x = \langle x_l\rangle_{l \in \boldsymbol{N}}$ is a sequence of elements of G we write $x^{\underline{i}} = \ldots x_n^{e_n} \ldots x_1^{e_1} \in G$ and $(x^{\underline{i}})^{-1} = x_1^{-e_1} \ldots x_n^{-e_n} \ldots \in G$. If $\underline{j} = \langle d_l\rangle \in I_A$, then

$a^{\underline{i}+\underline{j}} = \dots a_n^{e_n+d_n}\dots a_1^{e_1+d_1}$. In particular, when $\underline{i}=\underline{j}$ we write $a^{2\underline{i}}$ instead of $a^{\underline{i}+\underline{i}}$.

The following is a generalization to a non-commutative case of the condition: $R_s(X, x) = 1$ for $s\in \boldsymbol{N}$, $x\in X$; $R_s(X, e) = 1$ when $s = 0$ and $R_s(X, e) = 0$ for $s\in \boldsymbol{N}$ (cf. [5], p. 124).

CONDITION (C). *We say that a sequence $x_1, x_2, \dots$ of elements of G satisfies condition* (C) *if no two elements of it are conjugated in G and for every natural N and $\underline{i}, \underline{j}\in I_{\{N,\dots,1\}}$*

$$x^{\underline{i}}(x^{\underline{j}})^{-1}\in \begin{cases} \{e\} \Rightarrow \underline{i}=\underline{j}, \\ Ox_k \Rightarrow \underline{i}=\underline{j} \text{ on } \{N\dots 1\}\setminus\{k\} \text{ and } e_k = 1,\ d_k = 0, \end{cases}$$

where $Ox_k = \{yx_ky^{-1}\in G\colon y\in G\}$.

THEOREM 2. *Suppose that G is a discrete amenable group, X a subset of G whose elements form a sequence which satisfies* (C) *and, moreover, there is a constant c such that $|Ox|\leqslant c$ for all $x\in X$. Then by replacing every $x\in X$ by a suitable $y\in Ox$ we obtain a Sidon set Y in G.*

Proof. For sake of simplicity for a function f in $L^1(G)$ we write

$$f = \sum_{g\in G} f(g)g.$$

Let $X = \{x_1, x_2, \dots\}\subset G$. We consider the products

$$P_N = \prod_{k=N}^{1}(a_ke+b_kx_k)\prod_{l=1}^{N}(a_le+b_lx_l^{-1}), \tag{3}$$

where $\langle a_l\rangle_{l\in \boldsymbol{N}}$ and $\langle b_l\rangle_{l\in \boldsymbol{N}}$ are the sequences of real numbers with $a_l^2+b_l^2 = 1$ for every $l\in \boldsymbol{N}$ and we see that

$$P_N = f*f^*$$

and so P_N is positive-definite.

We call the P_N the Riesz products since if the x's commute

$$P_N = \prod_{k=N}^{1}[a_k^2e+a_kb_k(x_k+x_k^{-1})+b_k^2e] = \prod_{k=N}^{1}\left[e+c_k\left(\frac{x_k+x_k^{-1}}{2}\right)\right],$$

where $c_k = 2a_kb_k$ and so P_N is the Fourier transform of the ordinary Riesz product

$$\prod_{k=N}^{1}(1+c_k\cos x_kt)$$

if $G = \boldsymbol{Z}$ (cf. [6], p. 208, v. I).

We have

$$P_N = \sum_{\underline{i}\in I_{\{N\dots 1\}}} a^{\underline{i}'}b^{\underline{i}}x^{\underline{i}}\sum_{\underline{j}\in I_{\{N\dots 1\}}} a^{\underline{j}'}b^{\underline{j}}(x^{\underline{j}})^{-1} = \sum_{\underline{i},\underline{j}\in I_{\{N\dots 1\}}} a^{\underline{i}'+\underline{j}'}b^{\underline{i}+\underline{j}}x^{\underline{i}}(x^{\underline{j}})^{-1}$$

and so

$$P_N(e) = \sum_{\underline{i},\underline{j}} a^{\underline{i}'+\underline{j}'}b^{\underline{i}+\underline{j}},$$

where the summation is over the $\underline{i}$'s and $\underline{j}$'s from $I_{\{N\dots 1\}}$ such that $x^{\underline{i}}(x^{\underline{j}})^{-1} = e$.

