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The generalized almost periodic part of an ergodic function

by

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Abstract. Let G be a locally compact non-compact abelian group. Denote the dual of G by Γ and the Bohr compactification of G by $\bar{G}$. Let $C(G)$ be the space of all bounded complex-valued uniformly continuous functions on G under the supremum norm. Let $\mathrm{IM}(G)$ be the set of all positive translation-invariant linear functionals on $C(G)$ with norm 1. A $\varphi \in C(G)$ is ergodic if $\mathscr{F}(\varphi)(\gamma) \equiv M(\varphi\bar{\gamma})$ is independent of $M \in \mathrm{IM}(G)$ for each $\gamma \in \Gamma$. Given an ergodic $\varphi \in C(G)$ define $F(\varphi) \in L^\infty(\bar{G})$ by $F(\varphi)^\wedge = \mathscr{F}(\varphi)$ and set $\mathscr{E}(\bar{G}) = \{F(\varphi): \varphi \text{ is ergodic on } G\}$. The theory of averaging kernels is used to give a more accessible definition of ergodic functions and to define the class of Weyl almost periodic functions in $C(G)$. $\mathscr{E}(\bar{G})$ can be characterized in terms of kernels which are absolutely continuous with respect to Haar measure on G. A similar result holds for Weyl almost periodic functions. Using these results when $G = \boldsymbol{R}$ or Z, one can construct a large class of functions in $L^\infty(\bar{G}) \cap \mathscr{E}(\bar{G})^c$. Moreover there exists ergodic φ such that $F(\varphi)^2 \notin \mathscr{E}(\bar{G})$. Such φ's as given here are Besicovitch almost periodic ergodic functions which are necessarily not Weyl almost periodic.

Introduction. If φ is weakly almost periodic [2], [3] on G, then $F(\varphi)$ is almost periodic. Weyl almost periodic functions φ ([1], Chapter II, Section 4) are not so well behaved, but it is true that $F(\varphi)^2 = F(\varphi^2)$. Very little has been published concerning the generalized almost periodic part $F(\varphi)$ of an ergodic function φ on G. However, Kahane [5] has some interesting results concerning null ergodic functions ($\mathscr{F}(\varphi) \equiv 0$). We will use his Theorem 2 together with several constructive lemmas to prove that $\mathscr{E}(\bar{G})$ is not closed under pointwise products.

Our interest in the generalized almost periodic part of an ergodic function φ arises from the study of ergodic sets in Fourier analysis [7]. For example it would be useful if $F(\varphi) = F(\omega)$ for some Weyl almost periodic function ω on G. Our main theorem (Theorem 6) implies that this is not true in general.

The definitions and first few results are of interest for the general locally compact non-compact abelian group G and are no more complicated to deal with than when $G = \boldsymbol{R}$ or Z. So let G be such a group. Let Γ be the dual of G, $\bar{G}$ the Bohr compactification of G, and Γ_d the dual of $\bar{G}$. Denote by $P(G)$ and $\mathrm{AP}(G)$ the spaces of trigonometric polynomials and almost periodic functions on G, respectively, under the supremum norm $\|\cdot\|_\infty$. Recall that Γ_d is Γ under the discrete topology and that $\mathrm{AP}(G)$ is identified with $C(\bar{G})$ in a natural way. We denote the x-translate of a $\varphi \in C(G)$ by φ_x, where $\varphi_x(y) \equiv \varphi(y-x)$ for $x, y \in G$. The space of bounded regular Borel measures on G is denoted by $M(G)$.

A continuous linear functional M on $C(G)$ is an *invariant mean* on G if

(a) $\|M\| = M(1)$,

(b) $M(\varphi_x) = M(\varphi)$, for all $\varphi \in C(G)$ and $x \in G$.

Here $\|\cdot\|$ denotes the norm of a linear functional. The set of all such means is denoted by $\mathrm{IM}(G)$. Note that (a) implies that each $M \in \mathrm{IM}(G)$ is positive.

