and

$$\sigma_r(A_1, A_2, A_3) \subset \sigma_{[A_1,A_2,A_3]}(A_1, A_2, A_3),$$

which hold true for any triple A_1, A_2, A_3 of pairwise commuting operators in $L(X)$, and from the condition (ii) of the previous theorem. So we have

3.4. THEOREM. *Let X be a complex Banach space. Then the joint spectra σ_π, σ_l, σ_r, and σ defined on $\mathfrak{c}(X)$ possess the spectral mapping property with respect to polynomial mappings.*

As we mentioned before, the part of this theorem concerning the spectra σ_l, σ_r, and σ is due to Harte [5] and [6].

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Domains of attraction of stable measures on a Hilbert space

by

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Abstract. We characterize all probability measures in the domain of attraction of a stable measure defined on the Borel subsets of a real separable Hilbert space H.

1. Introduction and notation. Let E be a topological vector space and $\mathscr{B}(E)$ the class of Borel subsets of E. We say that a probability measure on $\mathscr{B}(E)$ is in the domain of attraction of a probability measure μ on $\mathscr{B}(E)$ if there exists real numbers $b_n > 0$ and vectors a_n in $E(n = 1, 2, \ldots)$ such that $\mathscr{L}\left(\frac{X_1 + \ldots + X_n}{b_n} - a_n\right)$ converges weakly to μ where $X_1, X_2, \ldots$ are independent identically distributed random variables with $\mathscr{L}(X_i) = P$ $(i = 1, 2, \ldots)$. and P is a Borel probability measure. The b_n's are called *norming constants*. In case E is a real separable Banach space it is shown in [6] that stable measures and only stable measures have non-empty domains of attraction. When E is a real separable Hilbert space H, a detailed Levy–Khinchine representation of the stable measures analogous to the one-dimensional case [3] is obtained in [5] and it is used here to characterize probability measures P which lie in the domain of attraction of a non-degenerate stable measure on H. Our results will include and generalize the work of Rvaćeva [9] when H is finite-dimensional. The difficulty in the infinite-dimensional case results from the fact that the conditions for weak convergence of infinitely divisible measures ([4] and [7]) involve certain compactness criteria and these are attacked by using the concept of regular variation and modifications of some of the elegant ideas in [1].

Let μ be a finite Borel measure on a topological space X. Then A in $\mathscr{B}(X)$ is called a *continuity set of* μ if $\mu(\partial A) = 0$ where ∂A denotes the boundary of A. A set S_μ is called the *support of* μ if

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(i) S_μ is closed and the complement of S_μ has μ-measure zero, and

(ii) if $x \in S_\mu$ and if U is open in X with $x \in U$ then $\mu(U) > 0$. In case X is a separable metric space, it is easy to see that S_μ exists and is unique, i.e. $x \notin S_\mu$ iff there exists an open set U such that $x \in U$ and $\mu(U) = 0$ and since X is separable and metric the complement of S_μ is a countable union of open sets of μ-measure zero so it is of measure zero.

A finite measure μ on a topological vector space E is called *non-degenerate* if the closed linear subspace generated by $S_\mu = E$.

If μ is a measure on a set X and f maps X into Y then μ^f is the measure on Y defined by $\mu^f(A) = \mu(f^{-1}(A))$ for all A such that $\mu(f^{-1}(A))$ exists.

The Fourier transform of a probability measure μ on a real Hilbert space H is defined by

$$\hat{\mu}(x) = \int_H e^{i(x,y)} \mu(dy) \qquad (x \in H).$$

When H is separable it is shown in [5] that μ is a stable measure on H iff:

(1.1) μ is a Gaussian measure on H and

$$\hat{\mu}(x) = \exp\{i(x, \beta) - 1/2(Tx, x)\}$$

where $\beta \in H$ is called the *mean vector* and T is an *S-operator* ([7], p. 164) on H which is the covariance operator of μ,

(1.2) there exists a constant α $(0 < \alpha < 2)$, a finite Borel measure Γ on $S = \{x \in H: \|x\| = 1\}$, and a vector $\beta \in H$ such that

$$\hat{\mu}(x) = \exp\{i(x, \beta) - \int_S |(x, s)|^\alpha \Gamma(ds) + iC(\alpha, x)\}$$

where

$$C(\alpha, x) = \begin{cases} \tan\frac{\pi\alpha}{2} \int_S (x, s)|(x, s)|^{\alpha-1}\Gamma(ds) & (\alpha \neq 1), \\ \frac{2}{\pi}\int_S (x, s)\log|(x, s)|\Gamma(ds) & (\alpha = 1). \end{cases}$$

We call the number α $(0 < \alpha < 2)$ the type of the stable law μ and if μ is Gaussian we say μ is of type 2. For the sake of simplicity, the representations (1.1) and (1.2) will be denoted by $\mu = [\beta, T]$ and $\mu = [\alpha, \Gamma, \beta]$, respectively. We remark that in case μ is Gaussian the representation $[\beta, T]$ can be alternatively thought of as $[2, \Gamma, \beta]$ where Γ is the discrete measure on S sitting at the normalized eigenvectors of T with the amount of mass at each eigenvector equal to the corresponding eigenvalue divided by two. With this interpretation we can denote the representations (1.1) and (1.2) by $\mu = [\alpha, \Gamma, \beta]$ for $0 < \alpha \leq 2$.

