PWN (Polish Scientific Publishers). However, most of his heart and energy went into the editorial work of Polish mathematical journals and books. He was an active member of many editorial boards, such as *Acta Arithmetica* (secretary), *Wiadomości Matematyczne* (editor), *Studia Mathematica* (secretary), *Matematyka Stosowana* (editor), *Dissertationes Mathematicae*, and the series of advanced textbooks *Biblioteka Matematyczna* (Chairman).

At least as important as his formal activities were his other contributions to the mathematical community. Having no family, he put all his time, energy and knowledge at the service of Polish mathematics. He was irreplaceable as a source of information of any kind, and he encouraged many young people to write monographs, textbooks and articles.

Marceli Stark did very much for Studia Mathematica. He became secretary of the Editorial Board soon after the war, and supervised the publication of all postwar volumes (begining with Volume 10 published in 1948) until his death in 1974. The present state of our journal is due in great part to the contributions of Marceli Stark.

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Best possible bounds of the norms of inverses adjoined to normed algebras

by

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Abstract. The note is a sequel to [4], where the possibility of adjoining inverses to a commutative normed algebra was investigated. Here we examine the problem of how small norms one can choose for the inverses. In a certain sense the following result gives a complete solution to this problem. Let $n \geqslant 1$ and let $A_1, \ldots, A_n$, $d_1, \ldots, d_n$, $B_1, B_2, \ldots, B_n$ be positive numbers such that $0 < d_i < 1$ and $A_i d_i B_i > 1$, $i = 1, \ldots, n$. Then the following two conditions are equivalent.

(i) If $\mathscr{A}$ is any commutative normed algebra and $a_1, \ldots, a_n \in \mathscr{A}$ are such that $|a_i| \leqslant A_i$ and $|a_i x| \geqslant d_i |x|$ for all $x \in \mathscr{A}$, then there is an extension $\mathscr{B}$ of $\mathscr{A}$ such that $a_i^{-1} \in \mathscr{B}$, $|a_i^{-1}| \leqslant B_i$, $i = 1, \ldots, n$.

(ii) $\displaystyle\sum_{i=1}^{n} \frac{\log d_i}{\log A_i B_i} \geqslant -1.$

We also deduce some consequences of this result.

1. Introduction. All algebras appearing in this note are *commutative unital normed algebras* and the norm is always denoted by $|\cdot|$. The problem we discuss in this note is the following. Given a set of elements of an algebra, when can one adjoin the inverses of all of them to the algebra and how small can the norms be chosen?

Shilov's classical result [7] states that the inverse of an element $a \in \mathscr{A}$ can be adjoined to the algebra iff a is not a topological zero-divisor, i.e. if $\inf\{|ax|: x \in \mathscr{A},\ |x| = 1\} > 0$. Consequently throughout the note we shall be interested in sets consisting of elements which are not topological divisors of zero. To get a somewhat firmer grip on these elements, let us introduce the following terminology. If $\mathscr{A}$ is an algebra, denote by $I(\mathscr{A})$ the set of *norm-increasing* elements in $\mathscr{A}$, i.e.

$$I(\mathscr{A}) = \{a: a \in \mathscr{A},\ |ax| \geqslant |x| \text{ for all } x \in \mathscr{A}\}.$$

Clearly an element is a *topological divisor of zero* iff no scalar multiple of it is norm-increasing. To measure how far an element $a \in \mathscr{A}$ is from being a topological zero-divisor, introduce the number

$$d(a) = d_{\mathscr{A}}(a) = \inf\left\{\frac{|ax|}{|a|\,|x|}: x \in \mathscr{A},\ x \neq 0\right\}.$$

Thus $0 \leqslant d(a) \leqslant 1$, a is a topological zero-divisor iff $d(a) = 0$ and d is norm-increasing iff $|a|\, d(a) \geqslant 1$.

Arens [2] showed that if $a \in \mathscr{A}$, then there is an extension $\mathscr{B}$ of $\mathscr{A}$ in which $|a^{-1}| \leqslant B$ if and only if $(|a|\, d(a))^{-1} \leqslant B$. In other words, a^{-1} can be chosen to have norm at most 1 in some extension of the algebra iff a is norm-increasing. Thus the following would be the ideal situation. "If $S \subset \mathscr{A}$ is such that the function d is positive on S, then there is an extension $\mathscr{B}$ of $\mathscr{A}$ such that $a^{-1} \in \mathscr{B}$ and $|a^{-1}| = (|a|\, d(a))^{-1}$ for all $a \in S$".

