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On joint spectra of commuting families of operators

by

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Abstract. In this paper we study basic properties of joint spectra of mutually commuting families of operators of a Banach space X. The spectra in question are the bicommutant spectrum σ'', the commutant spectrum σ', the left spectrum σ_l, the right spectrum σ_r, the spectrum σ and the approximate point spectrum σ_π (the study of the last one is the main goal of this paper). Among other results we obtain here the projection property and the spectral mapping property for the spectra σ_π, σ_l, σ_r, σ, and we disprove these properties for the spectra σ' and σ''.

Introduction. Let $\mathscr{A}$ be a commutative complex Banach algebra with unit element I. If $A_1, \ldots, A_n \in \mathscr{A}$, then the joint spectrum is defined by the relation

$$\sigma(A_1, \ldots, A_n) = \{(f(A_1), \ldots, f(A_n)) \in C^n \colon f \in \mathfrak{M}(\mathscr{A})\},$$

where $\mathfrak{M}(\mathscr{A})$ is the set of all non-zero multiplicative linear functionals defined on $\mathscr{A}$. Another formula giving the same set is

$$\sigma(A_1, \ldots, A_n) = \{(\lambda_1, \ldots, \lambda_n) \in C^n \colon \text{the elements } A_i - \lambda_i I,\ i = 1, \ldots, n, \text{ generate a proper ideal in } \mathscr{A}\}.$$

The last formula makes sense also in a non-commutative algebra (with an ideal replaced by a left or a right ideal) giving the concept of the left, resp. right joint spectrum.

The joint spectrum is a basic concept for one of most important chapters of the theory of commutative Banach algebras, namely for the operational (symbolic) calculus of analytic functions of several complex variables. The basic property of the joint spectrum permitting to formulate a theorem on functional calculus in such a way that the obtained calculus is unique is the projection property of the joint spectrum. The projection property is given by the relation

$$P\sigma(A_1, \ldots, A_n) = \sigma(A_1, \ldots, A_k),$$

where $k < n$ and P is the projection of C^n onto C^k given by $P(\lambda_1, \ldots, \lambda_n) = (\lambda_1, \ldots, \lambda_k)$.

For a non-commutative Banach algebra $\mathscr{A}$ it is not very clear which concept should be taken for the joint spectrum, even if $\mathscr{A} = L(X)$ is the algebra of all bounded operators of a Banach space X. In recent years the attention of several authors was drain to joint spectra of several mutually commuting operators of a Banach or Hilbert space (some authors considered also non-commuting n-tuples of operators, but joint spectra of such n-tuples may be empty). Various authors accepted various concepts of a joint spectrum, and so Bonsall and Duncan suggest in the book [1] the union of the left and the right spectrum. The same concept is accepted in the papers of Harte [5], [6]. Dash in [3] proposes the bicommutant spectrum σ'', while Taylor in [10], before introducing his very interesting concept, starts with the commutant joint spectrum σ'. All these spectra coincide in the case of a single operator with its usual spectrum. Some special subsets of joint spectra have also been studied, e.g. the left spectrum, or, particularly, the joint approximate point spectrum (cf. e.g. [2], [3], [6], [13]).

In this paper we discuss the basic properties of the following types of spectra: the commutant and bicommutant joint spectra σ' and σ'', the left and the right joint spectrum σ_l and σ_r, the spectrum $\sigma = \sigma_r \cup \sigma_l$ and the joint approximate point spectrum σ_π, all for an arbitrary family of pairwise commuting operators of a Banach space X. In the first section we give all necessary definitions, we establish relations between all those joint spectra and we prove that they always are non-void compact subsets of a suitable product of the complex planes. In the second section we discuss the projection property of these spectra. First we show that if a joint spectrum is defined only for finite families of mutually commuting operators of a Banach space X and if this spectrum possesses the projection property, then it can be uniquely extended to a joint spectrum defined on the family of all subsets of $L(X)$ consisting of pairwise commuting operators, and possessing there the projection property too. This result permits e.g. to define the Taylor spectrum on the set $\mathfrak{c}(X)$ consisting of all families of mutually commuting endomorphisms of X. Then we prove that the joint approximate point spectrum possesses the projection property. This is a generalization of a result of Bunce given in [2], where it was shown that the joint approximate point spectrum defined on finite subsets consisting of pairwise commuting operators of a Hilbert space possesses the projection property. Using this result we prove that the left spectrum, the right spectrum and the spectrum also possess the projection property. These results were obtained earlier by Harte in [5] and [6], we decided, however, not to withdraw the proofs when we learned about the results of Harte after submitting the first version of this paper for publication since our proofs are different and clarify the relations between different types of joint spectra. Finally, using an example constructed by Taylor in [10] we disprove the projection property for the commutant and bicommutant joint spectra. In the third section, added in the second version of this paper, we discuss the spectral mapping theorem for all joint spectra considered above. We prove a theorem giving some conditions equivalent to the spectral mapping property and applying this theorem we obtain the spectral mapping theorem for σ, σ_l, σ_r and σ_π. Here only the result on joint approximate point spectra is new, the other results are due to Harte [5] and [6]. The spectral mapping theorem fails for the commutant and for the bicommutant spectrum. This fact rather eliminates these concepts as possible generalizations of a joint spectrum and gives one more argument towards considering the Taylor spectrum as the proper generalization. This is resonable also since the Taylor spectrum not only possesses the projection property and the spectral mapping property, but also, as shown in [11], it is suitable for building up an operational calculus.

The methods used in this paper are based on the concept of an ideal consisting of joint topological divisors of zero. This shows that there are strong relations between this concept and the concepts of joint approximate point spectrum and other types of joint spectra.

§1. Basic definitions and properties of the joint spectra. Let X be a complex Banach space. We denote by $L(X)$ the Banach algebra of all continuous endomorphisms of X. If S is a non-void subset of $L(X)$ then its commutant S' consists of all operators $B \in L(X)$ such that $AB = BA$ for all $A \in S$. Clearly $S_1 \subset S_2$ implies $S_2' \subset S_1'$. Also $S = \bigcup S_\alpha$ implies $S' = \bigcap S_\alpha'$. The bicommutant S'' of S is defined as $(S')'$.

If S consists of pairwise commuting operators then $S \subset S'$ and so $S'' \subset S'$. We have also $S \subset S''$ since each element of S commutes with each element of S'. If we denote by $M(S)$ the family of all maximal commutative subalgebras of $L(X)$ containing S, we have

$$S' = \bigcup M(S) \tag{1.1}$$

and

$$S'' = \bigcap M(S), \tag{1.2}$$

where $\bigcup M(S) = \bigcup \{\mathscr{A}\colon \mathscr{A} \in M(S)\}$, and similarly for $\bigcap M(S)$.

For any non-void set S of pairwise commuting operators of X the bicommutant S'' is a commutative Banach subalgebra of $L(X)$, while, in general, the commutant S' is a non-commutative Banach algebra containing S in its center.

1.1. Definition. Let X be a complex Banach space and $S = (A_t)_{t \in T}$ a family of pairwise commuting elements of $L(X)$, where T is a set of indices. The *bicommutant joint spectrum* $\sigma''(S)$ of the family S is defined as the set of all points $\lambda \in C^T$, $\lambda = (\lambda_t)_{t\in T}$, such that the closed ideal generated by the set $\{A_t - \lambda_t I\colon t \in T\}$ is a proper ideal in the algebra S''.

In the above definition and in the sequel I denotes the identity operator.

Thus a point $\lambda \in C^T$ is not in the spectrum $\sigma''(S)$ if and only if there exist a finite set $t_1, \ldots, t_n$ of indices and a finite number of operators $B_1, \ldots, B_n \in S''$ such that

$$\sum_{i=1}^{n} B_i(A_{t_i} - \lambda_{t_i} I) = I. \tag{1.3}$$

If S consists of a single operator A, or $S = (A_1, \ldots, A_n)$, then we also write $\sigma''(A)$, or $\sigma''(A_1, \ldots, A_n)$ instead of $\sigma''(S)$.

The above defined bicommutant spectrum is sometimes called a spectrum (e.g. in the paper [3] in the case when X is a Hilbert space and S is a finite set). It is clear that if S consists of a single operator A, then $\sigma''(A)$ coincides with the usual spectrum of A.