For a non-degenerate stable measure μ, the non-degeneracy of Γ plays an important role in the study of the problem of the domain of attraction.

2. The support of a stable measure. Throughout we consider measures μ defined on the Borel subsets of a real separable Hilbert space H, S_μ denotes the support of μ, and $L(S_\mu)$ denotes the closed linear subspace generated by S_μ.

2.1. LEMMA. *If* $\mu = [\alpha, \Gamma, 0]$ *is a stable measure on* H *of type* α $(0 < \alpha \leq 2)$, *then* $L(S_\mu) = L(S_\Gamma)$.

Proof. If $y \notin L(S_\Gamma)$ and $y \neq 0$, then there is an $f \in H'$ (the topological dual of H) such that $(f, y) > 0$ and $f(L(S_\Gamma)) = 0$. Now $f(L(S_\Gamma)) = 0$ implies $\hat{\mu}(f) = 1$ by (1.1) and (1.2) so $f(\cdot) = 0$ almost everywhere with respect to μ. Thus $\{x: |f(x)| > 0\}$ is an open set which has μ measure zero and contains y, giving $y \notin S_\mu$ and hence $L(S_\mu) \subseteq L(S_\Gamma)$. Conversely, for $y \notin L(S_\mu)$, $y \neq 0$, there exists $g \in H'$ such that $g(y) > 0$ and $g(L(S_\mu)) = 0$. Therefore $g = 0$ almost everywhere with respect to μ and hence from (1.1) or (1.2) and the interpretation for the Gaussian case given at the end of Section 1 we have $\hat{\mu}(g) = 1$ implying $\int_S |(g, s)|^\alpha \Gamma(ds) = 0$. Hence $(g, s) = 0$ on S_Γ and due to the linearity and continuity of $g(\cdot)$ we have that $g(L(S_\Gamma)) = 0$. This implies $y \notin L(S_\Gamma)$ completing the proof.

The following corollary is now immediate

2.2. COROLLARY. *If* $\mu = [\alpha, \Gamma, 0]$, $0 < \alpha \leq 2$, *is a non-degenerate stable law on* H, *then every non-zero linear functional on* H *has a non-degenerate stable distribution with parameter* α.

For a Borel probability measure μ on H, define for each $a \in H$ $\mu_a(A) = \mu(A - a)$ for $A \in \mathscr{B}(H)$. We say that a is an admissible translate of μ if μ_a is absolutely continuous with respect to μ and we denote the set of admissible translates of μ by A_μ. Clearly 0 is in A_μ.

2.3. Remarks. (i) Let $\mu = [\alpha, \Gamma, 0]$. In case $\alpha = 2$, Γ is a discrete measure (see Section 1) with positive mass only on the eigenvectors of the covariance operator T of μ. Denote these eigenvectors by $\{e_j\}$. Hence S_Γ equals this discrete set and since it is easy to see that A_μ contains all finite linear combinations of the e_j's, we get A_μ dense in $L(S_\Gamma)$. If V is an open neighborhood of zero such that $\mu(V) = 0$, then $\mu(V - a) = 0$ $\forall a \in A_\mu$. Hence $1 = \mu(L(S_\mu)) = \mu(L(S_\Gamma)) \leq \sum_{j=1}^\infty \mu(V - a_j) = 0$ for any countable subset $[a_j]$ of A_μ which is dense in $L(S_\mu)$. This is impossible and hence $\mu(V) > 0$.

In view of Lemma 2.1, $S_\mu \subseteq L(S_\Gamma)$ so if S_μ is a proper subset of $L(S_\Gamma)$, we can find $x \in L(S_\Gamma) - S_\mu$ and $U = \{y: \|y - x\| < \varepsilon\}$ such that $U \cap S_\mu = \emptyset$ and $\mu(U) = 0$. Since U is open and A_μ is dense in $L(S_\mu)$

$= L(S_\Gamma)$, we can choose $a \in U \cap A_\mu$. Then $U - a$ is an open neighborhood of zero and hence $\mu_a(U) > 0$ giving $\mu(U) > 0$ since $a \in A_\mu$. Thus $S = L(S_\Gamma)$.

Now putting $H = L_2[0,1]$ and $\alpha = 2$ we get Theorem 2.1 of [2] for Gaussian processes.

(ii) For $0 < \alpha < 1$ and μ non-symmetric, it can be shown that S_μ can be a cone in H. It is not known, to us anyway, whether $S_\mu = L(S_\Gamma)$ is case μ is symmetric.

3. The domain of attraction for a Gaussian measure ($\alpha = 2$). The following theorem generalizes Theorem 4.1 of ([9]), p. 194). We note that the methods of [9] depend on the finiteness of the dimension of H. The main techniques used here deal with the properties of regularly varying functions as given in ([1], pp. 275–284). A function U on $[0, \infty)$ varies regularly with exponent ϱ $(-\infty < \varrho < \infty)$ if

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\varrho$$

for each $x > 0$. In the case $\varrho = 0$ we say the function varies slowly.