In view of the result just mentioned this would hold if, given $a, b \in \mathscr{A}$, one could find an extension $\mathscr{B}$ of $\mathscr{A}$ in which $|a^{-1}| = (|a|\, d(a))^{-1}$ and $d_{\mathscr{A}}(b) = d_{\mathscr{B}}(b)$, i.e. $d(b)$ does not decrease when $\mathscr{A}$ is replaced by the bigger algebra $\mathscr{B}$. This ideal situation was expected by Arens [2], but in [4] it was shown to be over-optimistic. On the other hand, it was also shown in [4] that if $a_1, \ldots, a_n$ are norm-increasing, $|d_i| = A_i$, and for every i, $1 \leqslant i \leqslant n$, B_i is bounded by a certain function of $A_1, \ldots, A_i$, then there is an extension in which $|a_i^{-1}| \leqslant B_i$, $1 \leqslant i \leqslant n$. The existence of these functions enabled the author to prove in [4] that if S is a *countable* set consisting of elements which are not topological zero-divisors, then in an extension of the algebra *every* element of S is invertible. This result indicates that if the norms of $a_1, \ldots, a_n \in \mathscr{A}$ are given then the *larger* the $d(a_i)$'s are, the *smaller* norms of the inverses a_i^{-1} we can *guarantee* in some extension. The aim of this note is to make this statement as accurate as possible.

Call a set of three sequences $A_1, \ldots, A_n$, $d_1, \ldots, d_n$, $B_1, \ldots, B_n$ *accessible* if it has the following property:

if $\mathscr{A}$ is any algebra, $a_1, \ldots, a_n \in \mathscr{A}$, $|a_i| = A_i$ and $d(a_i) = d_i$, then there is an extension of $\mathscr{A}$ in which each a_i is invertible and $|a_i^{-1}| \leqslant B_i$, $1 \leqslant i \leqslant n$.

As the main result of this note we will determine all accessible sets.

Arens's expectation was supported by his result that if $a \in \mathscr{A}$ and $d(a) = 1$, then there is an extension $\mathscr{B}$ of $\mathscr{A}$, which contains a^{-1} with minimal norm ($|a^{-1}| = |a|^{-1}$) and $d_{\mathscr{A}}(b) = d_{\mathscr{B}}(b)$ for all $b \in \mathscr{A}$. This shows that when investigating accessible sets we can confine ourselves without loss of generality to the case $d_i < 1$ for all i. (In fact, one can consider this case first and the general case, including this result of Arens, follows immediately.)

As a simple case of a corollary of the main result, we obtain that there exist an algebra $\mathscr{A}$, elements $a_1, a_2 \in \mathscr{A}$ and numbers B_1, B_2 such that for each i, $i = 1, 2$, there is an extension $\mathscr{B}_i$ of $\mathscr{A}$ that contains a_j^{-1} $(j \neq i)$ with $|a_j^{-1}| \leqslant B_j$ but if $\mathscr{B}_i$ is *any* extension with this property, then a_j is a topological zero-divisor. Thus the situation is considerably worse than it used to be expected: if one is too greedy with the norm of a_1^{-1} and requires it to be minimal, then not only must $d(a_2)$ decrease but it must become zero.

Some results and problems related to the ones discussed here can be found in [1], [3], [5], [6] and [8].

All the results of this note are closely connected to the results in [4]. In fact, this note is a natural continuation of [4]; by taking more care we sharpen the results of [4] and by showing that the bounds we obtain are best possible we answer some natural questions.

2. The main result. As we remarked, when investigating whether a set $A_1, \ldots, A_n$, $d_1, \ldots, d_n$, $B_1, \ldots, B_n$ is accessible or not it suffices to examine the sets satisfying $0 < d_i < 1$. Furthermore, it also follows from our remarks that if there is an index i such that $A_i d_i B_i < 1$, then this set cannot be accessible. Thus one can suppose without loss of generality that $A_i d_i B_i \geqslant 1$ for all i. Our main result is as follows.