By Condition (C) we have

$$P_N(e) = \sum_{\underline{i}\in I_{\{N\dots 1\}}} a^{2\underline{i}'}b^{2\underline{i}} = \prod_{l\in\{N\dots 1\}}(a_l^2+b_l^2) = 1.$$

Similarly

$$P_N(y_k) = \sum_{\underline{i},\underline{j}} a^{\underline{i}'+\underline{j}'}b^{\underline{i}+\underline{j}} \quad \text{for } y_k\in Ox_k,$$

where the summation is over the $\underline{i}$'s and $\underline{j}$'s from $I_{\{N\dots 1\}}$ such that $x^{\underline{i}}(x^{\underline{j}})^{-1} = y_k$. Thus we have

$$P_N(y_k) = \sum_{\underline{i}_1,\underline{i}_2} a^{2\underline{i}_1'}b^{2\underline{i}_1}\cdot a_kb_ka^{2\underline{i}_2'}b^{2\underline{i}_2},$$

where the summation is over all $\underline{i}_1$'s and $\underline{i}_2$'s such that

$$\underline{i}_1\in I_{\{N\dots k+1\}},\ \underline{i}_2\in I_{\{k-1\dots 1\}} \quad \text{and} \quad x^{\underline{i}_1}x_k(x^{\underline{i}_1})^{-1} = y_k \tag{4}$$

and

$$P_N(y_k) = a_kb_k\sum_{\underline{i}_1} a^{2\underline{i}_1'}b^{2\underline{i}_1}\sum_{\underline{i}_2} a^{2\underline{i}_2'}b^{2\underline{i}_2} = a_kb_k\sum_{\underline{i}_1} a^{2\underline{i}_1'}\cdot b^{2\underline{i}_1}; \tag{5}$$

here $\underline{i}_1, \underline{i}_2$ run over all sequences which satisfy (4). The sum of the values of P_N at the points belonging to the same conjugate class is

$$\sum_{x\in Ox_k} P_N(x) = \sum_{\underline{i},\underline{j}} a^{\underline{i}'+\underline{j}'}b^{\underline{i}+\underline{j}} = \sum_{\underline{i}_1,\underline{i}_2} a^{2\underline{i}_1'}b^{2\underline{i}_1}a_kb_ka^{2\underline{i}_2'}b^{2\underline{i}_2},$$

where the summations are over all $\underline{i},\underline{j}\in I_{\{N\dots 1\}}$ such that $x^{\underline{i}}(x^{\underline{j}})^{-1}\in Ox_k$ and $\underline{i}_1\in I_{\{N\dots k+1\}}$, $\underline{i}_2\in I_{\{k-1\dots 1\}}$. Thus

$$\sum_{x\in Ox_k} P_N(x) = a_kb_k\prod_{l\neq k,\, l=1}^{N}(a_l^2+b_l^2) = a_kb_k.$$

Consider the positive-definite and normalized functions P_N, $N = 1, 2, \dots$, as a subset of the unit ball in the $L^\infty(G)$. It is a precompact set in the weak-star topology of $L^\infty(G)$.

Now we consider the Riesz products as defined in (3) with all the coefficients a_l and b_l equal to $2^{-1/2}$. We denote them by p_N, $N = 1, 2, \dots$

Let $\langle p_{N_l}\rangle$ be a subsequence of $\langle p_N\rangle$ $*$-weak convergent to a positive-definite and normalized function p_0. We denote by S the corresponding subset $\{N_l\}_{l\in N}$ of natural numbers. We have $\sum_{x\in Ox_k} p_{N_l}(x) = \frac{1}{2}$ for every $N_l\in S$ and so $\sum_{x\in Ox_k} p_0(x) = \frac{1}{2}$. Since $p_{N_l}(x)\geqslant 0$ for all $x\in Ox_k$, $N_l\in S$, also $p_0(x)\geqslant 0$ on Ox_k. Consequently there exists $y_k\in Ox_k$ such that $p_0(y_k)\geqslant \dfrac{1}{2c}$.