3.1. Theorem. *Let H be a real separable Hilbert space. Then a Borel probability measure P on H is in the domain of attraction of a non-degenerate Gaussian measure μ with mean vector zero and covariance operator T iff*

(a) $$\lim_{R\to\infty} \frac{R^2 \int_{\|x\|>R} P(dx)}{\int_{\|x\|<R} \|x\|^2 P(dx)} = 0,$$

(b) $$\lim_{R\to\infty} \frac{\int_{\|x\|<R} (y, x)^2 P(dx)}{\int_{\|x\|<R} (z, x)^2 P(dx)} = \frac{(Ty, z)}{(Tz, z)}$$

for $z \neq 0$ provided $\int_H \|x\|^2 P(dx) = \infty$, or

$$\lim_{R\to\infty} \frac{\int_{\|x\|<R} (y, x-a)^2 P(dx)}{\int_{\|x\|<R} (z, x-a)^2 P(dx)} = \frac{(Ty, y)}{(Tz, z)}$$

for $z \neq 0$ provided $\int_H \|x\|^2 P(dx) < \infty$ where $a = \int_H x P(dx)$ in the sense of Bochner.

(c) $$\lambda_m = \overline{\lim_{R\to\infty}} \frac{\int_{\|x\|<R} \|\pi_m x\|^2 P(dx)}{\int_{\|x\|<R} \|x\|^2 P(dx)} > 0$$

for $m = 1, 2, \ldots$ where, for each m, $\pi_m(x) = \sum_{i\geq m} (x, e_i) e_i$ for some complete orthonormal system $\{e_i\}$ in H, and

$$\lim_m \lambda_m = 0.$$

Proof. First assume P is a Borel probability measure on H with

(3.2) $$\int_H \|x\|^2 P(dx) < \infty.$$

Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $\mathscr{L}(X_k) = P$ and set $Z_k = X_k - a$ where $a = \int_H x P(dx)$. Then by [8], p. 173, we have that

$$\mathscr{L}\left(\frac{Z_1 + \ldots + Z_n}{\sqrt{n}}\right) = \mathscr{L}\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}} - \sqrt{n}a\right)$$

converges weakly to the Gaussian measure with mean zero and covariance operator

(3.3) $$(Sy, z) = \int_H (y, x-a)(z, x-a) P(dx).$$

Hence if P is in the domain of attraction of a non-degenerate Gaussian measure μ with mean zero and covariance operator T, then by ([6], Theorem 1.5) we get that $T = \lambda S$ where $\lambda > 0$. Now condition (a) holds since we are assuming (3.2). Since $T = \lambda S$ with $\lambda > 0$ and μ is non-degenerate we have T and S vanishing only at zero and hence (3.3) implies (b). Now (c) follows since

$$\lambda_m = \frac{\int_H \|\pi_m x\|^2 P(dx)}{\int_H \|x\|^2 P(dx)}.$$

Further, $\lambda_m > 0$ since

$$\int_H \|\pi_m x\|^2 P(dx) = \sum_{i\geq m} \int_H (x, e_i)^2 P(dx) = \sum_{i\geq m} [(Se_i, e_i) + (a, e_i)^2]$$

and the last term is a positive number as S is a positive trace class operator which, as mentioned above, only vanishes at zero since μ is non-degenerate.

Now assume (a), (b), (c) and (3.2). By [8] we have P in the domain of attraction of the mean zero Gaussian measure with covariance operator λS where S is as in (3.3) and λ is any positive number. Since (b) holds, we have $T = \lambda S$ for some $\lambda > 0$, and since $(Tz, z) > 0$ for $z \neq 0$, we have $\mu = [0, T]$ non-degenerate and P is the domain of attraction of μ.

Now assume

(3.4) $$\int \|x\|^2 P(dx) = \infty.$$

From [4], p. 331, we have P in the domain of attraction of the Gaussian measure $\mu = [0, T]$ with norming constants $[b_n]$ iff

$$(3.5)\quad \begin{cases} \text{(i)} \ \lim\limits_n nP\{\|x-b_n\gamma_n\| > Rb_n\} = 0 \text{ for every } R>0 \\ \qquad\qquad \text{where } \gamma_n = \int\limits_{\|x\|<b_n} x/b_n P(dx). \\ \text{(ii)} \ \sup\limits_n nb_n^{-2} \int\limits_{\|x-b_n\gamma_n\|<\varepsilon b_n} \|x-b_n\gamma_n\|^2 P(dx) < \infty \text{ for some } \varepsilon>0. \\ \text{(iii)} \ \lim\limits_m \sup\limits_n nb_n^{-2} \int\limits_{\|x-b_n\gamma_n\|<\varepsilon b_n} \|\pi_m(x-b_n\gamma_n)\|^2 P(dx) = 0 \text{ for some} \\ \qquad \varepsilon>0 \text{ and } \pi_m(x) = \sum\limits_{i\geqslant m} (x, e_i)e_i \text{ for some CONS } \{e_i\} \text{ in } H. \\ \text{(iv)} \ \lim\limits_{\varepsilon\downarrow 0} \overline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x-b_n\gamma_n\|<\varepsilon b_n} (y, x-b_n\gamma_n)^2 P(dx) = (Ty, y) \\ \qquad = \lim\limits_{\varepsilon\downarrow 0} \underline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x-b_n\gamma_n\|<\varepsilon b_n} (y, x-b_n\gamma_n)^2 P(dx). \end{cases}$$