THEOREM. *A set $A_1, \ldots, A_n$, $d_1, \ldots, d_n$, $B_1, \ldots, B_n$, where $0 < d_i < 1$ and $A_i d_i B_i \geqslant 1$ for all i, is accessible if and only if*

$$-\sum_{i=1}^{n} \frac{\log d_i}{\log A_i B_i} \leqslant 1.$$

In order to simplify the notations, let us formulate this result in terms of norm-increasing elements. Furthermore, as the necessity and sufficiency will be proved independently, we state the two parts separately.

THEOREM 1. *Suppose $\mathscr{A}$ is a normed algebra, $a_1, \ldots, a_n \in I(\mathscr{A})$, $|a_i| \leqslant A_i > 1$ and $B_1, \ldots, B_n$ $(1 \leqslant B_i)$ satisfy*

$$\sum_{i=1}^{n} \frac{\log A_i}{\log A_i B_i} \leqslant 1.$$

Then there is an extension $\mathscr{B}$ of $\mathscr{A}$ such that $a_i^{-1} \in \mathscr{B}$ and $|a_i^{-1}| \leqslant B_i$ for all i, $1 \leqslant i \leqslant n$.

THEOREM 2. *Suppose A_i, B_i $(1 \leqslant i \leqslant n)$, $1 < A_i$, $B_i \geqslant 1$, satisfy*

$$\sum_{i=1}^{n} \frac{\log A_i}{\log A_i B_i} > 1.$$

Then there is an algebra $\mathscr{A}$ containing elements $a_1, \ldots, a_n \in I(\mathscr{A})$ with the property that $|a_i| = A_i$ and no extension $\mathscr{B}$ of $\mathscr{A}$ is such that $a_i^{-1} \in \mathscr{B}$ and $|a_i^{-1}| \leqslant B_i$ for all i.

Theorem 1 is a somewhat clearer form of [4], Theorem 3.4, whose proof is indicated in [4]. (When writing that note the author did not expect [4], Theorem 3.4 to be best possible.) For the sake of completeness we shall give a proof here. This proof is formulated in a slightly different way from the one indicated in [4]. Before turning to the proofs of the main results we state and prove a simple technical lemma which allows us to replace the inequalities of the theorems by more readily applicable conditions.

LEMMA. *Suppose* $A_i > 1$ *and* $B_i \geqslant 1$, $i = 1, 2, \ldots, n$. *Then*

$$\sum_{i=1}^{n} \frac{\log A_i}{\log A_i B_i} \leqslant 1$$

if and only if whenever $k_1, \ldots, k_n$ *are non-negative integers, there is an index* i *for which*

$$(A_i B_i)^{k_i} \geqslant \prod_{j=1}^{n} A_j^{k_j}. \tag{2}$$

Proof. Suppose (1) holds and $k_1, \ldots, k_n$ are non-negative integers, not all zero. Put $K = \sum_{j=1}^{n} k_j \log A_j$. Then we cannot have

$$k_j < \frac{K}{\log A_j B_j} \quad \text{for all } j,$$

since that would imply

$$K = \sum_{j=1}^{n} k_j \log A_j < \sum_{j=1}^{n} \frac{K \log A_j}{\log A_j B_j} \leqslant K.$$

Consequently, there is an index i, $1 \leqslant i \leqslant n$, for which $k_i \log A_i B_i \geqslant K$ and so (2) holds.

Suppose now that whenever $k_1, \ldots, k_n$ are non-negative integers, (2) holds for an index i, i.e.

$$k_i \log A_i B_i \geqslant \sum_{j=1}^{n} k_j \log A_j.$$

This implies that whenever $\alpha_1, \ldots, \alpha_n$ are non-negative real numbers, there is an index i for which

$$\alpha_i \log A_i B_i \geqslant \sum_{j=1}^{n} \alpha_j \log A_j.$$

In particular, choosing $\alpha_j = (\log A_j B_j)^{-1}$, one obtains that (1) holds.