1.2. Definition. Let X and S be as above. The *commutant spectrum* $\sigma'(S)$ is the set of all points in C^T, such that the set $\{A_t - \lambda_t I\colon t \in T\}$ is contained in a proper (two-sided) ideal of the Banach algebra S'.

Thus $\lambda \in C^T$ is not in $\sigma'(S)$ if and only if there exist indices $t_1, \ldots, t_n \in T$ and operators $B_1, \ldots, B_n \in S'$ such that relation (1.3) is satisfied. As before we write $\sigma'(A_1, \ldots, A_n)$ instead of $\sigma'(S)$ for a finite $S = (A_1, \ldots, A_n)$ and in this notation $\sigma'(A)$ coincides with the usual spectrum of an operator A.

The following proposition is an immediate consequence of the definitions.

1.3. Proposition. *For any family S of mutually commuting operators in $L(X)$ the following inclusion holds true*:

$$\sigma'(S) \subset \sigma''(S). \tag{1.4}$$

1.4. Definition. Let X and S be as above. The *joint approximate point spectrum* $\sigma_\pi(S)$ is defined as the set of all $\lambda \in C^T$ such that there exists a net $(x_a) \subset X$, $\|x_a\| = 1$ for all α, with

$$\lim_{a} (A_t - \lambda_t I) x_a = 0 \tag{1.5}$$

for each $t \in T$.

We have $\sigma_\pi(S) \subset \sigma'(S)$, since if $\lambda \in \sigma_\pi(S) \setminus \sigma'(S)$, then we can find indices $t_1, \ldots, t_n \in T$ and operators $B_1, \ldots, B_n \in S'$ such that formula (1.3) holds true. Acting with both sides of this formula on elements x_a we obtain $\sum_{i=1}^{n} B_i(A_{t_i} - \lambda_{t_i} I) x_a = x_a$, what is impossible since the norms of the left-hand side elements tend to 0 with α, while the norms of the right-hand side are always equal 1.

In the case when $S = \{A\}$ we have, in general, $\sigma_\pi(A) \neq \sigma(A)$, but only $\sigma_\pi(A) \subset \sigma(A)$.

1.5. Definition. Let X and S be as above. The *left* (*right*) *joint spectrum* $\sigma_l(S)\,(\sigma_r(S))$ is defined as the set of all $\lambda \in C^T$ such that the family $(A_t - \lambda_t I)_{t \in T}$ generates in the algebra $L(X)$ a proper left (right) ideal.

Thus $\lambda \in C^T \setminus \sigma_l(S)$ means that there exist indices $t_1, \ldots, t_n \in T$ and operators $B_1, \ldots, B_n \in L(X)$ such that relation (1.3) holds true. Similaryl $\lambda \in C^T \setminus \sigma_r(S)$ if there are indices $t_1, \ldots, t_n \in T$ and operators $B_1, \ldots, B_n \in L(X)$ such that

$$\sum_{i=1}^{n} (A_{t_i} - \lambda_{t_i} I) B_i = I. \tag{1.6}$$

Clearly, $\sigma_l(S) \subset \sigma'(S)$ and $\sigma_r(S) \subset \sigma'(S)$.

1.6. Definition. Let X and S be as above. The *joint spectrum* $\sigma(S)$ is defined as

$$\sigma(S) = \sigma_l(S) \cup \sigma_r(S).$$

Such a definition is given e.g. in the book of Bonsall and Duncan [1], or in the papers of Harte [5] and [6]. For a single operator A this spectrum coincides with the usual spectrum of A, while, in general, the left or right spectrum does not.

Let us also remark that in the paper [3] the expression "joint approximate point spectrum", for X a Hilbert space and finite S, is defined as our left spectrum. In this case, however, we have $\sigma_\pi(S) = \sigma_l(S)$. This relation, proved in [3], is also true for infinite families S, what immediately follows from formulas (1.5) and (1.6). We shall see that for an arbitrary Banach space X the set $\sigma_\pi(S)$ need not be equal to $\sigma_l(S)$, even if S consists of a single operator.

1.7. Proposition. *Let X and S be as above. We have*

$$\sigma_\pi(S) \subset \sigma_l(S) \tag{1.7}$$

and no other relation of inclusion holds in general among $\sigma_\pi(S)$, $\sigma_r(S)$ and $\sigma_l(S)$.

Proof. Let A be a left-invertible but non-invertible element of X (this can be realised e.g. if X is a Hilbert space, A is a non-unitary isometry of X and $B = A^*$). We have $BA = I$, while the closures $\overline{L(X)B}$ and $\overline{AL(X)}$ are proper respectively left and right ideals in $L(X)$. Thus we have $0 \in \sigma_r(A)$, $0 \in \sigma_l(B)$, $0 \notin \sigma_l(A)$ and $0 \notin \sigma_r(B)$. Consequently neither the relation $\sigma_r(S) \subset \sigma_l(S)$, nor $\sigma_l(S) \subset \sigma_r(S)$ is true in general, even if S consists of a single operator and X is a Hilbert space. Since for a Hilbert space X it is always $\sigma_\pi(S) = \sigma_l(S)$, we see that neither $\sigma_r(S) \subset \sigma_\pi(S)$, nor $\sigma_\pi(S) \subset \sigma_r(S)$ is true for all X and S.

We shall now show that there exists a Banach space X such that the relation $\sigma_l(A) \subset \sigma_\pi(A)$ is not satisfied for every $A \in L(X)$. Suppose, to the contrary, that for every X and every $A \in L(X)$ it is always $\sigma_l(A) \subset \sigma_\pi(A)$, and in particular $0 \notin \sigma_\pi(A)$ implies $0 \notin \sigma_l(A)$. It means that an operator A possesses a left inverse $B \in L(X)$, whenever A is an isomorphism into. So suppose that A is an isomorphism into and put $X_0 = AX$. It is a closed subspace of X, isomorphic to X and $P = AB$ is a projection of X onto X_0. So our assumption implies that whenever there is a subspace $X_0 \subset X$, isomorphic to X, then there exists a projection P of X onto X_0. But this is false e.g. if $X = C(0, 1)$ (cf [8]). Thus relation $\sigma_l(S) \subset \sigma_\pi(S)$ cannot be always true.

Finally we prove relation (1.7). Let $\lambda \in C^T \setminus \sigma_l(S)$. So there are indices $t_1, \ldots, t_n \in T$ and elements $B_1, \ldots, B_n \in L(X)$ such that relation (1.3) is satisfied. If there exists a net $(x_a) \subset X$, $\|x_a\| = 1$, satisfying relation (1.5) for each $t \in T$, then from (1.3) we obatin

$$\sum_{i=1}^{n} B_i(A_{t_i} - \lambda_{t_i} I) x_a = x_a,$$

what is a nonsense since the left-hand net tends to 0, while the norms on the right-hand side are equal 1. The contradiction shows that $\lambda \in C^T \setminus \sigma_\pi(S)$ and thus we obtain relation (1.7).

The following proposition gives a useful in the sequel characterization of the spectrum $\sigma_\pi(S)$.

1.8. Proposition. *Let X and S be as above. We have $\lambda \in \sigma_\pi(S)$ if and only if for every finite subset $(t_1, \ldots, t_n) \subset T$ it is*

$$\inf\Big\{\sum_{i=1}^{n} \|(A_{t_i} - \lambda_{t_i} I)x\| \colon x \in X, \|x\| = 1\Big\} = 0. \tag{1.8}$$

Proof. If $\lambda \in \sigma_\pi(S)$, then by (1.5) we have

$$\lim_a \sum_{i=1}^{n} \|(A_{t_i} - \lambda_{t_i} I)x_a\| = 0$$

for each finite subset $(t_1, \ldots, t_n) \subset T$, and so relation (1.8) holds true. On the other hand, suppose that for a fixed $\lambda \in C^T$ relation (1.8) is satisfied for every choice of a finite number of indices $t_1, \ldots, t_n \in T$. First we form a directed set consisting of pairs of the form $a = (\tau, k)$ where τ is a finite subset of T and k is a positive integer. For $a_1 = (\tau_1, k_1)$ we write $a_1 \geqslant a$ in the case when $\tau_1 \supset \tau$ and $k_1 \geqslant k$. We obtain in this way a directed set suitable for building up the desired net. We now construct the net (x_a) in the following way. If $a = (\tau, k)$ with $\tau = (t_1, \ldots, t_n)$, then by (1.8) there exists an element $x \in X$, $\|x\| = 1$, such that

$$\sum_{i=1}^{n} \|(A_{t_i} - \lambda_{t_i} I)x\| \leqslant \frac{1}{k},$$

and we choose such an element x as x_a. It is clear that for the obtained net (x_a) relation (1.5) is satisfied for each $t \in T$.