Now (3.4) and P in the domain of attraction of a non-degenerate normal law implies that $\int\limits_H (y, x)^2 P(dx) = \infty$ for all non-zero $y \in H$. Hence by the argument used in ([3], p. 173) and since $\lim\limits_n \gamma_n = 0$, we have (3.5) equivalent to

$$(3.6)\quad \begin{cases} \text{(i)} \ \lim\limits_n nP(\|x\| > Rb_n) = 0 \text{ for each } R>0. \\ \text{(ii)} \ \sup\limits_n nb_n^{-2} \int\limits_{\|x\|<\varepsilon b_n} \|x\|^2 P(dx) < \infty \text{ for some } \varepsilon>0. \\ \text{(iii)} \ \lim\limits_m \sup\limits_n nb_n^{-2} \int\limits_{\|x\|<\varepsilon b_n} \|\pi_m x\|^2 P(dx) = 0 \text{ for some } \varepsilon>0 \\ \qquad\qquad \text{and } \pi_m \text{ as in (3.5).} \\ \text{(iv)} \ \lim\limits_{\varepsilon\downarrow 0} \overline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x\|<\varepsilon b_n} (y, x)^2 P(dx) = (Ty, y) \\ \qquad = \lim\limits_{\varepsilon\downarrow 0} \underline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x\|<\varepsilon b_n} (y, x)^2 P(dx). \end{cases}$$

Hence our theorem is proved if we show (3.6) is equivalent to conditions (a), (b) and (c) of the theorem.

First we observe that for $y \neq 0$ and for each $\varepsilon < 1$ we have

$$(3.7)\quad \underline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x\|<b_n} \|x\|^2 P(dx) \geqslant \frac{1}{\|y\|^2} \underline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x\|<\varepsilon b_n} (y, x)^2 P(dx).$$

Now assume (3.6) holds. Then from (3.7), (3.6)(iv), and the non-degeneracy of μ we obtain

$$(3.8)\quad \underline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x\|<b_n} \|x\|^2 P(dx) \geqslant \frac{1}{\|y\|^2} (Ty, y) > 0.$$

Thus for R such that $b_n \leqslant R \leqslant b_{n+1}$ we have

$$\frac{R^2 P(\|x\| > R)}{\int\limits_{\|x\|<R} \|x\|^2 P(dx)} \leqslant \frac{nb_{n+1}^2 P(\|x\| > b_n)}{n \int\limits_{\|x\|<b_n} \|x\|^2 P(dx)} \xrightarrow[n\to 0]{} 0$$

by (3.6)(i), (3.8), and that $\lim\limits_n \frac{b_{n+1}}{b_n} = 1$ since the b_n's are norming constants. Thus (a) holds. To obtain (b) suppose $b_n \leqslant R \leqslant b_{n+1}$ and note that

$$\frac{\int\limits_{\|x\|<b_n} (y, x)^2 P(dx)}{\int\limits_{\|x\|<b_{n+1}} (z, x)^2 P(dx)} \leqslant \frac{\int\limits_{\|x\|<R} (y, x)^2 P(dx)}{\int\limits_{\|x\|<R} (z, x)^2 P(dx)} \leqslant \frac{\int\limits_{\|x\|<b_{n+1}} (y, x)^2 P(dx)}{\int\limits_{\|x\|<b_n} (z, x)^2 P(dx)}.$$

Hence from (3.6)(iv) and since $\frac{b_{n+1}}{b_n} \to 1$ (b) follows since we are assuming $\int\limits_H \|x\|^2 P(dx) = \infty$. To show $\lambda_m > 0$ we first note that if $b_n \leqslant R \leqslant b_{n+1}$ then

$$\frac{\int\limits_{\|x\|<R} \|\pi_m x\|^2 P(dx)}{\int\limits_{\|x\|<R} \|x\|^2 P(dx)} \geqslant \frac{nb_n^{-2} \int\limits_{\|x\|<b_n} \|\pi_m x\|^2 P(dx)}{nb_n^{-2} \int\limits_{\|x\|<b_{n+1}} \|x\|^2 P(dx)}.$$

Hence, since $\frac{b_{n+1}}{b_n} \to 1$, there is a constant $c > 0$ such that

$$(3.9)\quad \lambda_m \geqslant \frac{\underline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x\|<b_n} \|\pi_m x\|^2 P(dx)}{c \cdot \sup\limits_n nb_n^{-2} \int\limits_{\|x\|<b_n} \|x\|^2 P(dx)}.$$

Now P^{π_m} in the domain of attraction of μ^{π_m} with the same norming constants $\{b_n\}$, and the non-degeneracy of μ^{π_m} (μ non-degenerate implies μ^{π_m} non-degenerate) along with an inequality of the type in (3.8) implies

$$\underline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x\|<b_n} \|\pi_m x\|^2 P(dx) > 0.$$

Thus (3.9) implies $\lambda_m > 0$ as μ non-degenerate implies the denominator is positive. Similarly,

$$\lambda_m \leqslant \frac{\sup\limits_n nb_n^{-2} \int\limits_{\|x\|<b_n} \|\pi_m x\|^2 P(dx)}{\underline{\lim\limits_n} nb_n^{-2} \int\limits_{\|x\|\leqslant b_n} \|x\|^2 P(dx)},$$

so by (3.8) and (3.6) (iii), (c) holds.

