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On the potential operators associated with a semigroup

by

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Abstract. We discuss two different ways of defining the potential operator for an equicontinuous semigroup of class (C_0).

0. In [3] Yosida studied the potential operator $(V, D(V))$ (defined below) of an equicontinuous semigroup of class (C_0) on a locally convex, sequentially complete, Hausdorff topological vector space. We shall here discuss another potential operator $(N, D(N))$, which is in general a restriction of $(V, D(V))$, but which is equal to V whenever it is densely defined.

1. Let E be a locally convex, sequentially complete, Hausdorff topological vector space, and let $(P_t)_{t\geqslant 0}$ be an equicontinuous semigroup of class (C_0) (cf. [2]). The infinitesimal generator and the resolvent are denoted respectively $(A, D(A))$ and $(V_\lambda)_{\lambda>0}$.

The two potential operators $(N, D(N))$ and $(V, D(V))$ are defined by

$$Nf = \lim_{t\to\infty}\int_0^t P_s f\,ds, \qquad Vf = \lim_{\lambda\to 0} V_\lambda f,$$

and their domains $D(N)$ and $D(V)$ are the set of elements f in E for which the limits under consideration exist.

We first remark that $N \subseteq V$, for if $f \in D(N)$ we find by partial integration that

$$V_\lambda f = \int_0^\infty e^{-\lambda s} P_s f\,ds = \lambda \int_0^\infty e^{-\lambda s}\varphi(f, s)\,ds,$$

where

$$\varphi(f, s) = \int_0^s P_u f\,du,$$

thus $\lim_{\lambda\to 0} V_\lambda f = Nf$.

As usual we denote the range of an operator S by $R(S)$.

LEMMA. (i) *The potential operator N is injective, $R(N) \subseteq D(A)$, and $A(Nf) = -f$ for all $f \in D(N)$.*

(ii) $\overline{R(N)} \supseteq D(N)$.

(iii) *We have* $\lim_{t\to\infty} P_t f = 0$ *for all* $f \in \overline{R(N)}$.

THEOREM. *The following conditions concerning the semigroup* $(P_t)_{t\geq 0}$ *are equivalent*:

(a) $D(N)$ *is dense in* E.

(b) $R(N)$ *is dense in* E.

(c) $\lim_{t\to\infty} P_t f = 0$ *for all* $f \in E$.

If conditions (a)–(c) *are verified,* A *is injective and* $N = V = -A^{-1}$.

Proof of lemma. For $f \in D(N)$ we have $P_t f \in D(N)$ and $N(P_t f) = P_t(Nf)$. Furthermore we have the formula

$$P_t(Nf) - Nf = -\int_0^t P_s f ds, \quad f \in D(N). \tag{1}$$

By (1) follows that

$$\lim_{t\to 0} \frac{1}{t}(P_t(Nf) - Nf) = -f$$

for any $f \in D(N)$, so (i) is proved.

By (1) follows also that

$$\lim_{t\to 0} N\left(\frac{1}{t}(f - P_t f)\right) = f,$$

and this implies (ii).

Finally by (1) follows that $\lim_{t\to\infty} P_t f = 0$ for all $f \in R(N)$, and this extends to all $f \in \overline{R(N)}$ as a consequence of the equicontinuity property of the semigroup.

Proof of Theorem. Suppose that condition (c) is verified. Then the following condition ((5) of [3]) is verified:

$$\lim_{\lambda\to 0} \lambda V_\lambda f = 0 \quad \text{for all } f \in E, \tag{2}$$

but this condition implies that $R(A)$ is dense in E (cf. [3]).

If we let $t\to\infty$ in the formula

$$P_t f - f = \int_0^t P_s(Af) ds,$$

valid for $f \in D(A)$, we get from (c) that $Af \in D(N)$ and that $N(Af) = -f$. This together with (i) of the lemma prove that $D(N) = R(A)$ and that $N = -A^{-1}$, and we have proved that (c)$\Rightarrow$(a).

If conditions (a)–(c) are verified, we have already proved that $N = -A^{-1}$, and since (2) is verified, we know from the result in [3] that $V = -A^{-1}$.

2. The following special case was studied in [1].

Let G be a locally compact abelian group, and let $E = C_0(G)$ be the Banach space of the continuous functions vanishing at infinity on G. As semigroups we consider convolution semigroups, $P_t f = \mu_t * f$, where $(\mu_t)_{t>0}$ is a family of positive measures on G satisfying

$$\int d\mu_t \leqslant 1, \quad \mu_t * \mu_s = \mu_{t+s}, \ \lim_{t\to 0} \mu_t = \varepsilon_0 \ \text{(vaguely)}.$$

The Fourier transform of μ_t has the form

$$\hat{\mu}_t(\gamma) = e^{-t\psi(\gamma)}, \quad t > 0, \gamma \in \hat{G},$$

where $\psi: \hat{G} \to C$ is the so-called negative definite function associated with $(\mu_t)_{t>0}$. It was proved that conditions (a)–(c) of the theorem are verified if and only if $\operatorname{Re}\psi > 0$ locally almost everywhere with respect to Haar measure on $\hat{G}$. Furthermore $D(V)$ is dense in $C_0(G)$ if and only if $\psi \neq 0$ locally almost everywhere with respect to Haar measure on $\hat{G}$ (cf. [1]).

The analogous results hold if we consider the same convolution semigroups on $L^2(G)$ instead of $C_0(G)$.

References

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[3] — *The existence of the potential operator associated with an equicontinuous semigroup of class* (C_0), Studia Math. 31 (1968), pp. 531–533.

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