We say that $\varphi \in C(G)$ is *ergodic at* $\gamma \in \Gamma$ if $\mathscr{F}(\varphi)(\gamma) \equiv M(\varphi\bar{\gamma})$ is independent of $M \in \mathrm{IM}(G)$. Such a φ is *ergodic* if and only if it is ergodic at each $\gamma \in \Gamma$. Denote by $\mathscr{E}(G)$ the space of ergodic functions on G with the supremum norm. It is a Banach space. Since $\bar{G}$ has only one invariant mean, $\mathrm{AP}(G) \subset \mathscr{E}(G)$ and $M(q\bar{\gamma}) = \hat{q}(\gamma)$ for $q \in \mathrm{AP}(G)$ and $\gamma \in \Gamma$. Here $\hat{q}$ denotes the Fourier transform of q considered as an element in $L^1(\bar{G})$.

Given $p = \sum a_\gamma \gamma \in P(G)$ and $\varphi \in \mathscr{E}(G)$, define the *mean convolution* $(p\circ\varphi)(x) = M_y\big(p(x-y)\varphi(y)\big)$ for some $M \in \mathrm{IM}(G)$. Since $p\circ\varphi = \sum a_\gamma \mathscr{F}(\varphi)(\gamma)\gamma$, this definition is independent of the choice of M. Since $P(G)$ is uniformly dense in $\mathrm{AP}(G)$, the function $q\circ\varphi(x) = M_y\big(q(x-y)\varphi(y)\big)$ is independent of $M \in \mathrm{IM}(G)$ and almost periodic for each $q \in \mathrm{AP}(G)$. Moreover, $(q\circ\varphi)\hat{}\,(\gamma) = \hat{q}(\gamma)\mathscr{F}(\varphi)(\gamma)$ for each $\gamma \in \Gamma$. Note that if one also has $\varphi \in \mathrm{AP}(G)$, then $q\circ\varphi = q*\varphi$, the convolution in $L^1(\bar{G})$.

Let $\{p_a\} \subset P(G)$ be an $L^1(\bar{G})$ bounded approximate identity. Then for $\varphi \in \mathscr{E}(G)$, the net $\{p_a\circ\varphi\}$ is bounded in $L^\infty(\bar{G})$. Since $\{(p_a\circ\varphi)\hat{}\,\}$ converges pointwise to $\mathscr{F}(\varphi)$ on Γ_d, $\{p_a\circ\varphi\}$ has exactly one weak* cluster point $\psi \in L^\infty(\bar{G})$ and $\hat{\psi} = \mathscr{F}(\varphi)$. Set $F(\varphi) = \psi$ and let $\mathscr{E}(\bar{G}) = \{F(\varphi) \mid \varphi \in \mathscr{E}(G)\}$. The map F is continuous and linear, but not injective (the image of $C_0(G)$ is $\{0\}$).

There are three rather basic facts concerning the $M \in \mathrm{IM}(G)$ which we use:

1. For $f \in L^1(G)$ and $\varphi \in C(G)$, $M(f*\varphi) = \hat{f}(o)\, M(\varphi)$. Similarly, $f*(q\circ\varphi) = (f*q)\circ\varphi = q\circ(f*\varphi)$, whenever $q \in \mathrm{AP}(G)$ and $\varphi \in \mathscr{E}(G)$.

2. For $q \in \mathrm{AP}(G)$, $\varphi \in \mathscr{E}(G)$, and $x, y \in G$,

$$\|(q\circ\varphi)_x - (q\circ\varphi)_y\|_\infty \leqslant M(|q|\,\|\varphi_x-\varphi_y\|_\infty) = \|q\|_1\|\varphi_x-\varphi_y\|_\infty,$$

where $\|q\|_1$ is the norm of q in $L^1(\bar{G})$.

3. $\{M(\varphi) \mid M \in \mathrm{IM}(G)\} = \bigcap \{\overline{\mathrm{co}}\big(R(\psi)\big) \mid \psi \in \mathrm{co}(\mathrm{Tr}\,\varphi)\}$. Here co denotes convex hull, $R(\psi)$ the range of ψ, and $\mathrm{Tr}\,\varphi$ the set of translates of φ. O course $\varphi \in C(G)$.

Here 1 is a consequence of the fact that $f*\varphi$ is the uniform limit of a sequence of particular linear sums of translates of φ; 2 is immediate from the definition of M; and 3 can be found in [4], proof of Theorem 2.1.

An *averaging kernel* $\{k_a\}$ on G is a net in $L^1(G)$ satisfying

(a) $\lim_a \|k_a\|_1 = \lim_a \hat{k}_a(o) = 1$,

(b) $\lim_a \|k_a - (k_a)_x\|_1 = 0$ for all $x \in G$.