Now we show (a), (b), (c) imply (3.6). From (a) and [1], p. 283, we have that $U(R) = \int_{\|x\|<R} \|x\|^2 P(dx)$ varies slowly (recall that we assume $U(\infty) = \infty$). Hence there exist constants $b_n > 0$ such that $\lim \frac{b_{n+1}}{b_n} = 1$, $b_n \to \infty$ such that

$$\frac{n}{b_n^2} U(b_n) \to 1.$$

Thus U slowly varying implies $nb_n^{-2} U(b_n R) \xrightarrow[n\to\infty]{} 1$ for each $R > 0$ and hence (3.6)(i) follows from this and condition (a). Further (3.6)(ii) holds by definition of the b_n's with $\varepsilon = 1$. With $U_m(R) = \int_{\|x\|<R} \|\pi_m x\|^2 P(dx)$, (c) and our choice of b_n imply that

$$\overline{\lim_n}\, nb_n^{-2} U_m(b_n) = \overline{\lim_n}\, nb_n^{-2} U(b_n) \frac{U_m(b_n)}{U(b_n)} = \lambda_m$$

and since $\lambda_m \to 0$, we get (3.6)(iii) with $\varepsilon = 1$.

Let $M_n(A) = nP(b_n^{-1}A)$ for $A \in \mathscr{B}(H - \{0\})$. Then by Theorem 5.1 of ([7], p. 186), and (3.6)(i), (ii) and (iii) we have that the sequence of infinitely divisible measures μ_n with Levy-Khinchine representation $[0, 0, M_n]$ (see [7] for details) is weakly conditionally compact. From (3.6)(i) and Theorem 5.4 of [7], p. 189, we see that all limit points of μ_n are Gaussian. If $\mu_0 = [0, S_0]$ is a limit point of $\{\mu_n\}$ and $\{\mu_{n_k}\}$ is a subsequence converging weakly to μ_0, then by ([4], p. 328) we have for all $y \in H$

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_k}\, n_k b_{n_k}^{-2} \int_{\|x\|<\varepsilon b_{n_k}} (y, x)^2 P(dx)$$

$$= \lim_{\varepsilon\downarrow 0} \underline{\lim_k}\, n_k b_{n_k}^{-2} \int_{\|x\|<\varepsilon b_{n_k}} (y, x)^2 P(dx) = (S_0 y, y).$$

However, then by (b) we have $(S_0 y, y) = (Ty, y)$ and hence all limit points μ_0 of $\{\mu_n\}$ coincide with the non-degenerate Gaussian measure $\mu = [0, T]$. Again applying [4], p. 328, we obtain 3.6 (iv) so the proof is complete.

4. The domain of attraction for $0 < \alpha < 2$. In this section we consider the domain of attraction of a non-Gaussian stable measure. The main techniques used deal with the properties of regularly varying functions as defined in Section 3.

We start with a lemma which will be needed to prove the main theorem of the section.

4.1. Lemma. *Let P be a Borel probability measure on a real separable Hilbert space H, Γ a finite non-degenerate Borel measure on the unit sphere S of H, and assume $0 < \alpha < 2$. Then the conditions*

(4.2)
$$\begin{cases} \text{(a)}\ \dfrac{P(\|x\| > R)}{P(\|\pi_m x\| > kR)} \xrightarrow[R\to\infty]{} \dfrac{c_1 k^\alpha}{c_m} \quad \text{where for each } m \geq 1, \\ \qquad c_m > 0,\ \pi_m(x) = \sum_{k=m}^{\infty} (x, e_k) e_k \text{ for some C.O.N.S. } \{e_k\}, \text{ and} \\ \qquad \lim_{m\to\infty} c_m = 0, \\ \text{(b)}\ \dfrac{P(\|x\| > R,\ x/\|x\| \in A)}{P(\|x\| > R,\ x/\|x\| \in A^*)} \xrightarrow[R\to\infty]{} \dfrac{\Gamma(A)}{\Gamma(A^*)} \quad \text{for all continuity} \\ \qquad \text{sets } A, A^* \in \mathscr{B}(S) \text{ with } \Gamma(A^*) \neq 0, \end{cases}$$

imply that there exists a sequence of $b_n > 0$ such that $b_n \to \infty$, $\frac{b_n}{b_{n+1}} \to 1$ and

(4.3)
$$\begin{cases} \text{(a)}\ \lim_n nP\{\|x\| > Rb_n,\ x/\|x\| \in A\} = R^{-\alpha} \dfrac{\Gamma(A)}{\Gamma(S)} \dfrac{(2-\alpha)}{\alpha} \\ \qquad \text{for continuity sets } A \text{ of } \Gamma \text{ and } R > 0, \\ \text{(b)}\ \sup_n nb_n^{-2} \int_{\|x\|<\varepsilon b_n} \|x\|^2 P(dx) < \infty \text{ for some } \varepsilon > 0, \\ \text{(c)}\ \lim_m \sup_n nb_n^{-2} \int_{\|x\|<\varepsilon b_n} \|\pi_m x\|^2 P(dx) = 0 \quad \text{for some } \varepsilon > 0 \text{ and} \\ \qquad \text{some projections of the form given in (4.2)(a),} \\ \text{(d)}\ \lim_{\varepsilon\downarrow 0} \overline{\lim_{n\to\infty}}\, nb_n^{-2} \int_{\|x\|<\varepsilon b_n} (y, x)^2 P(dx) = 0. \end{cases}$$