1.9. Proposition. *Let X and S be as above. All the sets: $\sigma_\pi(S)$, $\sigma_l(S)$, $\sigma_r(S)$, $\sigma(S)$, $\sigma'(S)$, $\sigma''(S)$ are compact subsets of C^T.*

Proof. It is clear that all considered spectra are contained in $K = \prod_{t \in T} K_t$, where $K_t = \{\lambda_t \in C, |\lambda_t| \leqslant \varrho_t\}$ with $\varrho_t = \lim_n \|A_t^n\|^{1/n}$ being the spectral radius of the operator A_t; $\prod K_t$ denotes the cartesian product.

Since K is a compact subset of C^T, it is sufficient to prove that all considered spectra are closed in C^T.

Take a point $\lambda \notin \sigma_\pi(S)$. By Proposition 1.8 there exists a finite subset $(t_1, \ldots, t_n) \subset T$ such that

$$\inf\Big\{\sum_{i=1}^{n} \|(A_{t_i} - \lambda_{t_i} I)x\| \colon x \in X, \|x\| = 1\Big\} = \delta > 0.$$

We take a neighbourhood V of λ in C^T given by

$$V = \{\mu \in C^T \colon |\mu_{t_i} - \lambda_{t_i}| < \delta/2n, i = 1, \ldots, n\}.$$

For any $\mu \in V$ and $x \in X$ with $\|x\| = 1$ we have

$$\sum_{i=1}^{n} \|(A_{t_i} - \mu_{t_i} I)x\| \geqslant \sum_{i=1}^{n} \big(\|(A_{t_i} - \lambda_{t_i} I)x\| - \|(\lambda_{t_i} - \mu_{t_i})x\|\big)$$
$$= \sum_{i=1}^{n} \|(A_{t_i} - \lambda_{t_i} I)x\| - \sum_{i=1}^{n} |\lambda_{t_i} - \mu_{t_i}| \geqslant \delta - \delta/2 = \delta/2,$$

and so, by Proposition 1.8 we obtain $\mu \notin \sigma_\pi(S)$, what implies that $C^T \setminus \sigma_\pi(S)$ is an open subset of C^T.

If $\lambda \notin \sigma_l(S)$, then there exist indices $t_1, \ldots, t_n \in T$ and operators $B_1, \ldots, B_n \in L(X)$ such that formula (1.3) is satisfied. We can find a $\delta > 0$ in such a way that for any $\mu \in C^T$ with $|\mu_{t_i} - \lambda_{t_i}| < \delta$ for $i = 1, \ldots, n$ it is

$$\Big\|\sum_{i=1}^{n} B_i(A_{t_i} - \mu_{t_i} I) - I\Big\| < 1. \tag{1.9}$$

This implies that the element

$$\sum_{i=1}^{n} B_i(A_{t_i} - \mu_{t_i} I) \tag{1.10}$$

has in $L(X)$ an inverse Q, and so

$$\sum_{i=1}^{n} QB_i(A_{t_i}-\lambda_{t_i}I) = I,$$

which means, by (1.3), that $\mu \notin \sigma_1(S)$, and thus the set $C^T \setminus \sigma_1(S)$ is open in C^T. Similarly one can prove that $\sigma_r(S)$ is a compact subset of C^T and so is the set $\sigma(S)$.

In order to prove the compactness of $\sigma''(S)$ or $\sigma'(S)$ we find in S'' or in S' elements $B_1, \ldots, B_n$ satisfying relation (1.9) for μ sufficiently close to λ in C^T. It is now sufficient to observe that the inverses of elements (1.10) also belong to S'' or to S' what gives the desired conclusion.

Applying the same method as in the paper [13] we shall prove that the joint spectra in question never are void for an arbitrary family S of mutually commuting operators in $L(X)$, where X is an arbitrary Banach space. First we recall some concepts which will be useful in the proof.

1.10. Definition. Let $\mathscr{A}$ be a commutative complex Banach algebra with unit element I. A subset $S \subset \mathscr{A}$ is said *to consist of joint topological divisors of zero* if there exists a net $(Q_a) \subset \mathscr{A}$, $\|Q_a\| = 1$, such that

$$\lim_a \|AQ_a\| = 0 \tag{1.11}$$

for all $A \in S$.

The main result of [13] states that if M is a maximal ideal of $\mathscr{A}$ belonging to the Shilov boundary $\Gamma(\mathscr{A})$, then M consists of joint topological divisors of zero.

Applying this result we obtain the following theorem.

1.11. Theorem. *Let X be a complex Banach space and $S = (A_t)_{t\in T}$ a family of pairwise commuting operators in $L(X)$. Then each of the sets $\sigma_\pi(S)$, $\sigma_1(S)$, $\sigma_r(S)$, $\sigma(S)$, $\sigma'(S)$, $\sigma''(S)$ is a non-void compact subset of C^T.*

Proof. By Proposition 1.9 all considered spectra are compact subsets of C^T. Since $\sigma_\pi(S) \subset \sigma_1(S) \subset \sigma(S) \subset \sigma'(S) \subset \sigma''(S)$, it is sufficient to prove that $\sigma_\pi(S)$ and $\sigma_r(S)$ are non-void subsets of C^T. Denote by $\mathscr{A}$ the smallest closed subalgebra of $L(X)$ containing the set S. It is a commutative Banach algebra. Let f be a multiplicative-linear functional in $\mathscr{A}$ whose kernel belongs to the Shilov boundary $\Gamma(\mathscr{A})$. By the main result of [13] this kernel consists of joint topological divisors of zero. Define an element $\lambda \in C^T$ by setting $\lambda_t = f(A_t)$ for each $t \in T$. We shall show that $\lambda \in \sigma_\pi(S) \cap \sigma_r(S)$. Since the set $\{A_t - \lambda_t I\colon\ t \in T\}$ consists of joint topological divisors of zero, there is a net $(Q_a) \subset \mathscr{A}$, $\|Q_a\| = 1$, such that

$$\lim_a \|(A_t-\lambda_t I)Q_a\| = 0 \tag{1.12}$$

for each $t \in T$. We now choose elements $y \in X$ such that $\|y_a\| \leqslant 2$ and $\|Q_a y_a\| = 1$ for all a. Setting $x_a = Q_a y_a$ we have $\|x_a\| = 1$ and

$$\|(A_t-\lambda_t I)x_a\| = \|(A_t-\lambda_t I)Q_a y_a\| \leqslant 2\|(A_t-\lambda_t I)Q_a\|$$

for each $t \in T$. By formulas (1.12) and (1.5) we see that $\lambda \in \sigma_\pi(S)$.

If $\lambda \notin \sigma_r(S)$, then there exist indices $t_1, \ldots, t_n \in T$ and operators $B_1, \ldots, B_n \in L(X)$ such that formula (1.6) holds. Multiplying both sides of this formula by Q_a from the left we obtain

$$\sum_{i=\lambda}^{n} Q_a(A_{t_i}-\lambda_{t_i}I)B_i = Q_a,$$

what is a nonsense, since the left-hand side net tends to 0 by (1.12), while the norms of operators occuring on the right-hand side are always equal 1. The contradiction shows that $\lambda \in \sigma_r(S)$.

The above proof is short and easy, but it gives only the existence of some elements in the joint spectra. Another proof for the joint approximative point spectrum was given in [2], in the case when X is a Hilbert space and S is a finite set of mutually commuting operators. In this proof it was shown that the joint approximative point spectrum has the finite projection property (cf. Definition 2.2). It implies not only that the spectrum in question is never void, but also gives an important information about the spectrum itself. In the next section we discuss this property for considered above joint spectra of arbitrary families of mutually commuting operators defined on an arbitrary Banach space.