One particularly nice averaging kernel can be defined as follows. Let $\{U_a\}$ be a neighborhood base for $o \in \Gamma$ which consist of compact symmetric sets. Denote the measure of U_a by $|U_a|$ and the characteristic function of U_a by χ_a. Define k_a by $\hat{k}_a = |U_a|^{-1}\chi_a*\chi_a$. With the partial ordering $a \leqslant \beta$ if and only if $U_\beta \subseteq U_a$, $\{k_a\}$ is a net of positive functions in $L^1(G)$ which satisfies (a). To prove (b), fix $x \in G$, an index a, and choose $g \in L^1(G)$ so that $\hat{g} \equiv 1$ on U_a. Since $(g-g_x)\hat{}\,(o) = 0$ and since points are spectral sets ([6], Section 7.2.5), there is for each n an $h_n \in L^1(G)$ with $\hat{h_n} \equiv 0$ on a neighborhood of zero such that $\|(g-g_x)-h_n\|_1 < n^{-1}$. We then have for $\beta > a$ and sufficiently large

$$\begin{aligned}\|k_\beta - (k_\beta)_x\|_1 &= \|(k_\beta-(k_\beta)_x)*g\|_1\\ &= \|k_\beta*(g-g_x)\|_1\\ &\leqslant \|k_\beta\|_1\|(g-g_x)-h_n\|_1 + \|k_\beta*h_n\|_1\\ &< n^{-1}.\end{aligned}$$

The classical averaging kernels can also be described in this context. For instance let $G = \boldsymbol{R}$ or Z and let $\{x_a\}$ be a positive increasing unbounded net in G. Let χ_a be the characteristic function of the interval $[o, x_a)$ in G and set $u_a = x_a^{-1}\chi_a$. Clearly, u_a satisfies (a) and (b).

The connection between averaging kernels and invariant means is summarized in the following theorem and corollary whose proofs we shall sketch (a more detailed analysis of averaging kernels and their applications is the subject of another paper). (See also [2], Part I.)

THEOREM. *Let $\{k_a\}$ be an averaging kernel on G and suppose $\varphi \in C(G)$. Then*

$$s = \sup\{|M(\varphi)| \mid M \in \mathrm{IM}(G)\} = \limsup_a \|k_a*\varphi\|_\infty.$$

Assume each k_a is positive with $\hat{k}_a(o) = 1$. Then $k_a*\varphi \in \overline{\mathrm{co}}(\mathrm{Tr}\,\varphi)$ Hence $s \leqslant \limsup \|k_a*\varphi\|_\infty$ by Fact 3. Conversely if $\psi \in \mathrm{co}(\mathrm{Tr}\,\varphi)$, then property (b) of $\{k_a\}$ implies $\|k_a*\psi - k_a*\varphi\|_\infty \to 0$. Hence $s \geqslant \limsup \|k_a*\varphi\|_\infty$, again by Fact 3. For the general k_a we need only note that property (a) implies the positive part k_a^+ of k_a satisfies $\|c_a k_a^+ - k_a\|_1 \to 0$, where $c_a^{-1} = (k_a^+)\hat{}\,(o)$.

Corollary. *Under the hypothesis of the theorem, φ is ergodic at $\gamma \in \Gamma$ if and only if $\{k_a * (\varphi\bar{\gamma})\}$ is uniformly Cauchy and converges to $c = \mathscr{F}(\varphi)(\gamma)$.*

If $\{k_a * (\varphi\bar{\gamma})\}$ is uniformly Cauchy, then property (b) implies it must converge to some constant c. If φ is ergodic at γ set $c = \mathscr{F}(\varphi)(\gamma)$. Now apply the theorem to the function $\varphi\bar{\gamma} - c$ and use property (a). This corresponds to Theorem 3.1 of [2].

Let $\{u_a\}$ be the previously defined averaging kernel for $G = \boldsymbol{R}$. According to the corollary, a $\varphi \in C(\boldsymbol{R})$ is ergodic precisely when

$$u_a * (\varphi e^{-iy(\cdot)})(x) = x_a^{-1} \int_{x-x_a}^{x} \varphi(t) e^{-iyt} dt$$

converges uniformly in x to some constant as $a \to \infty$, for each $y \in \boldsymbol{R}$. Thus $\mathscr{E}(\boldsymbol{R})$ consists exactly of the uniformly continuous Ryll-Nardzewski almost periodic functions on $\boldsymbol{R}$ as defined in [5].