Proof. Let $Z(t) = P(\|x\| > t)$ for $t > 0$. Then (4.2)(a) with $m = 1$ implies $Z(t)$ is regularly varying with exponent $-\alpha$. Hence [1], p. 281, Theorem 1 implies

$$\lim_{R\to\infty} \frac{R^2 Z(R)}{\int_0^R tZ(t)dt} = 2 - \alpha$$

and since

$$\int_{\|x\|\leq R} \|x\|^2 P(dx) = -\int_0^R t^2 Z(dt) = -R^2 Z(R) + 2\int_0^R tZ(t)\, dt,$$

we have

(4.4)
$$\lim_{R\to\infty} \frac{R^2 P(\|x\| > R)}{\int_{\|x\|<R} \|x\|^2 P(dx)} = \frac{2-\alpha}{\alpha}.$$

Thus $0 < a < 2$ and [1], p. 283, Theorem 2 implies $U(t) = \int_{\|x\|<t} \|x\|^2 P(dx)$ $(t > 0)$ is a regularly varying function and hence there is a sequence $\{b_n\}$ such that $b_n \to \infty$, $\frac{b_n}{b_{n+1}} \to 1$, and $\lim_n n b_n^{-2} U(b_n) = 1$. Thus (4.3)(b) follows with $\varepsilon \leqslant 1$. Further, from (4.4) we see that $\lim_n nP(\|x\| > b_n) = \frac{2-a}{a}$ and hence by (4.2)(a) with $m = 1$ that

$$\lim_n nP(\|x\| > b_n R) = \frac{2-a}{a} R^{-a} \quad (R > 0). \tag{4.5}$$

Thus (4.2)(b) and (4.5) imply (4.3)(a).

We now seek to establish (4.3)(c) and (4.3)(d). Let $Z_m(t) = P(\|\pi_m x\| > t)$. Then (4.2)(a) implies for $t > 0$

$$\lim_n \frac{Z(b_n t)}{Z_m(b_n t)} = \frac{c_1}{c_m},$$

and hence along with (4.3)(a) with $A = S$ one has

$$\lim_n nP(\|\pi_m x\| > b_n R) = c_m R^{-a} \frac{2-a}{a c_1}. \tag{4.6}$$

Since $0 < a < 2$ and c_m, c_1 are positive, [1], p. 277, Lemma 3 implies that $Z_m(t)$ is regularly varying with exponent $-a$. The arguments used to obtain (4.4) can now be repeated to conclude

$$\lim_n \frac{nP(\|\pi_m x\| > b_n)}{n b_n^{-2} \int_{\|\pi_m x\| < b_n} \|\pi_m x\|^2 P(dx)} = \frac{2-a}{a}. \tag{4.7}$$

But $\int_{\|x\|<b_n} \|\pi_m x\|^2 P(dx) \leqslant \int_{\|\pi_m x\|<b_n} \|\pi_m x\|^2 P(dx)$ so (4.6) and (4.7) gives (4.3)(c) with $\varepsilon \leqslant 1$ as $\lim_m c_m = 0$.

Let $M_n(A) = nP(b_n^{-1} A)$ and $M(A) = \int_A \frac{dr}{r^{1+a}} \Gamma(ds)$ for $A \in \mathscr{B}(H)$. Condition (4.3)(a) implies that the sequence of finite measures $\{M_n\}$ converges weakly to $M \cdot \frac{2-a}{\Gamma(S)}$ when both are restricted outside some neighborhood of zero. Hence we have for every $y \in H$ $(y \neq 0)$ that

$$\lim_n nP(|(y, x)| > t b_n) = \frac{2-a}{\Gamma(S)} M(|(y, x)| > t) \tag{4.8}$$

for t in a dense set of positive real numbers. Now for any $A \in \mathscr{B}(H - \{0\})$ we have $M(A/a) = a^a M(A)$ so (4.8) implies

$$\lim_n nP(|(y, x)| > t b_n) = t^{-a} \frac{2-a}{\Gamma(S)} M(|(y, x)| > 1). \tag{4.9}$$

Now $M(|(y, x)| > 1) > 0$ or otherwise by the definition of M we would have $\Gamma(s\colon |(y, s)| > 0) = 0$, and thus y is orthogonal to $L(S_\Gamma)$ which is a contradiction since $y \neq 0$ and Γ non-degenerate implies $L(S_\Gamma) = H$. Thus (4.9) holds on a dense set of t and the limit is positive so by [1], p. 277, Lemma 3, $U(t) = P\{|(y, x)| > t\}$ is regularly varying with exponent $-a$. Using the argument as in (4.4) we have

$$\lim_n \frac{\varepsilon^2 nP(|(y, x)| > \varepsilon b_n)}{n b_n^{-2} \int_{\{|(y,x)| < \varepsilon b_n\}} (y, x)^2 P(dx)} = \frac{2-a}{a}. \tag{4.10}$$

From (4.9), (4.10), and (4.3)(a) we get for each $\varepsilon > 0$

$$\lim_n n b_n^{-2} \int_{\{|(y,x)| < \varepsilon b_n\}} (y, x)^2 P(dx) = \varepsilon^{2-a} \frac{M(|(y, x)| > 1)}{\Gamma(S)}.$$

Since $\{\|x\| < \varepsilon b_n\} \subseteq \{|(y, x)| < \varepsilon \|y\| b_n\}$, we get for each $y \neq 0$ that

$$\overline{\lim_n}\, n b_n^{-2} \int_{\{\|x\| < \varepsilon b_n\}} (y, x)^2 P(dx) \leqslant (\varepsilon \|y\|)^{2-a} \frac{M(|(y, x)| > 1)}{\Gamma(S)}$$

and hence (4.3)(d) holds for $y \neq 0$. For $y = 0$ (4.3)(d) is obvious so the lemma is proved.