Finally we would like to make some comments on the definitions of joint spectra considered in this paper. The convention assumed here can be called a *spatial convention*, i.e. we relate all concepts to a Banach space X. Another possible convention is a Banach algebra convention (considered e.g. in the book of Bonsall and Duncan [1], or in the papers of Harte [5] and [6]), it has same advantages and some disadvantages with respect to the spatial one. All definitions in this section can be easily reformulated into definitions dealing with a Banach algebra only. So if $\mathscr{A}$ is an arbitrary Banach algebra with unit I, then for $S \subset \mathscr{A}$ consisting of pairwise commuting elements of $\mathscr{A}$ its commutant S' and bicommutant S'' are suitable subsets of $\mathscr{A}$ and we obtain more general versions of σ' and σ'' which give the concepts defined by 1.1 and 1.2 for $\mathscr{A} = L(X)$. Also the left and right joint spectra $\sigma_{1,\mathscr{A}}$ and $\sigma_{r,\mathscr{A}}$ in $\mathscr{A}$ can be defined similarly as in 1.4, when taking B_i from $\mathscr{A}$. We have, clearly,

$$\sigma_1(S) = \sigma_{1,\mathscr{A}}(S) \tag{1.13}$$

and

$$\sigma_r(S) = \sigma_{r,\mathscr{A}}(S) \tag{1.14}$$

if $\mathscr{A} = L(X)$. In the Banach algebra convention it is possible to express the left spectrum by the right one (we shall use it in the next section). Namely, for a given Banach algebra $\mathscr{A}$ we define $\tilde{\mathscr{A}}$ as a Banach algebra with the same elements, the same norm, and the same linear structure as the Banach algebra $\mathscr{A}$ but with a new multiplication defined as $A \circ B = BA$. We obtain in this way a Banach algebra which is antyisomorphic to $\mathscr{A}$ and for any n-tuple of mutually commuting elements $(A_1, \ldots, A_n) \subset \mathscr{A} (= \tilde{\mathscr{A}}$ as the set) we have

$$\sigma_{l,\mathscr{A}}(A_1, \ldots, A_n) = \sigma_{r,\tilde{\mathscr{A}}}(A_1, \ldots, A_n). \tag{1.15}$$

While for σ', σ'', σ_r, σ_l, and σ the Banach algebra convention is less restrictive then the spatial one, it turns out that for the joint approximate point spectrum the spatial convention is more general. We obtain here the Banach algebra definition, or rather two of them assuming $X = \mathscr{A}$ and interpreting the elements of $\mathscr{A}$ as operators of left multiplication $B \to AB$, or right multiplication $B \to BA$. We obtain in this way the concepts of a left and of a right approximate point spectrum as it is done e.g. in the paper [6]. Here the Banach algebra convention is more restrictive and since the main subject of our article is the study of the joint approximate point spectrum, we decided, also for the unicity of exposition, to use the spatial convention. The reader will observe that all presented here results are valid also in the Banach algebra convention essentially with the same proofs.

§ 2. The projection property.

2.1. Definition. Let X be a complex Banach space. We denote by $\mathfrak{c}(X)$ the family of all subsets $S \subset L(X)$ consisting of pairwise commuting operators, and by $\mathfrak{c}_0(X)$ the family consisting of all finite members of $\mathfrak{c}(X)$. A spectral system on $\mathfrak{c}(X)$, or on $\mathfrak{c}_0(X)$ is a map $S \to \sigma^*(S)$, where $S = (A_t)_{t \in T}$ belongs to $\mathfrak{c}(X)$ or $\mathfrak{c}_0(X)$, and $\sigma^*(S)$ is a subset of $\boldsymbol{C}^T$, satisfying the following conditions:

(a) For each S the set $\sigma^*(S)$ is a compact subset of $\boldsymbol{C}^T$, where T is the index set for S.

(b) If S consists of a single operator A, then $\sigma^*(A) = \sigma^*(S)$ is a non-void subset of the spectrum $\sigma(A)$.

2.2. Definition. Let X be a complex Banach space and let $S \to \sigma^*(S)$ be a spectral system on $\mathfrak{c}(X)$ or $\mathfrak{c}_0(X)$. We say that the spectral system $S \to \sigma^*(S)$, or the spectrum $\sigma^*(S)$ has a *projection property* if for each S in the domain of definition of the spectrum and for each $S' \subset S$ it is

$$\sigma^*(S') = P\sigma^*(S), \tag{2.1}$$

where P is the natural projection of $\boldsymbol{C}^T$ onto $\boldsymbol{C}^{T'}$, provided $S = (A_t)_{t \in T}$ and $S' = (A_t)_{t \in T'}$ with $T' \subset T$. In the case when $S \to \sigma^*(S)$, restricted to $\mathfrak{c}_0(X)$, has the projection property we say that the spectrum $\sigma^*(S)$ has a *finite projection property* (for spectra defined only on $\mathfrak{c}_0(X)$ this property coincides with the projection property).

The projection property is an important property for many reasons. One, mentioned in the introduction, is that a spectrum possessing this property may be suitable for building up an operational calculus of analytic functions. Another related reason is that the projection property is a necessary condition for having the spectral mapping property (cf. Section 3). Yet another is that a spectrum possessing the projection property is never void, since $\sigma^*(S)$ is a subdirect product of non-void sets $\sigma^*(A_t)$, $t \in T$.

We shall now show that a spectral system $S \to \sigma^*(S)$ defined on $\mathfrak{c}_0(X)$ and having the projection property can be uniquely extended to a spectral system defined on $\mathfrak{c}(X)$ and having this property too.

2.3. Theorem. *Let X be a complex Banach space and let $S \to \sigma^*(S)$ be a spectral system defined on $\mathfrak{c}_0(X)$ and possessing the projection property. There exists a unique spectral system defined on $\mathfrak{c}(X)$, possessing the projection property, whose restriction to $\mathfrak{c}_0(X)$ coincides with the given system.*

Proof. We fix $S \in \mathfrak{c}(X)$, $S = (A_t)_{t \in T}$, and for any finite subset $T' \subset T$, $T' = (t_1, \ldots, t_n)$, we put $P_{T'}(\lambda) = (\lambda_{t_1}, \ldots, \lambda_{t_n})$, $\lambda \in \boldsymbol{C}^T$.

We define

$$\tilde{\sigma}(S) = \{\lambda \in \boldsymbol{C}^T\colon P_{T'}(\lambda) \in \sigma^*(A_{t_1}, \ldots, A_{t_n}) \text{ for all finite } T' \subset T\}. \tag{2.2}$$

Thus $\tilde{\sigma}(S)$ is the projective limit of the family of compact sets $\sigma^*(A_{t_1}, \ldots, A_{t_n})$ indexed by finite subsets $(t_1, \ldots, t_n) \subset T$ and ordered by inclusion of indices. From the well-known properties of projective limits it follows that $\tilde{\sigma}(S)$ is a non-void subset of $\boldsymbol{C}^T$ (cf. e.g. [4]). Moreover, the map $S \to \tilde{\sigma}(S)$ satisfies conditions (a) and (b) of Definition 2.1 and also condition (2.1) of Definition 2.2. Since for any finite set S we have clearly $\tilde{\sigma}(S) = \sigma^*(S)$, we see that $S \to \tilde{\sigma}(S)$ is an extension of $S \to \sigma^*(S)$ onto $\mathfrak{c}(X)$.