In a similar manner we can use the theorem to give a satisfying definition of Weyl almost periodic functions ([1], pages 71, 72, 82). First the classical case $G = \boldsymbol{R}$. A $\varphi \in C(\boldsymbol{R})$ is Weyl almost periodic if and only if there exist a sequence $\{p_n\} \subset P(\boldsymbol{R})$ such that

$$(*) \qquad \lim_{T\to\infty} \sup_{-\infty<x<\infty} \frac{1}{T} \int_{x-T}^{x} |\varphi(t) - p_n(t)|\, dt = o(1), \qquad \text{as } n \to \infty.$$

But

$$\frac{1}{T} \int_{x-T}^{x} |\varphi(t) - p_n(t)|\, dt = u_T * |\varphi - p_n|(x),$$

where $\{u_T\}$ is as before. Thus $(*)$ is equivalent to

$$(**) \qquad \limsup_{T} \big\| u_T * |\varphi - p_n| \big\|_\infty = o(1), \qquad \text{as } n \to \infty,$$

which must remain valid when $\{u_T\}$ is replaced by any averaging kernel on G by the theorem. So we define, for the general G: $\varphi \in C(G)$ is Weyl *almost periodic* on G if $(**)$ holds for some $\{p_n\} \subset P(G)$ and some (therefore every) averaging kernel $\{u_T\}$ on G. Let $W(G)$ denote the class of Weyl almost periodic functions in $C(G)$ and $W(\bar{G}) = \{F(\varphi) \mid \varphi \in W(G)\}$. Of course, $W(G) \subset \mathscr{E}(G)$; hence $W(\bar{G}) \subset \mathscr{E}(\bar{G})$.

Where possible we use the standard notation in Fourier analysis. However, the reader must be slightly careful since the same notation is used to denote action on G as well as $\bar{G}$. For example, if $q \in \mathrm{AP}(G)$, then $\hat{q}$ and $\|q\|_1$ refer to $q \in L^1(\bar{G})$.

The space $\mathscr{E}(\bar{G})$. Our first result gives a useful characterization of $\mathscr{E}(\bar{G})$ for the general non-compact G. For technical reasons however, we will restrict our attention to those $\psi \in L^\infty(\bar{G})$ whose spectra are precompact in Γ. We define the isometric homomorphism $\varrho\colon M(G) \to M(\bar{G})$ by $\int q\, d\varrho(\mu) \equiv \int q\, d\mu$ for $\mu \in M(G)$ and $q \in \mathrm{AP}(G)$. Since each $f \in L^1(G)$ determines a unique measure in $M(G)$, $\varrho(f)$ is well defined.

Proposition 1. *Let $\{k_a\}$ be an averaging kernel on G and suppose that $\psi \in L^\infty(\bar{G})$ has spectrum which is precompact in Γ. Then $\psi \in \mathscr{E}(\bar{G})$ if and only if for each $\gamma \in \Gamma$*

$$(1) \qquad \limsup_{a} \|\varrho(k_a) * (\psi\bar{\gamma}) - \hat{\psi}(\gamma)\|_\infty = 0.$$

Proof. Let $\{p_\beta\} \subset P(G)$ be an $L^1(\bar{G})$ bounded approximate identity. By definition, $\psi \in \mathscr{E}(\bar{G})$ if and only if there is a $\varphi \in \mathscr{E}(G)$ such that $\mathscr{F}(\varphi) = \hat{\psi}$. Therefore, $p_\beta \circ \varphi = p_\beta * \psi$ for each β. This means that $\{p_\beta \circ \varphi\}$ converges to ψ both on the $L^1(\bar{G})$ norm topology and in the bounded $L^\infty(\bar{G})$ weak* topology. In particular,

$$\begin{aligned}
\|\varrho(k_a) * (\psi\bar{\gamma}) - \hat{\psi}(\gamma)\|_\infty &\leqslant \limsup_{\beta} \big\| k_a * \big(p_\beta \circ (\varphi\bar{\gamma})\big) - \mathscr{F}(\varphi)(\gamma) \big\|_\infty \\
&= \limsup_{\beta} \big\| p_\beta \circ \big(k_a * (\varphi\bar{\gamma})\big) - \mathscr{F}(\varphi)(\gamma) \big\|_\infty \\
&= \limsup_{\beta} \| p_\beta \circ [k_a * (\varphi\bar{\gamma}) - \mathscr{F}(\varphi)(\gamma)] \|_\infty \\
&\leqslant \limsup_{\beta} \|p_\beta\|_1 \|k_a * (\varphi\bar{\gamma}) - (\mathscr{F}\varphi)(\gamma)\|_\infty .
\end{aligned}$$