4.11. Theorem. *Let P be a Borel probability measure on a real separable Hilbert space H. Then P lies in the domain of attraction of a non-degenerate stable measure $\mu = [a, \Gamma, 0]$ where $0 < a < 2$ iff (4.2) holds.*

Proof. If $\mu = [a, \Gamma, 0]$ where $0 < a < 2$ and μ is non-degenerate then by Lemma 2.1 Γ is non-degenerate and by Lemma 4.1 (4.2) implies (4.3) where b_n is a sequence such that $b_n > 0$, $b_n \to \infty$, and $\frac{b_n}{b_{n+1}} \to 1$.

We now will show that for this sequence of b_n's (4.3) and (4.6) which follows from (4.2) imply

(4.12)

(a) $\lim_n nP\left(\|x - b_n \gamma_n\| > R b_n, \frac{x}{\|x\|} \in A\right) = R^{-a} \frac{\Gamma(A)}{\Gamma(S)} \frac{(2-a)}{a}$

for $R > 0$, A a continuity set of Γ, and $\gamma_n = \int_{\|x\|<b_n} x/b_n P(dx)$,

(b) $\sup_n n b_n^{-2} \int_{\|x - b_n\gamma_n\| < \varepsilon b_n} \|x - b_n \gamma_n\|^2 P(dx) < \infty$ for some $\varepsilon > 0$,

(c) $\lim_m \sup_n n b_n^{-2} \int_{\|x - b_n \gamma_n\| < \varepsilon b_n} \|\pi_m (x - b_n \gamma_n)\|^2 P(dx) = 0$

for some $\varepsilon > 0$ and some sequence of projections π_m as defined in (4.2)(a),

(d) $\lim_{\varepsilon \downarrow 0} \overline{\lim_n}\, n b_n^{-2} \int_{\|x - b_n\gamma_n\| < \varepsilon b_n} (y, x - b_n \gamma_n)^2 P(dx) = 0$, Since $\lim_n b_n = \infty$, it follows easily from the dominated convergence theorem that $\gamma_n = \int_{\|x\|<b_n} x/b_n P(dx) \to 0$ in H. Hence we see (4.12)(a) and (4.3)(a) are equivalent and further that the set integrated over in (4.12)(b), (c) and (d) can be replaced by the set $\{\|x\| < \varepsilon b_n\}$ for some $\varepsilon > 0$ as in (4.3)(b), (c) and (d). Further, if (4.3)(b),(c) hold for any $\varepsilon > 0$ then easy estimates along with (4.6) and $\lim_m c_m = 0$ imply that they hold for all $\varepsilon > 0$. Hence assuming (4.3)(b),(c) hold with $\varepsilon = 1$ we see that

$$nb_n^{-2} \int_{\|x\|<b_n} \|x - b_n\gamma_n\|^2 P(dx) \leqslant nb_n^{-2}\Big\{\int_{\|x\|<b_n} \|x\|^2 P(dx) - \Big\|\int_{\|x\|<b_n} xP(dx)\Big\|^2\Big\}$$

$$\leqslant nb_n^{-2} \int_{\|x\|<b_n} \|x\|^2 P(dx) < \infty.$$

so (4.12)(b) easily follows since $\gamma_n \to 0$. Similar estimates imply that (4.3)(c) gives (4.12)(c) and (4.3)(d) gives (4.12)(d).

Thus (4.2) implies (4.12) where $\{b_n\}$ is a sequence of positive numbers converging to infinity such that $\frac{b_n}{b_{n+1}} \to 1$. Since $b_n \to \infty$, the triangular array of probability measures

$$\{\mu_j^{(n)} = \mathscr{L}(X_j/b_n)\colon 1 \leqslant j \leqslant n,\ n \geqslant 1\}$$

where $X_1, X_2, \ldots$ are independent random variables with distribution P is uniformly asymptotically negligible, and hence by the Corollary of [4], p. 331, we have that P is in the domain of attraction of $\mu = [a, \Gamma, 0]$ iff (4.12) holds. Thus (4.2) implies P is in the domain of attraction of μ.