Now we are going to check that the set $Y = \{y_k\}_{k\in N}\subset G$ satisfies condition (b) of the Theorem 1. Let $d\colon E\to\{-1,1\}$. We put $a_k = d(y_k)2^{-1/2}$ and $b_k = 2^{-1/2}$ and we construct the Riesz products (3) with such $\langle a_k\rangle$ and $\langle b_k\rangle$ and denote them by P_N^d. Notice that (5) implies that

$$P_N^d(y_k) = d(y_k)p_N(y_k) \quad \text{for } k, N\in N.$$

Now we consider the functions $P_{N_l}^d$ with $N_l\in S$. For the $*$-weak limit point P_0^d of $\langle P_{N_l}^d\rangle$ thus we have

$$P_0^d(y_k) = d(y_k)p_0(y_k)$$

since $p_{N_l}(y_k)$ tends to $p_0(y_k)$ as l tends to ∞. We see that condition (b) of Theorem 1 is satisfied with $K=1$ and $\delta = \dfrac{1}{2c}$.

3. Construction of a set. Now we propose an inductive procedure which in some classes of non-abelian groups leads to a set $X = \{x_l\}_{l\in N}$ which satisfies Condition (C). We do not specify the precise conditions on a group which guarantee that the set thus obtained is infinite. In the abelian case this procedure is similar to the construction of a dissociate set (cf. [3], p. 400).

Construction 1. 1) Select $x_1\in G$ such that $e\notin x_1\cdot Ox_1$ or equivalently $Ox_1\cap Ox_1^{-1} = \emptyset$. If this is not possible the set X is void.

2) Assume that $x_1,\ldots,x_N$ have been selected. Let

$$A_N = \{x^{\underline{i}}(x^{\underline{j}})^{-1}\in G\colon \underline{i},\underline{j}\in I_{\{N\ldots1\}}\}$$

and

$$B_N = O_{A_N}\cup Ox_1\cdot A_N\cup Ox_1^{-1}\cdot A_N\cup\ldots\cup Ox_N\cdot A_N\cup Ox_N^{-1}\cdot A_N,$$

where $O_A = \bigcup_{a\in A} Oa$ and $A\cdot B = \{ab\in G\colon a\in A,\, b\in B\}$ for $A, B\subset G$. We select x_{N+1} in $G\setminus B_N$ in a way such that

$$A_N\cap x_{N+1}^{-1}\cdot Ox_{N+1} = \{e\} \tag{6}$$

and

$$A_N\cap x_{N+1}\cdot Ox_{N+1} = \emptyset. \tag{7}$$

(If this is not possible, then $X = \{x_1,\ldots,x_N\}$.)

Proposition 1. *The set X obtained from the construction above satisfies Condition* (C).

Proof. 1° Let $x^{\underline{i}}(x^{\underline{j}})^{-1} = e$, $\underline{i},\underline{j}\in I_{\{N\ldots1\}}$ and let $n\in\{N\ldots1\}$ be the greatest number such that $e_n = \underline{i}|_n = 1$ or $d_n = \underline{j}|_n = 1$, where $\underline{i} = \langle e_n\rangle$, $\underline{j} = \langle d_n\rangle$. There are then three possibilities:

$$e = x_n a,\qquad e = bx_n^{-1},\qquad e = x_n c x_n^{-1}$$

with $x_n\in X$, $a, b, c\in A_{n-1}$ (here $A_0 = \{e\}$) and only the third one is possible because x_n was chosen from $G\setminus A_{n-1}$. We obtain $\underline{i}|_m = \underline{j}|_m$ for $N\geqslant m\geqslant n$ and $x^{\underline{i}_1}(x^{\underline{j}_1})^{-1} = c = e$, $\underline{i}_1,\underline{j}_1\in I_{\{n-1\ldots1\}}$ and we apply the same reasoning to $N = n-1$ and finally get $\underline{i} = \underline{j}$.