We shall now show that such an extension is unique in the family of all spectral systems satisfying the projection property. Suppose then that there exists another spectral system $S \to \bar{\sigma}(S)$ defined on the whole of $\mathfrak{c}(X)$ and satisfying there the projection property, which is an extension of the system $S \to \sigma^*(S)$. We have to show that $\bar{\sigma}(S) = \tilde{\sigma}(S)$ for each $S \in \mathfrak{c}(X)$. From the projection property of the spectrum $\bar{\sigma}$ and from formula (2.2) it follows that $\bar{\sigma}(S) \subset \tilde{\sigma}(S)$ for each $S \in \mathfrak{c}(X)$. Suppose that for some $S \in \mathfrak{c}(X)$ it is $\bar{\sigma}(S) \neq \tilde{\sigma}(S)$, so that there is a $\lambda \in \tilde{\sigma}(S) \setminus \bar{\sigma}(S)$. Since both sets $\tilde{\sigma}(S)$ and $\bar{\sigma}(S)$ are compact subsets of $\boldsymbol{C}^T$, there exists a neighbourhood V of λ in C^T which is disjoint from $\tilde{\sigma}(S)$. We can assume

$$V = \{\mu \in C^T\colon\ |\mu_{t_i} - \lambda_{t_i}| < \varepsilon,\ i = 1, 2, \ldots, n\}$$

for some positive ε. In particular, if $P_{T'}(\lambda) = P_{T'}(\mu)$, where $T' = (t_1, \ldots, t_n)$, then $\mu \in V$ and so $\mu \notin \tilde{\sigma}(S)$. But it is impossible since $\lambda \in \tilde{\sigma}(S)$, and $P_{T'}(\tilde{\sigma}(S)) = P_{T'}(\tilde{\sigma}(S)) = \sigma^*(A_{t_1}, \ldots, A_{t_n})$.

Let us remark that by the above theorem we can extend the definition of spectrum, which is defined on $\mathfrak{c}_0(X)$, onto the whole of $\mathfrak{c}(X)$, provided it has the projection property. Such a situation holds e.g. for the Taylor spectrum $\sigma_{T'}(S)$ (cf. [10]), defined on $\mathfrak{c}_0(X)$ for an arbitrary Banach space X.

Let us also remark that if we want to know whether a spectral system $S \to \sigma^*(S)$ defined on $\mathfrak{c}(X)$ possesses the projection property, we have only to verify whether it has the finite projection property and whether $\sigma^*(S) = \tilde{\sigma}(S)$ for all $S \in \mathfrak{c}(X)$, where $\tilde{\sigma}(S)$ is given by formula (2.2). We shall use this remark in proving that some of joint spectra considered in the preceding section possess the projection property. To this end we observe that from Proposition 1.8 and from the remarks after Definition 1.5 it follows immediately the following lemma.

2.4. Lemma. *For the spectra $\sigma_\pi(S)$, $\sigma_1(S)$, $\sigma_T(S)$ and $\sigma(S)$ relation (2.2) holds true.*

2.5. Theorem. *For an arbitrary complex Banach space X the joint approximate point spectrum $\sigma_\pi(S)$, $S \in \mathfrak{c}(X)$, possesses the projection property.*

This is a generalization of a result of Bunce, given in [2] for $\sigma_\pi(S)$ defined on $\mathfrak{c}_0(H)$, where H is a Hilbert space, onto arbitrary Banach spaces and onto infinite families of pairwise commuting operators.

In view of the remarks preceding Lemma 2.4 and by the lemma itself it is sufficient to prove that the joint approximate point spectrum possesses the finite projection property. The proof will be preceded by a few lemmas. The method applied here is the same as that in the paper [9], where it was used for establishing an important property of ideals consisting of joint topological divisors of zero.

2.6. Lemma. *Let $\mathscr{A}$ be a commutative complex algebra with unit element I. Let p be a seminorm on $\mathscr{A}$ such that the multiplication in $\mathscr{A}$ is separately continuous with respect to this seminorm,* i.e. *for each $A \in \mathscr{A}$ there exists a positive constant C_A such that*

$$p(AR) \leqslant C_A p(R) \tag{2.3}$$

for all $R \in \mathscr{A}$. Then

$$q(A) = \inf\{C\colon\ p(AR) \leqslant Cp(R) \text{ for all } R \text{ in } \mathscr{A}\}$$

is a submultiplicative seminorm on $\mathscr{A}$. Moreover, if $p(I) \neq 0$ then $q(I) = 1$.

Proof. Clearly q is a non-negative homogeneous functional defined on $\mathscr{A}$ and $p(AR) \leqslant q(A)p(R)$ for all $A, R \in \mathscr{A}$. So for any $A, B \in \mathscr{A}$ we have

$$p(ABR) \leqslant q(A)p(BR) \leqslant q(A)q(B)p(R),$$

what implies

$$q(AB) \leqslant q(A)q(B).$$

Similarly

$$p((A+B)R) \leqslant p(AR) + p(BR) \leqslant q(A)p(R) + q(B)p(R)$$

and consequently

$$q(A+B) \leqslant q(A) + q(B)$$

for all $A, B \in \mathscr{A}$. It is also obvious that $p(I) \neq 0$ implies $q(I) = 1$.

2.7. Lemma. *Let $\mathscr{A}$ be as above and let q be a submultiplicative seminorm on $\mathscr{A}$. Suppose that for an element $A \in \mathscr{A}$ the relation*

$$q(R(cI - A)) \geqslant \varepsilon_c q(R) \tag{2.4}$$

holds for each complex number c and each $R \in \mathscr{A}$, where ε_c is a positive constant depending on c only. Then $q(I) = 0$.

Proof. Suppose that $q(I) \neq 0$. Then $J = \{R \in \mathscr{A}\colon q(R) = 0\}$ is a proper ideal in $\mathscr{A}$, and $\mathscr{A}/J$ is a unital normed algebra with the norm q. Denote by $\bar{\mathscr{A}}$ the completion of $\mathscr{A}/J$ in this norm. If Φ is the natural map of $\mathscr{A}$ into $\bar{\mathscr{A}}$, then (2.4) implies

$$q\big(\Phi(R)(c\bar{I} - \Phi(A))\big) \geqslant \varepsilon_c q(\Phi(R)),$$

where $\bar{I} = \Phi(I)$ is the unit element of $\bar{\mathscr{A}}$. This inequality means that $\Phi(A) - c\bar{I} \neq 0$ for every complex c and $\Phi(A) - c\bar{I}$ is never a topological divisor of zero in $\bar{\mathscr{A}}$. But such a situation is impossible, e.g. for c belonging to the boundary of the spectrum of $\Phi(A)$ in $\bar{\mathscr{A}}$ (cf. e.g. [14]).

2.8. Lemma. *Let X be a complex Banach space and let $(A_1, \ldots, A_n, A_0)$ be an $(n+1)$-tuple of pairwise commuting operators in $L(X)$ such that*

$$\inf\Big\{\sum_{i=1}^{n} \|A_i x\|\colon\ \|x\| = 1\Big\} = 0. \tag{2.5}$$

Then there exists a complex scalar c such that

$$\inf\Big\{\sum_{i=1}^{n} \|A_i x\| + \|(A_0 - cI)x\|\colon\ \|x\| = 1\Big\} = 0.$$

Proof. Suppose that the conclusion does not hold. Thus there exist a Banach space X and pairwise commuting operators $A_0, A_1, \ldots, A_n$ in

$L(X)$ such that for each complex number c

$$\inf\Big\{\sum_{i=1}^{n}\|A_i x\|+\|(A_0-cI)x\|:\ \|x\|=1\Big\}=\varepsilon_c>0 \tag{2.6}$$

and for the n-tuple $(A_1,\ldots,A_n)$ formula (2.5) is satisfied. So there exists a sequence $(x_k)\subset X$, $\|x_k\|=1$ and $\lim_k\|A_i x_k\|=0$ for $i=1,\ldots,n$. Let $\mathscr{A}$ be the unital algebra of operators on X generated by the operators $A_0,\ldots,A_n$. It is a commutative subalgebra of $L(X)$. For $A\in\mathscr{A}$ we put

$$p_k(A)=\|Ax_k\|, \tag{2.7}$$

for $k=1,2,\ldots$ By (2.6) we have

$$\sum_{i=1}^{n}\|A_i x\|+\|(A_0-cI)x\|\geqslant\varepsilon_c\|x\|$$

for all $x\in X$, what implies

$$\sum_{i=1}^{n}p_k(A_iB)+p_k\big((A_0-cI)B\big)\geqslant\varepsilon_c p_k(B) \tag{2.8}$$

for all $B\in\mathscr{A}$, $k=1,2,\ldots$

All seminorms p_k satisfy relation (2.3), since

$$p_k(AB)\leqslant p_k(A)\|B\| \tag{2.9}$$

for all $A,B\in\mathscr{A}$.