Since $\varphi \in \mathscr{E}(G)$, we conclude that (1) holds for any $\psi \in \mathscr{E}(\bar{G})$. Conversely suppose ψ satisfies (1) and has precompact spectrum in Γ. Then $\{p_\beta * \psi\} \subset P(G)$ is equicontinuous and bounded on G; hence it has a $C(G)$ cluster point φ in the topology of uniform convergence on compact subsets of G. Since

$$\begin{aligned}
\|k_a * (\varphi\bar{\gamma}) - \hat{\psi}(\gamma)\|_\infty &\leqslant \limsup_{\beta} \big\| k_a * \big(p_\beta * (\psi\bar{\gamma})\big) - \hat{\psi}(\gamma) \big\|_\infty \\
&= \|\varrho(k_a) * (\psi\bar{\gamma}) - \hat{\psi}(\gamma)\|_\infty ,
\end{aligned}$$

we conclude that $\varphi \in \mathscr{E}(G)$ and that $F(\varphi) = \psi$. ∎

Part of our interest in Proposition 1 lies in the following corollary whose validity follows directly from the proposition and its proof. The τ-topology is the topology on $C(G)$ of uniform convergence on compact sets.

Corollary 2. *Assume $\psi \in \mathscr{E}(\bar{G})$.*

(i) *If $\{u_\beta\} \subset M(\bar{G})$ is norm bounded, then the weak* cluster points of $\{u_\beta * \psi\}$ are in $\mathscr{E}(\bar{G})$.*

(ii) *If $\{u_\beta\} \subset L^1(\bar{G})$ is norm bounded, then the τ-cluster points of $\{u_\beta * \psi\}$ are in $\mathscr{E}(G)$.*

(iii) *In either case $\lim_a \|\varrho(k_a) * (u_\beta * \psi\bar{\gamma}) - \hat{u}_\beta(\gamma)\hat{\psi}(\gamma)\|_\infty = 0$, uniformly in β whenever $\{k_a\}$ is an averaging kernel on G.*

An analogous result holds for Weyl almost periodic functions. Simply replace $\mathscr{E}(G)$, $\mathscr{E}(\bar{G})$ with $W(G)$, $W(\bar{G})$.

The techniques used in proving Proposition 1 can also be used to prove: $\psi\in L^\infty(\bar{G})$ with precompact spectrum in Γ satisfies $\psi = F(\varphi)$ for some $\varphi\in W(G)$ if and only if

$$\limsup_{\alpha}\limsup_{\beta}\big\|\varrho(k_\alpha)*|p_\beta*\psi-\psi|\big\|_\infty = 0,$$

where $\{k_\alpha\}$ is an averaging kernel on G. This extends slightly a classical result ([1], Remark, page 107).

The following is a simple and interesting construction concerning the generalized almost periodic part $F(\varphi)$ of a $\varphi\in W(G)$. We present it here because it sets the tone for our later work. Let $1 = k_1 < k_2 < \ldots$ be a sequence of integers satisfying $k_i \equiv 0 \bmod (k_{i-1})$ for $i\geqslant 2$ and let $\{a_n\}$ be a sequence of bounded complex numbers. Define the sequence $\{\varphi_n\}$ of periodic functions on the group Z of integers as follows: if $k\equiv 0 \bmod (k_n)$, set $\varphi_n(k) = a_n$; if $k\equiv 0 \bmod (k_j)$ and $k\not\equiv 0 \bmod (k_{j+1})$ for some $1\leqslant j<n$, set $\varphi_n(k) = a_j$. Let $\varphi(k) = \lim\varphi_n(k)$ for $k\neq 0$ and set $\varphi(0) = b$, where b is a cluster point of $\{n^{-1}\sum_1^n a_k\}$. Define the averaging kernel $\{f_m\}$ on Z by $f_m(k) = k_m^{-1}$ if $0\leqslant k<k_m$, and $f_m(k) = 0$ otherwise. Since $\varphi(k)-\varphi_n(k)\neq 0$ only if $k\equiv 0 \bmod (k_{n+1})$, it is easy to see that