2° Let $x^{\underline{i}}(x^{\underline{j}})^{-1}\in Ox_k$, $\underline{i},\underline{j}\in I_{\{N\ldots1\}}$ and let n be such as in 1°. The case $k>n$ is impossible because $x_k\notin O_{A_n}$ implies $O_{x_k}\cap A_n = \emptyset$. If $k = n$ we have three possibilities:

$$x_n a\in Ox_n,\quad bx_n^{-1}\in Ox_n,\quad x_n c x_n^{-1}\in Ox_n \quad \text{with } a, b, c\in A_{n-1},$$

that is,

$$a\in x_n^{-1}\cdot Ox_n,\qquad b\in x_n\cdot Ox_n,\qquad c\in Ox_n$$

and conditions (6), (7) and $x_n\in G\setminus O_{A_{n-1}}$ imply that only the first case is possible with $a = e$, that is, $\underline{i}|_n = 1$, $\underline{j}|_n = 0$, $\underline{i} = \underline{j}$ on $\{N\ldots n+1\}$ and $x^{\underline{i}_1}(x^{\underline{j}_1})^{-1} = a = e$ $\underline{i}_1,\underline{j}_1\in I_{\{n-1\ldots1\}}$. We apply 1° to get $\underline{i} = \underline{j}$ on $\{n-1,\ldots,1\}$.

If $k<n$, then $x_n\cdot a\in Ox_k$, $bx_n^{-1}\in Ox_k$ or $x_n c x_n^{-1}\in Ox_k$, that is, $x_n\in Ox_k\cdot A_{n-1}$, $x_n\in Ox_k^{-1}\cdot A_{n-1}$, $c\in Ox_k$ so the only possible is the third situation and we get $\underline{i} = \underline{j}$ on $\{N\ldots n\}$ and $x^{\underline{i}_1}(x^{\underline{j}_1})^{-1}\in Ox_k$, $\underline{i}_1,\underline{j}_1\in I_{\{n-1\ldots1\}}$ and we apply 2° again.

Choosing x_{N+1} outside O_{A_N} yields that no two elements of X are conjugated in G.

Example 1. Suppose that the set $[G, G] = \{ghg^{-1}h^{-1}\in G\colon g\in G,\, h\in G\}$ is finite and the set $S = \{g^2\colon g\in G\}$ is not. Then there is an element $x_1\in G\setminus[G,G]$ with $x_1^2\notin[G,G]$. Assume we have chosen $x_1,\ldots,x_N$, and select $x_{N+1}\in G$ such that neither x_{N+1} nor x_{N+1}^2 belong to the finite set $B_N\cup[G,G]\cdot A_N$. We shall show that the conditions (6) and (7) above are satisfied. If (6) does not hold then there are $\underline{i},\underline{j}\in I_{\{N\ldots1\}}$ such that $\underline{i}\neq\underline{j}$ and

$$x^{\underline{i}}(x^{\underline{j}})^{-1}\in x_{N+1}^{-1}\cdot Ox_{N+1} = [x_{N+1}^{-1}, G]\subset[G,G].$$

Let n be the greatest number $n\in\{N\ldots1\}$ such that $\underline{i}|_n\neq\underline{j}|_n$. We have

$$x^{\underline{i}}(x^{\underline{j}})^{-1} = cx_n\cdot ac^{-1}\quad\text{or}\quad x^{\underline{i}}(x^{\underline{j}})^{-1} = cbx_n^{-1}c^{-1},\quad a, b\in A_{n-1}$$

hence $x_n\in[G,G]\cdot A_{n-1}$ which is a contradiction. If (7) is not satisfied, then there is $a\in A_N$ which also belongs to the $x_{N+1}\cdot Ox_{N+1} = Ox_{N+1}\cdot x_{N+1} = [G, X_{N+1}]\cdot x_{N+1}^2$ and consequently $x_{N+1}^2\in[G,G]\cdot A_N$ which is not true.

From Proposition 1 we conclude that the set $\{x_1, \dots, x_N, \dots\} = X$ satisfies Condition (C). Obviously $|[G, G]| = c < \infty$ implies $|Ox| \leqslant c$ for every $x \in G$ and we can apply Theorem 2 to the set X. Of course, being every FC group, that is, a group with finite classes of conjugated elements, G is amenable.

Remark. When the commutator subgroup G' of the group G is finite and the set S in the example above is infinite, we can perform the selection of the set X less carefully. That is take x_1 from $G \setminus G'$ with $x_1^2 \notin G'$ and x_{N+1} such that neither x_{N+1} nor x_{N+1}^2 belong to the set $G' \cdot B_N$. None of the two elements of the set X obtained in that way are congruent modulo G' and it is not hard to see that the subset $\{\bar{x}_1, \dots, \bar{x}_N, \dots\}$ of the abelian group G/G', where $\bar{x}_N = x_N G'$, is a Sidon set in G/G'.