From formula (2.7) we see that $p_k(A)\leqslant\|A\|$ for each $A\in\mathscr{A}$, and so each p_k can be regarded as an element of the cartesian product of closed segments

$$P=\prod_{A\in\mathscr{A}}[0,\|A\|],$$

which is a compact space. Consequently we can take a convergent subnet (p_a) of the sequence (p_k), converging to an element $p\in P$. We have $0\leqslant p(A)\leqslant\|A\|$ and $p(A)=\lim_a p_a(A)$ for each $A\in\mathscr{A}$. It implies that p is a seminorm on $\mathscr{A}$, which by (2.9) satisfies

$$p(AB)\leqslant p(A)\|B\| \tag{2.10}$$

for all $A,B\in\mathscr{A}$. We note also that

$$\lim_k p_k(BA_i)=\lim_k\|BA_ix_k\|=0,$$

what implies $p(BA_i)=0$ for all $B\in\mathscr{A}$ and $i=1,\ldots,n$.

Relation (2.8) implies

$$\sum_{i=1}^{n}p(A_iB)+p\big((A_0-cI)B\big)\geqslant\varepsilon_c p(B),$$

or, since $p(A_iB)=0$,

$$p\big((A_0-cI)B\big)\geqslant\varepsilon_c p(B) \tag{2.11}$$

for all $B\in\mathscr{A}$. We have also $p_k(I)=\|x_k\|=1$ for all k and so $p(I)=1$.

By relation (2.10) we can apply Lemma 2.6 and consequently there exists on $\mathscr{A}$ a submultiplicative seminorm q satisfying

$$q(I)=1. \tag{2.12}$$

On the other hand for all $A,B\in\mathscr{A}$ we have

$$p\big((A_0-cI)AB\big)\leqslant q\big((A_0-cI)A\big)p(B),$$

and by (2.11) we obtain

$$p(AB)\leqslant\varepsilon_c^{-1}q\big((A_0-cI)A\big)p(B)$$

for all $B\in\mathscr{A}$. Thus, by the definition of the seminorm q, we have

$$q(A)\leqslant q\big((A_0-cI)A\big)\cdot\varepsilon_c^{-1}$$

for all $A\in\mathscr{A}$ and all complex numbers c. It means that formula (2.4) is satisfied. Applying now Lemma 2.7 we obtain $q(I)=0$, what is in contradiction with relation (2.12). The contradiction shows that for some complex number c it is $\varepsilon_c=0$. The definition of ε_c given in formula (2.6) yields the desired result.

Proof of Theorem 2.5. By the remarks after the formulation of this theorem we have only to show that the joint approximate point spectrum $\sigma_\pi(S)$ has the finite projection property.

Let $S=(A_1,\ldots,A_{n+1})$ be an $(n+1)$-tuple of pairwise commuting operators in $L(X)$. From Lemma 2.8 we see that if $0\in\sigma_\pi(A_1,\ldots,A_n)$, then there is a complex number c such that $(0,\ldots,0,c)\in\sigma_\pi(A_1,\ldots,A_{n+1})$, what in turn implies

$$\sigma_\pi(A_1,\ldots,A_n)\subset P\sigma_\pi(A_1,\ldots,A_{n+1}),$$

where P denotes the projection of C^{n+1} onto its first n coordinates subspace. On the other hand, the relation

$$P\sigma_\pi(A_1,\ldots,A_{n+1})\subset\sigma_\pi(A_1,\ldots,A_n)$$

follows immediately from the definition of the joint approximate point spectrum. Thus σ_π has the projection property with respect to projections onto subspaces of codimension one, for $S\in\mathfrak{c}_0(X)$. For projection onto subspaces of higher codimension we proceed by an easy induction.

From Theorem 2.5 we obtain results of Harte [5], [6] stating that the joint spectra $\sigma_1(S)$, $\sigma_r(S)$ and $\sigma(S)$ possess the projection property. First we introduce yet another spectrum. It has some interest of its own, in this paper, however, it serves only to auxiliary purposes.

2.9. DEFINITION. Let X be a complex Banach space and let $(A_1, \ldots, A_n)$ be an n-tuple of pairwise commuting endomorphisms of X. We put

$$\sigma_\tau(A_1, \ldots, A_n) = \Big\{\lambda \in C^n\colon \sum_{i=1}^{n}(A_i - \lambda_i I)X \neq X\Big\}, \tag{2.13}$$

where

$$\sum_{i=1}^{n} B_i X = \Big\{\sum_{i=1}^{n} B_i x_i\colon x_i \in X,\ i = 1, \ldots, n\Big\}.$$

The reader familiar with the papers of Taylor [10] and [11] will observe that $\sigma_\tau(A_1, \ldots, A_n)$ is a subset of the Taylor spectrum $\sigma_T(A_1, \ldots, A_n)$ related to the non-exactness of a suitable Koszul complex at one of its ends (it is also called a *defect spectrum*).

2.10. LEMMA. *Let X be a complex Banach space and $(A_1, \ldots, A_n)$ an n-tuple of pairwise commuting operators in $L(X)$. Then*

$$\sigma_\tau(A_1, \ldots, A_n) = \sigma_\pi(A_1^*, \ldots, A_n^*), \tag{2.14}$$

where $A_1^*, \ldots, A_n^*$ are the conjugate operators defined on the conjugate Banach space X^*.

Proof. First we remark that $A_1^*, \ldots, A_n^*$ are also pairwise commuting operators. Let $X^n = X \times \ldots \times X$ be the cartesian product of n copies of the space X with norm of an element $x = (x_1, \ldots, x_n)$ given by the formula

$$\|x\| = \sum_{i=1}^{n}\|x_i\|.$$

We now define a map from X^n into X setting

$$Tx = \sum_{i=1}^{n} A_i x_i.$$

This is a linear map and $0 \notin \sigma_\tau(A_1, \ldots, A_n)$ if and only if the corresponding map T maps the space X^n onto the whole of X. But T is a map onto if and only if its conjugate map $T^*\colon X^* \to (X^n)^*$ is an isomorphism into. Every element F of $(X^n)^*$ is of the form

$$F(x) = \sum_{i=1}^{n} f_i(x_i),$$

where $x = (x_1, \ldots, x_n) \in X^n$ and $f_i \in X^*$. Thus F can be identified with an element $(f_1, \ldots, f_n) \in (X^*)^n$, so that $(X^n)^*$ is topologically isomorphic to $(X^*)^n$. It is easily seen that under this identification the conjugate map T is of the form

$$T^* f = (A_1^* f, \ldots, A_n^* f).$$

Now, the fact that T^* is an isomorphism into means that there exists a positive constant c such that for every $f \in X^*$ it is $\|T^* f\| \geqslant c\|f\|$, or $\sum_{i=1}^{n}\|A_i^* f\| \geqslant c\|f\|$. Applying Proposition 1.8 we see that $0 \notin \sigma_\tau(A_1, \ldots, A_n)$ is equivalent to $0 \notin \sigma_\pi(A_1^*, \ldots, A_n^*)$, what in turn implies formula (2.14).

From this lemma we obtain the following proposition.

2.11. PROPOSITION. *Let X be a complex Banach space. Then the map $S \to \sigma_\tau(S)$ is the spectral system on $\mathfrak{c}_0(X)$ possessing the projection property.*

Proof. Applying Theorem 1.11 and formula (2.14) we see that for any n-tuple $(A_1, \ldots, A_n) \in \mathfrak{c}_0(X)$ the spectrum $\sigma_\tau(A_1, \ldots, A_n)$ is a non-void compact subset of C^n. Since for a single operator $A \in L(X)$ it is $\sigma(A)$ $\sigma(A^*)$, then $\sigma_\tau(A) = \sigma_\pi(A^*) \subset \sigma(A^*) = \sigma(A)$. Thus the map $S \to \sigma_\tau(S)$, $S \in \mathfrak{c}_0(X)$, fulfils the requirements of Definition 2.1. Finally, by the previous lemma and by Theorem 2.5 the spectrum σ_τ possesses the projection property.

Applying Theorem 2.3 and the previous proposition we obtain the following corollary.