3. Proof of Theorem 1. Let us introduce first some additional terminology. In what follows *n-tuple* always means an n-tuple of non-negative integers. If η is an n-tuple, $\eta = 0$ means that $\eta = (0, \ldots, 0)$. If $\varphi = (f_1, \ldots, f_n)$ and $\gamma = (g_1, \ldots, g_n)$ are n-tuples such that $g_k \leqslant f_k$ for all k, *define* $\varphi - \gamma$ *by* $\varphi - \gamma = (f_1 - g_1, \ldots, f_n - g_n)$. If $\gamma = (g_1, \ldots, g_n)$ and $\varphi = (f_1, \ldots, f_n)$ are arbitrary k-tuples, put $\gamma + \varphi = (g_1 + f_1, \ldots, g_n + f_n)$. If $1 \leqslant i \leqslant n$, denote by ε_i the n-tuple whose only non-zero term is at the ith place and it is 1. If k is a non-negative integer, denote by $k\varepsilon_i$ the n-tuple whose only non-zero term is at the ith place and it is k. If $\gamma = (g_1, \ldots, g_n)$, we write $a^\gamma = \prod_1^n a_i^{g_i}$, $b^\gamma = \prod_1^n b_i^{g_i}$, $A^\gamma = \prod_1^n A_i^{g_i}$ and $B^\gamma = \prod_1^n B_i^{g_i}$.

Let us turn now to the proof itself. Let $\mathscr{A}[b]$ be the commutative polynomial algebra (not normed!) over $\mathscr{A}$ in the variables $b_1, \ldots, b_n$, and let I be the ideal in $\mathscr{A}[b]$ generated by the elements $a_1 b_1 - 1$, $a_2 b_2 - 1, \ldots, a_n b_n - 1$. If $p(b) = \sum_{\gamma \in \Gamma} c_\gamma b^\gamma$, where Γ is a finite set of n-tuples and $c_\gamma \in \mathscr{A}$ for $\gamma \in \Gamma$, put

$$t(p) = \sum_{\gamma \in \Gamma} |c_\gamma| B^\gamma.$$

Denote by $\mathscr{B}$ the commutative normed algebra $\mathscr{A}[b]/I$ with norm

$$|p + I|' = \inf\{t(q) \colon q \in p + I\}.$$

It is obvious (see [2] and [4]) that if there exists an algebra $\mathscr{B}$ required by the theorem, then this algebra $\mathscr{B}$ we have just defined is one of them. (In fact, this is the extremal algebra with the given properties.) Thus, to complete the proof, we have to show only that if $c \in \mathscr{A}$, then $|c| = |c|'$. In other words, *supposing* that

$$c = \sum_{\gamma \in \Gamma} b^\gamma \tag{3}$$

holds in $\mathscr{B}$, *we hawe to show that*

$$|c| \leqslant \sum_{\gamma \in \Gamma} |c_\gamma| B^\gamma. \tag{4}$$

Let δ be an n-tuple for which $\delta \geqslant \gamma$ if $\gamma \in \Gamma$. By multiplying (3) by a^δ we obtain that (3) holds iff

$$c a^\delta = \sum_{\gamma \leqslant \delta} c_\gamma a^{\delta - \gamma} \tag{5}$$

holds in $\mathscr{A}$.

Construct a sequence $\gamma(0) = \delta, \gamma(1), \gamma(2), \ldots$ inductively as follows. Suppose $\gamma(k) = (k_1, \ldots, k_n)$ has been defined, it is an n-tuple $\gamma(k) \neq 0$. By the lemma there is an index i such that

$$B_i^{k_i} \geqslant \prod_{j \neq i} A_j^{k_j}. \tag{6}$$

Clearly, $k_i \geqslant 1$ since $A_j > 1$ for all j. Put $\gamma(k+1) = \gamma(k) - \varepsilon_i$. Thus we can obtain a sequence $\gamma(0), \gamma(1), \ldots, \gamma(p) = 0$.

Note that if $\gamma \leqslant \gamma(k)$ and $\gamma \nleqslant \gamma(k+1)$, then

$$A^{\gamma(k) - \gamma} \leqslant A^{\gamma(k) - k_i \varepsilon_i} \leqslant \prod_{j \neq i} A_j^{k_j}$$

and

$$B^\gamma \geqslant B_i^{k_i}.$$

Hence (6) implies that

(7) $$A^{\gamma(k)-\gamma} \leqslant B^\gamma.$$

As a_i is norm-increasing for each i, so is a^γ for each n-tuple γ. Consequently we have the following string of inequalities:

$$\begin{aligned}
|c| &\leqslant |c-c_{\gamma(p)}|+|c_{\gamma(p)}| \\
&\leqslant \Big|ca^{\gamma(p-1)} - \sum_{\gamma\leqslant\gamma(p-1)} c_\gamma a^{\gamma(p-1)-\gamma}\Big| + |c_{\gamma(p)}| + |c_{\gamma(p-1)}| \\
&\leqslant \Big|ca^{\gamma(p-2)} - \sum_{\gamma\leqslant\gamma(p-2)} c_\gamma a^{\gamma(p-2)-\gamma}\Big| + |c_{\gamma(p)}| + |c_{\gamma(p-1)}| + \sum_{\substack{\gamma\leqslant\gamma(p-2)\\ \gamma\,\mathrm{non}\leqslant\gamma(p-1)}} |c_\gamma a^{\gamma(p-2)-\gamma}| \\
&\leqslant \ldots \leqslant \Big|ca^{\gamma(l)} - \sum_{\gamma\leqslant\gamma(l)} c_\gamma a^{\gamma(l)-\gamma}\Big| + \sum_{k=0}^{p-l} \sum_{\substack{\gamma\leqslant\gamma(p-k)\\ \gamma\,\mathrm{non}\leqslant\gamma(p-k+1)}} |c_\gamma a^{\gamma(p-k)-\gamma}| \\
&\leqslant \sum_{k=0}^{p} \sum_{\substack{\gamma\leqslant\gamma(k)\\ \gamma\,\mathrm{non}\leqslant\gamma(k+1)}} |c_\gamma| A^{\gamma(k)-\gamma} \leqslant \sum_{\gamma\leqslant\gamma(0)} |c_\gamma| B^\gamma,
\end{aligned}$$

where the last inequality follows from (5) and (7). Thus (4) holds and the proof is complete.

4. The proof of Theorem 2. By the lemma the condition of Theorem 2 implies that there is an n-tuple $\delta = (k_1, \ldots, k_n)$ such that

$$(A_i B_i)^{k_i} < \prod_{j=1}^{n} A_j^{k_j} \quad \text{for all } i.$$

If necessary, by replacing δ by a multiple of it, *we can suppose* without loss of generality that

(8) $$n(A_i B_i)^{k_i} < \prod_{j=1}^{n} A^{k_j} \quad \text{for all } i.$$

As before, if $\gamma = (g_1, \ldots, g_n)$ is an n-tuple, we put $X^\gamma = \prod_{i=1}^{n} X_i^{g_i}$. Put $N = \{1, \ldots, n\}$.

If $M \subset N$, let M *also denote the n-tuple* whose ith term is k_i if $i \in M$ and zero otherwise. Thus $a^\delta = a^N$. Furthermore, we use the notations in the natural sense, e.g. if $\varphi = (f_1, \ldots, f_n)$ and $\gamma = (g_1, \ldots, g_n)$ are n-tuples, we put $X^{\varphi-\gamma} = (X^\varphi)(X^\gamma)^{-1} = \prod_{i=1}^{n} X_i^{f_i-g_i}$, $X^{M-N} = \prod_{i\in N-M} X_i^{-k_i}$, where the various products might be only formal products of symbols.

Let us define a number of normed spaces as follows. σ *always* denotes an n-tuple whose jth term is s_j: $\sigma = (s_1, \ldots, s_n)$.

a) Let $S(\sigma)$ be the one-dimensional normed space spanned by a^σ, where $|a^\sigma| = A^\sigma$. (In particular, $S(0)$ is just the scalar field with the ordinary norm.)

b) Let $M \subset N$, $M \neq N$ and let $\sigma = (s_1, \ldots, s_n)$ be an n-tuple such that $s_j < k_j$ if $j \notin M$. Let $T(M, \sigma)$ be the normed space spanned by the vectors $c_0 a^{M+\sigma}$, $c_i a^{M-\{i\}+\sigma}$, $i \in M$, where the norm is given as follows:

$$\Big|\lambda_0 c_0 a^{M+\sigma} + \sum_{i\in M} \lambda_i c_i a^{M-\{i\}+\sigma}\Big| = |\lambda_0| + \sum_{i\in M} |\lambda_i - \lambda_0| \prod_{j\notin M} A_j^{s_j-k_j}.$$

c) If $i \in N$, let $U(i, \sigma)$ be the one-dimensional normed space spanned by the *unit* vector $c_i a^{N-\{i\}+\sigma}$.