$$\lim_n\limsup_m\big\|f_m*|\varphi-\varphi_n|\big\|_\infty = 0.$$

Thus $\varphi\in W(Z)$. We claim that $F(\varphi)$ is almost periodic (i.e. continuous on $\bar{Z}$) if and only if the sequence $\{a_n\}$ converges. To see this define $\{p_n\}\subset P(Z)$ by $p_n(k) = k_n$ if $k\equiv 0 \bmod (k_n)$, and $p_n(k) = 0$ otherwise. Observe that $p_n*F(\varphi) = p_n\circ\varphi$. Now one can easily verify the equivalence of the following statements and hence prove the claim: $F(\varphi)\in C(\bar{Z})$; $\{p_n*F(\varphi)\}$ is uniformly Cauchy; $\{a_n\}$ converges; φ is almost periodic.

The basic difference between the previous construction and the one needed to establish our main result is in the choice of the bounded sequence $\{\varphi_n\}$ of periodic functions on Z. The "new" φ_n will converge in $L^1(\bar{Z})$ to a $\psi\in L^\infty(\bar{Z})$ and pointwise on Z to a bounded function φ. The construction of the φ_n will ensure that $\{p_n*\psi\}$ converges pointwise to φ for an appropriately chosen sequence $\{p_n\}\subset P(Z)$ which is bounded in $L^1(\bar{Z})$ and that (usually) $\varphi\notin W(Z)$; hence $\psi\notin W(\bar{G})$ according to the remark following Corollary 2. Finally we will give conditions on the φ_n which are sufficient to imply that $\psi\in\mathscr{E}(\bar{Z})$ and $\psi^2\notin\mathscr{E}(\bar{Z})$ (this also implies $\psi\notin W(\bar{G})$). It will follow from [5], Théorème 2 that these conditions are not vacuous. As indicated the result will be proved for $G = Z$ and then extended.

The construction. To begin, let $B>0$ and choose any sequence $\{k_n\}$ of positive integers satisfying (1) $\sum k_n/k_{n+1}<\infty$, (2) $k_{n+1}\equiv 0 \bmod (k_n)$, and (3) $(n+1)k_n<k_{n+1}$, for $n\geqslant 1$. Let $\{a_{ij}\}$ be an array of complex numbers with modulus bounded by B where (i,j) satisfies $2\leqslant i<\infty$ and $0\leqslant j<k_{i-1}$. Finally, for $n\geqslant 2$ set

$$I_n = \{k\in Z\mid -nk_{n-1}\leqslant k< -(n-1)k_{n-1}\}$$

and define $K_n = I_n+k_nZ$. Note that $I_n\subset(-k_n, -k_{n-1})$ and that $I_n+k_n\subset[k_{n-1}, k_n)$, for $n\geqslant 2$.

The sequence $\{\varphi_n\}$ is inductively defined. Let φ_1 be a periodic complex-valued function on Z whose period divides k_1 and which is bounded by B. Suppose $\varphi_1,\ldots,\varphi_{n-1}$ have been defined. Given $k\in K_n$, there is a unique $0\leqslant j<k_{n-1}$ such that $k\equiv(-nk_{n-1}+j) \bmod (k_n)$. Set $\varphi_n(k) = a_{nj}$; if $k\notin K_n$, then set $\varphi_n(k) = \varphi_{n-1}(k)$. It follows that φ_n has period dividing k_n and that φ_r differs from φ_n on at most $K_{r+1}\cup\ldots\cup K_n\equiv L_{r,n}$ for any $1\leqslant r<n$. In particular, $\{\varphi_n\}$ converges pointwise on Z to a function φ. Moreover, since $\|\varphi_n\|_\infty\leqslant B$, the sequence $\{\varphi_n\}$ has an $L^\infty(\bar{Z})$-weak* cluster point ψ (actually ψ is a limit point). The phrase *admissable triple* will refer to any triple $(\{\varphi_n\}, \varphi, \psi)$ that can be constructed as above.