2.12. COROLLARY. *The spectrum σ_τ can be extended to a spectrum, denoted also by σ_τ, defined on the whole of $\mathfrak{c}(X)$ and possessing the projection property.*

2.13. LEMMA. *Let $\mathscr{A}$ be a complex Banach algebra with unit I. Then the joint spectra $\sigma_{r,\mathscr{A}}(A_1, \ldots, A_n)$ and $\sigma_{l,\mathscr{A}}(A_1, \ldots, A_n)$ possess the finite projection property.*

Proof. The conclusion follows immediately from formula (1.15), from Proposition 2.11 and from the formula

$$\sigma_{r,\mathscr{A}}(A_1, \ldots, A_n) = \sigma_\tau(A_1, \ldots, A_n),$$

where the elements A_i in the right-hand spectrum are regarded as pairwise commuting operators on the Banach space $X = \mathscr{A}$, given by $B \to A_i B$, $B \in X = \mathscr{A}$, $i = 1, 2, \ldots, n$.

The finite version of the following theorem was obtained in the papers [5] and [6] of Robin Harte.

2.14. THEOREM. *The left spectrum, the right spectrum and the spectrum possess the projection property for an arbitrary Banach space X.*

Proof. In view of the remarks preceding Lemma 2.4 and the lemma itself, we have only to show that the spectra $\sigma_r(S)$, $\sigma_l(S)$ and $\sigma(S)$ possess the finite projection property. But this follows immediately from the definition of the spectrum $\sigma(S)$, from the previous lemma and from the formulas

$$\sigma_r(A_1, \ldots, A_n) = \sigma_{r,\mathscr{A}}(A_1, \ldots, A_n)$$

and

$$\sigma_1(A_1, \ldots, A_n) = \sigma_{1,\mathscr{A}}(A_1, \ldots, A_n),$$

where $\mathscr{A} = L(X)$.

We shall now show that for the commutant spectrum $\sigma'(S)$ and for the bicommutant spectrum $\sigma''(S)$ the finite projection property fails. In the introduction of the paper [10] Taylor mentioned that the projection property of the spectrum σ' follows from the theory developed in this paper. Actually, the use of an example constructed in the same paper disproves this statement. The example of Taylor is as follows:

Let D be a compact polydisc and U an open polydisc with $0 \in D \subset U \subset C^2$. Put $V = U \setminus D$ and $X = C^{(1)}(\overline{V}) \times C(\overline{V})$, where $\overline{V}$ is the closure of V, $C(\overline{V})$ is the sup-norm algebra of all continuous functions on $\overline{V}$ and $C^{(1)}(\overline{V})$ is the algebra of all continuously differentiable functions on $\overline{V}$. A 5-tuple of pairwise commuting operators on X is defined as follows: $A_1(f, g) = (z_1 f, z_1 g)$, $A_2(f, g) = (z_2 f, z_2 g)$, $A_3(f, g) = (0, \partial f/\partial \bar{z}_1)$, $A_4(f, g) = (0, \partial f/\partial \bar{z}_2)$, $A_5(f, g) = (0, f)$. It is shown in [10] that $0 \in \sigma'(A_1, \ldots, A_5)$, while $0 \notin \sigma'(A_1, A_2)$, and so the finite projection property fails for σ'.

This example shows also that the finite projection property fails for the bicommutant spectrum σ''. In fact, since $0 \notin \sigma'(A_1, A_2)$, there are B_1 and B_2 in the commutant $(A_1, A_2)'$ such that

$$A_1 B_1 + A_2 B_2 = I. \tag{2.15}$$

We can also have $B_1 B_2 = B_2 B_1$, what can be realized by setting

$$B_1(f, g) = \left(\frac{\bar{z}_1}{|z_1|^2 + |z_2|^2} f, \frac{\bar{z}_1}{|z_1|^2 + |z_2|^2} g\right)$$

and

$$B_2(f, g) = \left(\frac{\bar{z}_2}{|z_1|^2 + |z_2|^2} f, \frac{\bar{z}_2}{|z_1|^2 + |z_2|^2} g\right).$$

Thus A_1, A_2, B_1, B_2 form a mutually commuting family of elements of $L(X)$ and (2.15) together with the relation $(A_1, A_2, B_1, B_2) \subset (A_1, A_2, B_1, B_2)''$ implies

$$0 \notin P_{1,2}\sigma''(A_1, A_2, B_1, B_2), \tag{2.16}$$

where $P_{1,2}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda_1, \lambda_2)$.

On the other hand, it is

$$0 \in \sigma''(A_1, A_2), \tag{2.17}$$

since $0 \in \sigma'(A_1, \ldots, A_5) \subset \sigma''(A_1, \ldots, A_5)$ and since for an arbitrary $S \in \mathfrak{c}(X)$, $S = (A_t)_{t \in T}$, and for an arbitrary $S_1 \subset S$, $S_1 = (A_t)_{t \in T'}$ with $T' \subset T$, we have

$$P\sigma''(S) \subset \sigma''(S_1), \tag{2.18}$$

where P is the natural projection of C^T onto $C^{T'}$. Formula (2.18) follows immediately from the definition of the bicommutant spectrum: if $0 \notin \sigma''(S_1)$, then, by (1.3) and the fact that the bicommutant of S_1 is contained in S'', we obtain $0 \notin P\sigma''(S)$. Relations (2.16) and (2.17) show together that the finite projection property fails for the bicommutant spectrum.

The above remarks show that both the commutant and the bicommutant spectrum do not form suitable concepts for a spectral theory of operators, at least from the point of view of the functional calculus and the spectral mapping theorem (cf. the next section). The bicommutant spectrum could be perhaps taken into account since formula (2.18) can serve as a substitute for the projection property, but, on the other hand, it is bigger then the commutant spectrum, which in turn does not possess a property analogous to that given by formula (2.18). This is again an argument for the utility of the Taylor spectrum: it has all desired properties and it is smaller then the commutant spectrum.

§ 3. The spectral mapping theorem. In this section we discuss the spectral mapping theorem with respect to polynomial mappings for all joint spectra considered in previous sections. Such a theorem has been already obtained by Harte [5], [6] for the joint spectra σ, σ_1, and σ_r defined on $\mathfrak{c}_0(X)$. Here we prove a general theorem from which we obtain the results of Harte in a more general setting of infinite families of mutually commuting operators of X, as well as the spectral mapping theorem for the joint approximate point spectrum σ_π, what is a new result.

For the commutant spectrum σ' and for the bicommutant spectrum σ'' the spectral theorem fails.

3.1. DEFINITION. Let T and T' be non-void sets of indices. A polynomial map $p\colon C^T \to C^{T'}$ is a map $z \to p(z) = (p_{t'}(z))_{t' \in T'} \in C^{T'}$, $z \in C^T$, given by means of a family (denoted by the same letter) $p = (p_{t'})_{t' \in T'}$, each $p_{t'}$ depending only upon a finite number $z_{t_1}, \ldots, z_{t_n}$ of coordinates of z and being a polynomial with complex coefficients in these coordinates.

3.2. DEFINITION. Let X be a complex Banach space and let $S \to \sigma^*(S)$ be a spectral system defined on $\mathfrak{c}(X)$. We say that the spectrum $\sigma^*(S)$ has the *spectral mapping property with respect to polynomial mappings* (abbreviated as *SM*-property) if for each $S \in \mathfrak{c}(X)$, $S = (A_t)_{t \in T}$, and for each polynomial map $p = (p_{t'})_{t' \in T'}$ it is

$$\sigma^*(pS) = p\sigma^*(S). \tag{3.1}$$

Here pS is a member of $\mathfrak{c}(X)$ given by

$$pS = \big(p_{t'}(A_{t_1(t')}, \ldots, A_{t_n(t')})\big)_{t'\in T'},$$

where $p_{t'}$ depends only upon the coordinates $z_{t_1(t')}, \ldots, z_{t_n(t')}$.

If a spectrum $\sigma^*(S)$ possesses the SM-property, then it possesses also the projection property, what can be seen by taking $T' \subset T$ and setting $p_{t'}(z) = z_{t'}$, $t'\in T'$.

Thus, in particular, the SM-property fails for the spectra σ' and σ''.

The projection property is then a necessary condition for having the SM-property. It is, however, not a sufficient condition, what can be seen by setting $\sigma^*(S) = \prod_{t\in T}\sigma(A_t)$, $S = (A_t)_{t\in T}$, so that the relation (3.1) fails e.g. for $S = (A, A)$ with $\sigma(A) = [0, 1]$ and $p = (z_1+z_2, z_1-z_2)$.