Let V be the sum of all these normed spaces. I.e., V consists of all *finite formal sums*

$$x = \sum_\alpha x_\alpha + \sum_\beta Y_\beta + \sum_\gamma z_\gamma,$$

where $x_\alpha \in S(\sigma_\alpha)$, $Y_\beta \in T(M_\beta, \sigma_\beta)$, $z_\gamma \in U(i_\gamma, \sigma_\gamma)$, with norm

$$|x| = \sum_\alpha |x_\alpha| + \sum_\beta |y_\beta| + \sum_\gamma |z_\gamma|.$$

Let us turn V into *a commutative* algebra by putting

$$\begin{aligned}
(c_j a^\sigma)(c_k a^\tau) &= 0, \quad j, k = 0, 1, \ldots, n, \\
(c_j a^\sigma) a^\tau &= c_j a^{\sigma+\tau}, \quad j = 1, 2, \ldots, n, \\
(c_0 a^\sigma) a^\tau &= c_0 a^{\sigma+\tau} \quad \text{if } \delta \not\leqslant \sigma+\tau
\end{aligned}$$

and

$$(c_0 a^\sigma) a^\tau = -\sum_{i=1}^{n} c_i a^{\sigma+\tau} a_i^{-k_i} \quad \text{if } \sigma+\tau \geqslant \delta.$$

In other words, the multiplication is the natural multiplication of formal sums of formal products with the relations

$$c_i c_j = 0$$

and

$$c_0 a^N + \sum_{i\in M} c_i a^{N-\{i\}} = 0.$$

We claim that *with this multiplication V is a normed algebra.*

By symmetry and by the definition of the product *we have to check only that if $x \in V$, then $|a_1 x| \leqslant |x| A_1$.* To verify this, and also to prepare for the last part of the proof, let us enumerate where and how each of the summands of V is mapped under multiplication by a_1.

(i) $S(\sigma)$ is mapped into $S(\sigma+\varepsilon_1)$, the map is 1-1 and if $x \in S(\sigma)$, $|a_1 x| = A_1 |x|$.

(ii) Suppose $1 \in M$ or $1 \notin M$ and $s_1 < k_1 - 1$. Then $T(M, \sigma)$ is mapped into $T(M, \sigma+\varepsilon_1)$, the map is 1-1 and if $x \in T(M, \sigma)$, $|x| \leqslant |a_1 x| \leqslant A_1 |x|$.

(iii) Suppose $M \subset N-\{1\}$, $M \neq N-\{1\}$ and $s_1 = k_1 - 1$. Then $T(M, \sigma)$ is mapped into $T(M \cap \{1\}, \sigma - k_1 \varepsilon_1)$. Furthermore, it is immediate that if $x \in T(M, \sigma)$ then $|x| \leqslant |a_1 x| \leqslant A_1 |x|$.

(iv) Suppose $M = N-\{1\}$ and $s_1 = k_1 - 1$. Then $T(M, \sigma)$ is mapped into $\sum_{i=1}^{n} U(i, \sigma - s_1 \varepsilon_1)$. Namely, if

$$x = \Big(\lambda_0 c_0 a^M + \sum_{i \in M} \lambda_i c_i a^{M-\{i\}}\Big) a^{\sigma},$$

then

$$a_1 x = \Big(-\lambda_0 c_1 a^{N-\{1\}} + \sum_{i \in M} (\lambda_i - \lambda_0) c_i a^{N-\{i\}}\Big) \prod_{2}^{n} a_j^{s_j}.$$

Consequently,

$$|x| = |\lambda_0| + \sum_{i=2}^{n} |\lambda_i - \lambda_0| A_1^{-1} \leqslant |\lambda_0| + \sum_{i=2}^{n} |\lambda_i - \lambda_0| = |a_1 x| \leqslant A_1 |x|.$$

(v) $U(i, \sigma)$ is mapped isometrically onto $U(i, \sigma + \varepsilon_1)$.

Note that a space $U(i, \sigma - s_1 \varepsilon_1)$, appearing in (iv), does not coincide with a space $U(i, \sigma' + \varepsilon_1)$, appearing in (v), since the first term of $\sigma - s_1 \varepsilon_1$ is zero, while the first term of $\sigma' + \varepsilon_1$ is certainly non-zero. Thus under the multiplication by a_1 *different summands* are mapped into *sums of disjoint sets of summands*. Consequently, (i)–(v) imply *not only that* $|a_1 x| \leqslant A_1 |x|$ *for all* $x \in V$, *and so* V *is a normed algebra*, but *also that* $|a_1 x| \geqslant |x|$ *for all* $x \in V$.