EXAMPLE 2. Let G be a finite group and $g \in G$ such that

$$Og \cap Og^{-1} = \emptyset. \tag{8}$$

We consider the direct product

$$\prod_{n=1}^{\infty} G_n, \quad \text{where } G_n = G \text{ for } n = 1, 2, \dots$$

and its subset $X = \{x_1, x_2, \dots\}$, where every x_N is of the form $(e, \dots, h_N, e, \dots \dots, g, e, \dots)$ with g on the $N+1$-th axis, h_N on $2^p - l$-th axis, where p and l are such that $N = 2^p + l$, $0 \leqslant l < 2^p$, and e elsewhere. $e \neq h_N \in G$ is arbitrary. We are going to check that the selection of X agree with Construction 1. Because of (8) we have $Ox_1 \cap Ox_1^{-1} = \emptyset$ and $A_N \cap X_{N+1} \cdot Ox_{N+1} = \emptyset$. Clearly $x_{N+1} \notin B_N$. If $a \in A_N \cap x_{N+1}^{-1} \cdot Ox_{N+1}$, then it may differ from e only on the one axis and, by the following lemma, $a = e$.

LEMMA. *If $x^{\underline{i}}(x^{\underline{j}})^{-1} = (e, \dots s, e, \dots)$ for $\underline{i}, \underline{j} \in I_{\{N \dots 1\}}$ with s on the n-th axis, $n \leqslant N+1$, then $s = e$.*

Proof. Induction. For $N = 1$ it is obvious since $x_1 = (h_1, g, e, \dots)$ with $h_1 \neq e$. Suppose that lemma is true for $N-1$. For N we have:

$$x^{\underline{i}}(x^{\underline{j}})^{-1} = x_N^{e_N} x^{\underline{i}_1}(x^{\underline{j}_1})^{-1} x_N^{-d_N} = (e, \dots s, e, \dots)$$

here $\underline{i}, \underline{j} \in I_{\{N \dots 1\}}$; $\underline{i}_1, \underline{j}_1 \in I_{\{N-1, \dots 1\}}$. If s is on the $N+1$-th axis we put

$$x^{\underline{i}_1}(x^{\underline{j}_1})^{-1} = (e, \dots, s_1, e, \dots) \quad \text{with } s_1 \text{ on the } 2^p - l\text{-th axis} \tag{9}$$

and by induction hypothesis we have $s_1 = e$, so $x_N^{e_N}$ is equal to $x_N^{d_N}$ on the $2^p - l$-th axis, hence $e_N = d_N$ $(h_N \neq e)$ and $s = e$. If s is on the n-th axis with $n < N+1$, then $x_N^{e_N}$ is equal to $x_N^{d_N}$ on the $N+1$-th axis, so $e_N = d_N$ and we use the induction hypothesis to (9) with s_1 on n-th axis.

Since the direct product of finite groups is FC group thus amenable and $|Ox| < |G|^2$ for $x \in X$, we may obtain a Sidon set Y from X by Theorem 2.

Choosing a suitable sequence $\langle h_n \rangle$, we get "more or less commutative" set Y. For example, if we put $h_n = h$ for all n and add the condition $[h, Og] \not\ni e$, then the set Y will have a property: for every $y_0 \in Y$ there are infinitely many y's from Y which do not commute with y_0.

References

[1] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), pp. 181–236.

[2] A. Figà-Talamanca et M. Picardello, *Les multiplicateurs de $A(G)$ qui ne sont pas dans $B(G)$*, C. R. Acad. Sci. Paris, Sér. A, 277 (1973), pp. 117–119.

[3] E. Hewitt and H. S. Zuckerman, *Singular measures with absolutely continuous convolution squares*, Proc. Cambridge Philos. Soc. (1966), pp. 399–420.

[4] A. Hulanicki, *Groups whose regular representation weakly contains all unitary representations*, Studia Math. 24 (1964), pp. 37–59.

[5] W. Rudin, *Fourier analysis on groups*, 1962.

[6] A. Zygmund, *Trigonometric series*, 1968.

INSTYTUT MATEMATYCZNY
POLSKIEJ AKADEMII NAUK
WROCŁAW
MATHEMATICAL INSTITUTE
POLISH ACADEMY OF SCIENCES
WROCŁAW

Received May 30, 1973 (693)