We shall prove now a theorem giving necessary and sufficient conditions for having the SM-property. Applying this theorem to σ_r, σ_l, σ and σ_π we shall establish the SM-property for these spectra.

3.3. THEOREM. *Let X be a complex Banach space. Suppose that $\sigma^*(S)$ is a spectrum defined on $\mathfrak{c}(X)$ and possessing the projection property. Then the following conditions are equivalent:*

(i) *For any two commuting operators $A_1, A_2\in L(X)$ and for any two complex numbers α and β it is*

$$\sigma^*(A_1, A_2, \alpha A_1+\beta A_2) \subset \{(z_1, z_2, \alpha z_1+\beta z_2)\in C^3\colon (z_1, z_2)\in C^2\}. \tag{3.2}$$

(ii) *For any three mutually commuting operators $A_1, A_2, A_3\in L(X)$ it is*

$$\sigma^*(A_1, A_2, A_3) \subset \sigma_{[A_1,A_2,A_3]}(A_1, A_2, A_3), \tag{3.3}$$

where $[A_1, A_2, A_3]$ is the smallest unital Banach subalgebra of $L(X)$ containing the elements A_1, A_2, A_3 and $\sigma_{[A_1,A_2,A_3]}(S)$ is the joint spectrum of a subset S of the Banach algebra $[A_1, A_2, A_3]$.

(iii) *For an arbitrary $S\in\mathfrak{c}(X)$ and for an arbitrary polynomial mapping p it is*

$$p\sigma^*(S) \subset \sigma^*(pS). \tag{3.4}$$

(iv) *For an arbitrary $S\in\mathfrak{c}(X)$ and for an arbitrary polynomial mapping p it is*

$$p\sigma^*(S) = \sigma^*(pS),$$

i.e. the spectrum σ^* has the SM-property.

Proof. The implications (iv) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i) are obvious, so it is sufficient to prove that (i) implies (ii) and (iv).

First we prove (i) $\Rightarrow$ (ii). Let A_1, A_2, A_3 be three pairwise commuting elements of $L(X)$. The commutative Banach algebra $[A_1, A_2, A_3]$ can be regarded as an element of $\mathfrak{c}(X)$, say $[A_1, A_2, A_3] = (B_t)_{t\in T}$. We fix

an element $\lambda\in\sigma^*([A_1, A_2, A_3])$, $\lambda = (\lambda_t)_{t\in T}\in C^T$, and so we define on the algebra $[A_1, A_2, A_3]$ a functional given by $\lambda(B_t) = \lambda_t$. The condition (3.2) implies that $B\to\lambda(B)$ is a linear functional defined on the algebra $[A_1, A_2, A_3]$. The projection property of σ^* implies that for each $B\in[A_1, A_2, A_3]$ we have

$$\lambda(B)\in\sigma^*(B)\subset\sigma(B). \tag{3.5}$$

Since for a single element $B\in[A_1, A_2, A_3]$, we have clearly

$$\sigma(B)\subset\sigma_{[A_1,A_2,A_3]}(B),$$

the relation (3.5) shows that for all B in $[A_1, A_2, A_3]$ it is

$$\lambda(B)\in\sigma_{[A_1,A_2,A_3]}(B)$$

and so, by the main result of [7], $\lambda(B)$ is a multiplicative linear functional in the algebra $[A_1, A_2, A_3]$. Denote by $\mathfrak{M}$ the set of all multiplicative linear functionals defined in the algebra $[A_1, A_2, A_3]$ and by $\mathfrak{N}$ the subset of $\mathfrak{M}$ consisting of all functionals of the form $\lambda(B)$. With this notation we have

$$\sigma^*(S) = \big\{\big(f(A_t)\big)_{t\in T}\in C^T\colon f\in\mathfrak{N}\big\}, \tag{3.6}$$

where $S = (A_t)_{t\in T}\subset[A_1, A_2, A_3]$.

On the other hand, by the definition of the joint spectrum in a Banach algebra, we have

$$\sigma_{[A_1,A_2,A_3]}(S) = \big\{\big(f(A_t)\big)_{t\in T}\in C^T\colon f\in\mathfrak{M}\big\}$$

and so $\mathfrak{N}\subset\mathfrak{M}$ implies $\sigma^*(S)\subset\sigma_{[A_1,A_2,A_3]}(S)$ for each $S\subset[A_1, A_2, A_3]$. Thus the formula (3.3) follows.

The formula (3.6) also establishes the implication (i) $\Rightarrow$ (iv), since for $p = (p_{t'})_{t'\in T'}$ we have

$$\begin{aligned} p\sigma^*(S) &= p\,\big\{\big(f(A_t)\big)_{t\in T}\in C^T\colon f\in\mathfrak{N}\big\}\\ &= \Big\{\Big(p_{t'}\big(\big(f(A_t)\big)_{t\in T}\big)\Big)_{t'\in T'}\in C^{T'}\colon f\in\mathfrak{N}\Big\}\\ &= \Big\{\Big(f\big(p_{t'}\big((A_t)_{t\in T}\big)\big)\Big)_{t'\in T'}\in C^{T'}\colon f\in\mathfrak{N}\Big\}\\ &= \sigma^*\Big(\big(p_{t'}(S)\big)_{t'\in T'}\Big) = \sigma^*(pS), \end{aligned}$$

where $S = (A_t)_{t\in T}$.

Remark. The above result is also true for a spectrum σ^* defined on $\mathfrak{c}_0(X)$.

The spectral mapping theorem for the spectra σ_l, σ_r, σ, and σ_π follows from the relations

$$\sigma_\pi(A_1, A_2, A_3)\subset\sigma_l(A_1, A_2, A_3)\subset\sigma(A_1, A_2, A_3)\subset\sigma_{[A_1,A_2,A_3]}(A_1, A_2, A_3)$$

and

$$\sigma_r(A_1, A_2, A_3) \subset \sigma_{[A_1,A_2,A_3]}(A_1, A_2, A_3),$$

which hold true for any triple A_1, A_2, A_3 of pairwise commuting operators in $L(X)$, and from the condition (ii) of the previous theorem. So we have

3.4. THEOREM. *Let X be a complex Banach space. Then the joint spectra σ_π, σ_l, σ_r, and σ defined on $\mathfrak{c}(X)$ possess the spectral mapping property with respect to polynomial mappings.*

As we mentioned before, the part of this theorem concerning the spectra σ_l, σ_r, and σ is due to Harte [5] and [6].

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Domains of attraction of stable measures on a Hilbert space

by

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Abstract. We characterize all probability measures in the domain of attraction of a stable measure defined on the Borel subsets of a real separable Hilbert space H.

1. Introduction and notation. Let E be a topological vector space and $\mathscr{B}(E)$ the class of Borel subsets of E. We say that a probability measure on $\mathscr{B}(E)$ is in the domain of attraction of a probability measure μ on $\mathscr{B}(E)$ if there exists real numbers $b_n > 0$ and vectors a_n in $E (n = 1, 2, \ldots)$ such that $\mathscr{L}\left(\frac{X_1 + \ldots + X_n}{b_n} - a_n\right)$ converges weakly to μ where $X_1, X_2, \ldots$ are independent identically distributed random variables with $\mathscr{L}(X_i) = P$ $(i = 1, 2, \ldots)$. and P is a Borel probability measure. The b_n's are called *norming constants.* In case E is a real separable Banach space it is shown in [6] that stable measures and only stable measures have non-empty domains of attraction. When E is a real separable Hilbert space H, a detailed Levy–Khinchine representation of the stable measures analogous to the one-dimensional case [3] is obtained in [5] and it is used here to characterize probability measures P which lie in the domain of attraction of a non-degenerate stable measure on H. Our results will include and generalize the work of Rvačeva [9] when H is finite-dimensional. The difficulty in the infinite-dimensional case results from the fact that the conditions for weak convergence of infinitely divisible measures ([4] and [7]) involve certain compactness criteria and these are attacked by using the concept of regular variation and modifications of some of the elegant ideas in [1].

Let μ be a finite Borel measure on a topological space X. Then A in $\mathscr{B}(X)$ is called a *continuity set of* μ if $\mu(\partial A) = 0$ where ∂A denotes the boundary of A. A set S_μ is called the *support of* μ if

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