Choose V *as the normed algebra* $\mathscr{A}$, *required by the theorem*. As we have just seen that a_1 is norm-increasing, by symmetry $a_1, \ldots, a_n \in I(\mathscr{A})$. By definition, $|a_i| = A_i$. Suppose now that there is an extension $\mathscr{B}$ of $\mathscr{A}$ such that $b_i = a_i^{-1} \in \mathscr{B}$ and $|a_i^{-1}| \leqslant B_i$ for all i. Then by construction we have

$$c_0 a^N + \sum_{i=1}^{n} c_i a^{N-\{i\}} = 0,$$

so (8) implies

$$1 = |-c_0| = \Big|\sum_{i=1}^{n} c_i b_i^{k_i}\Big| \leqslant \sum_{i=1}^{n} \Big(\prod_{j \neq i} A_j^{-k_j}\Big) B_i^{k_i} < 1.$$

This contradiction completes the proof.

5. Additional results. Let us give first a corollary of Theorem 1, extending Theorem 1 and sharpening ([4], Theorem 3.4). To simplify the notations we again renorm the elements in question to make them norm-increasing.

COROLLARY 1. *Let* $\mathscr{A}$ *be a normed algebra and let* $S \subset I(\mathscr{A})$. *Let* $i(s) \geqslant 1$, $s \in S$, *be such that*

$$\sum_{\substack{s \in S \\ |s|>1}} \frac{\log |s|}{\log (|s|\, i(s))} \leqslant 1.$$

Then there is an extension $\mathscr{B}$ *of* $\mathscr{A}$ *such that* $s^{-1} \in \mathscr{B}$ *and* $|s^{-1}| \leqslant i(s)$ *for all* $s \in S$.

Proof. Let $\mathscr{A}[x]$ be the polynomial algebra in the variables $x(s)$, $s \in S$. If $p(x) = \sum c(s_1, \ldots, s_n; k_1, \ldots, k_n) x(s_1)^{k_1} \ldots x(s_n)^{k_n} \in \mathscr{A}[x]$, where $c(s_1, \ldots, s_n; k_1, \ldots, k_n) \in \mathscr{A}$ and the summation is over a finite set, put

$$t(p) = \sum |c(s_1, \ldots, s_n; k_1, \ldots, k_n)| \prod_{j=1}^{n} i(s_j)^{k_j}.$$

Let I be the ideal in $\mathscr{A}[x]$ generated by the elements $sx(s) - 1$, $s \in S$. Let $\mathscr{B} = \mathscr{A}[x]/I$ and define a norm $|\cdot|'$ in $\mathscr{B}$ by

$$|p + I|' = \inf \{t(q) \colon q \in p + I\}.$$

To show that $\mathscr{B}$ *is the required extension of* $\mathscr{A}$, we have to *check only* that $a \in \mathscr{A}$ then

$$|a + I|' = |a|.$$

In other words, we have to show that if p is a polynomial in $x(s_1), \ldots, x(s_n)$, then $t(a+p) \geqslant |a|$. However, this holds by Theorem 1, since there is an extension $\mathscr{C}$ of $\mathscr{A}$ such that $s_j^{-1} \in \mathscr{C}$ and $|s_j^{-1}| \leqslant i(s_j)$ for all j, $1 \leqslant j \leqslant n$.

The next result is a corollary of Theorem 2. It asserts the existence of certain algebras, showing that an element might have to become a topological zero-divisor in every extension in which another element has an inverse with small norm.

COROLLARY 2. *Let* $n \geqslant 2$ *be a natural number. Then there exists an algebra* $\mathscr{A}$ *that contains norm-increasing elements* $a_1, \ldots, a_n$, $|a_i| = 2$, $i = 1, \ldots, n$, *such that given* i, $1 \leqslant i \leqslant n$, *there is no extension* $\mathscr{B}_i$ *of* $\mathscr{A}$ *for which* $a_i^{-1} \in \mathscr{B}_i$ *and* $a_j^{-1} \in \mathscr{B}_i$, $|a_j^{-1}| \leqslant 2^{n-2}$, *if* $j \neq i$.