Now assume P is in the domain of attraction of $\mu = [a, \Gamma, 0]$. Since μ is non-degenerate, we have a sequence of positive constants $\{b_n\}$ such that $\lim_n b_n = \infty$, $\lim_n \frac{b_n}{b_{n+1}} = 1$, and as remarked above (4.12)(a) holds for these b_n's. Further, we then have (4.3)(a) as $\gamma_n \to 0$, and since $\frac{b_n}{b_{n+1}} \to 1$ some elementary inequalities as in [9], p. 197, make (4.2)(b) obvious. To obtain (4.2)(a) we first observe that P in the domain of attraction of μ implies P^{π_m} is in the domain of attraction of μ^{π_m} and that the same norming constants work for P^{π_m}. Now μ^{π_m} is stable and, in fact, $\mu^{\pi_m} = [a, \Gamma_m, 0]$ where Γ_m is a finite measure on $S_m = \{x\colon \|x\| = \|\pi_m x\| = 1\}$. Since μ^{π_m} converges to the unit mass at zero as m goes to infinity we have the Fourier transforms of μ^{π_m} as given in (1.2) converging to 1. Hence by [7], p. 189, we have

$$\lim_m \Gamma_m(S_m) = 0. \tag{4.13}$$

Now choose positive constants $\{b_n'\}$ and vectors $\{a_n\}$ such that $\lim_n b_n' = \infty$, $\lim_n \frac{b_{n+1}'}{b_n'} = 1$, and

$$\mathscr{L}\Big(\frac{X_1 + \ldots + X_n}{b_n'} - a_n\Big)$$

converges weakly to μ. Then by the Corollary of [4], p. 331, and arguing as above we have

$$\lim_n nP\Big(\|x\| > Rb_n', \frac{x}{\|x\|} \in A\Big) = \frac{\Gamma(A)}{a} R^{-a} \tag{4.14}$$

and

$$\lim_n nP^{\pi_m}(\|x\| > Rb_n', x/\|x\| \in A) = \frac{\Gamma_m(A)}{a} R^{-a} \tag{4.15}$$

for each $R > 0$ and each continuity set A of $\Gamma(\Gamma_m)$. Letting $c_m = \Gamma_m(S_m) = \Gamma_m(S)$ we have $c_m > 0$ for $m > 0$ since μ is non-degenerate and by (4.13) $\lim_m c_m = 0$. Further, $P^{\pi_m}(\|x\| > Rb_n') = P(\|\pi_m x\| > Rb_n')$ so (4.14) and (4.15) along with $\lim_n \frac{b_{n+1}'}{b_n'} = 1$ and some elementary inequalities (see, for example, [9], p. 197) imply (4.2)(a). Thus the theorem is proved.

Using the ideas involved in the proof of Theorem 4.11 we can easily establish the following fact which is analogous to the result [1], p. 313, when $H = R'$.

4.16. THEOREM. *If P is the domain of attraction of a non-degenerate stable law of type a $(0 < a < 2)$ on a real separable Hilbert space H, then*

$$P(\|x\| > t) \sim \frac{L(t)}{t^a}$$

as $t \to \infty$ where $L(t)$ is a slowly varying function. Furthermore, if μ is a stable law of type a $(0 < a < 2)$ on H then

$$\mu(\|x\| > t) \sim \frac{c}{t^a}$$

as $t \to \infty$ and hence

$$\int_H \|x\|^\gamma \mu(dx) < \infty \quad \textit{for each } 0 \leqslant \gamma < a.$$

4.17. **Remark.** After this work was completed, some work by M. Kłosowska on the domain of attraction of normal distribution on Hilbert space appeared in Studia Math. 43 (1972) (pp. 195–208). Clearly this work is related to our work in Section 3, in subject matter. However we note that our methods are entirely different.

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The moduli of smoothness and convexity and the Rademacher averages of trace classes S_p $(1 \leqslant p < \infty)$*

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Abstract. It is proved that the moduli of smoothness and convexity of the trace classes S_p have the same order as the corresponding moduli of L_p $(1 < p < \infty)$ and the Rademacher averages of S_p behave in the same manner as the corresponding averages of L_p $(1 \leqslant p < \infty)$. As a corollary some results on p-absolutely summing operators are obtained.

Let $1 \leqslant p < \infty$. By S_p we denote the Banach space of compact operators on a Hilbert space H such that

$$\|A\|_p = \left(\operatorname{tr}(A^*A)^{p/2}\right)^{1/p} < \infty.$$

In the present paper we investigate some geometric properties of these spaces. It is shown that several properties are similar to the corresponding properties of L_p spaces, despite of the fact that for $p \neq 2$ and the infinite-dimensional Hilbert space H, S_p is not isomorphic to any subspace of L_p (cf. [16]). In particular the moduli of smoothness and convexity of S_p have the same order as the corresponding moduli of L_p $(1 < p < \infty)$. This fact in the case of modulus of convexity and $p \geqslant 2$ was proved by Dixmier [1].

Furthermore the Rademacher averages of S_p behave in the same manner as the corresponding averages of L_p. Namely we prove the following inequalities: There exist constants C_p such that for arbitrary $A_0, \ldots, A_n$ in S_p $(n = 0, 1, \ldots)$ we have ([1])

$$\int_0^1 \Big\| \sum_{j=0}^{n} A_j r_j(t) \Big\|_p dt \leqslant C_p \Big(\sum_{j=0}^{n} \|A_j\|_p^2 \Big)^{1/2} \quad \text{for } p \geqslant 2, \tag{0.1}$$

$$\int_0^1 \Big\| \sum_{j=0}^{n} A_j r_j(t) \Big\|_p dt \geqslant C_p \Big(\sum_{j=0}^{n} \|A_j\|_p^2 \Big)^{1/2} \quad \text{for } p \leqslant 2. \tag{0.2}$$

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([1]) Further $\|\cdot\|_a^b$ denotes $(\|\cdot\|_a)^b$.