Proof. Let i and m be positive integers, $1 \leqslant i \leqslant n$. Let $\mathscr{B}(i, m)$ be the algebra constructed in the proof of Theorem 2, with $B_i = m$ and $B_j = 2^{n-2}$ if $j \neq i$. (Note that $\mathscr{B}(i, m)$ exists since $(n-1)\dfrac{\log 2}{\log (2 \cdot 2^{n-2})} + \dfrac{\log 2}{\log 2m} > 1$.) Denote by $\mathscr{B}_0$ the sum of the summands $S(\sigma)$ of $\mathscr{B}(i, m)$ and denote by $\mathscr{B}(i, m)'$ the sum of the other summands of $\mathscr{B}(i, m)$. Let $\mathscr{B}_1 = \sum_{m=1}^{\infty} \sum_{i=1}^{n} \mathscr{B}(i, m)'$ and let $\mathscr{B} = \mathscr{B}_0 + \mathscr{B}_1$. Extend the multiplication from the algebras $\mathscr{B}(i, m) = \mathscr{B}_0 + \mathscr{B}(i, m)'$ to the whole of $\mathscr{B}$ by putting the trivial multiplication on $\mathscr{B}_1$, i.e. if $x, y \in \mathscr{B}_1$, then let $xy = 0$. It is immediate that $\mathscr{B}$ has the required properties.

Our final result, which extends Theorem 2 and Corollary 2 of the present note and [4], Theorem 4.1, can be proved by sticking together in a slightly more sophisticated way a large set of algebras constructed in the proof of Theorem 2. The detailed proof is left to the reader.

COROLLARY 3. *Let T be an arbitrary index set and let A_t $(A_t \geqslant 1)$, $t \in T$, be real numbers. Then there is an algebra $\mathscr{A}$, containing norm-increasing elements a_t, $|a_t| = A_t$, $t \in T$, having the following property.*

If B_t $(\geqslant 1)$, $t \in S \subset T$, are real numbers, then there exists an extension $\mathscr{B}$ of $\mathscr{A}$, $a_t^{-1} \in \mathscr{B}$, $|a_t^{-1}| \leqslant B_t$, $t \in S$, if and only if $\sum_{\substack{t \in S \\ A_t > 1}} \frac{\log A_t}{\log A_t B_t} \leqslant 1.$

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Некоторые вопросы спектральной теории симметризуемых операторов в локально выпуклых пространствах

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Резюме. В работе изучаются свойства резольвенты и спектра некоторых линейных операторов в локально выпуклых пространствах и устанавливается справедливость теории Гильберта–Шмидта для симметризуемых операторов в таких пространствах.

Как известно, теория Фредгольма для линейных операторов в банаховом пространстве, построенная Ф. Риссом, была установлена Ж. Лере [1] для случая локально выпуклого пространства. Построенная Гильбертом и Шмидтом спектральная теория симметричных компактных операторов была обобщена на случай компактных операторов (в гильбертовом пространстве), симметризуемых ограниченным оператором, А. Зааненом [2] и В. Ридом [3], а на случай симметризуемых, вообще говоря, неограниченных операторов с дискретным спектром в предгильбертовых и банаховых пространствах — одним из авторов [4]-[7].

В настоящей работе устанавливается справедливость теории Гильберта-Шмидта для некоторых классов симметризуемых операторов в локально выпуклых пространствах. Эти классы, в частности, содержат симметризуемые компактные операторы и операторы, некоторая итерация которых компактна, а в случае нормированного пространства совпадают с ранее изученными в [5] и [7].

Для доказательства основных результатов мы устанавливаем ниже свойство голоморфности резольвенты некоторых линейных операторов в произвольных локально выпуклых пространствах. Насколько нам известно, впервые свойство голоморфности резольвенты супернепрерывного оператора (определение супернепрерывного оператора см. ниже) в счётно полном локально выпуклом пространстве доказал Х. Шефер [10] (см. также [11]). Это свойство для компактного оператора в произвольном локально выпуклом пространстве доказано в [12].

1. Некоторые определения и обозначения. В настоящее время в теории локально выпуклых пространств ещё нет согласованной терминологии, в связи с чем мы приведём определения, которых будем придерживаться в дальнейшем.