In the next three lemmas it is proved that $\psi\in\mathscr{E}(\bar{Z})$ if and only if $\varphi\in\mathscr{E}(Z)$. For $m\geqslant n$, set $L_{m,n} = \emptyset$ and define $A_{m,n} = L_{m,n}\cap[0, k_n)$ for any m, n. Denote the cardinality of a set K by $|K|$.

LEMMA 3. *Assume* $m<n$.

(i) $$|A_{m,n}\cap A^c_{m+1,n}| = k_nk_mk_{m+1}^{-1}\prod_{j=m+1}^{n-1}(1-k_jk_{j+1}^{-1}).$$

(ii) $$|A_{m,n}|\leqslant k_n\sum_{j=m}^{n-1}k_jk_{j+1}^{-1}.$$

Proof. Since $|A_{m,n}| = \sum|A_{j,n}\cap A^c_{j+1,n}|$ over $m\leqslant j<n$, (ii) follows immediately from (i). Thus we need only prove (i). To begin call each interval of the form $[sk_r, (s+1)k_r)$ an r-interval. Recall that each K_{r+1} consists of r-intervals, precisely one from each $r+1$-interval. Set $A = A_{m,n}\cap A^c_{m+1,n}$ and let α_r be the number of m-intervals in $[0, k_r)\cap A$, $r>m$. There are k_{r+1}/k_r r-intervals in $[0, k_{r+1})$. Each one contains α_r m-intervals in $K_{m+1}\cap(K_{m+2}\cup\ldots\cup K_r)^c$, but one of these r-intervals is in K_{r+1}; none are in $K_{r+2}, K_{r+3},\ldots$ Thus there are $\alpha_r(k_{r+1}k_r^{-1}-1)\equiv\alpha_{r+1}$ m-intervals in $[0, k_{r+1})\cap A$. This holds for $r = m+1,\ldots,n-1$. Therefore

$$\begin{aligned}|A_{m,n}\cap A^c_{m+1,n}| &= k_m\alpha_n = k_m\alpha_{n-1}(k_nk_{n-1}^{-1}-1)\\ &= \ldots = k_m(k_{m+2}k_{m+1}^{-1}-1)\ldots(k_nk_{n-1}^{-1}-1)\\ &= k_nk_mk_{m+1}^{-1}(1-k_{m+1}k_{m+2}^{-1})\ldots(1-k_{n-1}k_n^{-1}).\end{aligned}$$

LEMMA 4. $\lim_n \|\varphi_n - \psi\|_1 = 0$.

Proof. It is sufficient to prove that $\{\varphi_n\}$ is $L^1(\bar{Z})$-Cauchy. Let $m < n$ and recall that φ_m and φ_n both have period dividing k_n and that φ_m differs from φ_n on at most $L_{m,n}$. In particular

$$\|\varphi_n - \varphi_m\|_1 = k_n^{-1} \sum_{k=0}^{k_n-1} |\varphi_n(k) - \varphi_m(k)| \leqslant 2Bk_n^{-1}|A_{m,n}|,$$

where B is the bound for $\{\|\varphi_n\|_\infty\}$. Now apply Lemma 3 (ii).

Define $\{p_n\} \subset P(Z)$ by $p_n(k) = k_n$ if $k \in k_n Z$ and $p_n(k) = 0$ otherwise. We observe that $\|p_n\|_1 = 1$ and that $p_n * \varphi_m = \varphi_m$ for all $m \leqslant n$ because the period of such a φ_m divides k_n. Hence $\{p_n * \psi\}$ converges to ψ in $L^1(\bar{Z})$ by Lemma 4.

LEMMA 5.

(i) *$\{p_n * \psi\}$ converges pointwise to φ.*

(ii) *If $\varphi \in \mathscr{E}(Z)$, then $\psi = F(\varphi)$.*

(iii) *$\psi \in \mathscr{E}(\bar{Z})$ if and only if $\varphi \in \mathscr{E}(Z)$.*

Proof. We begin by computing $p_m * \varphi_n$ for $m < n$. Since p_m has period k_m, the period of $p_m * \varphi_n$ must divide k_m. Hence we need only compute $p_m * \varphi_n(k)$ for $0 \leqslant k < k_m$. Fix such a k. Then

$$p_m * \varphi_n(k) = k_n^{-1} \sum_{0 \leqslant s < k_n} p_m(k-s)\varphi_n(s)$$
$$= k_m k_n^{-1} \sum_{0 \leqslant s < l_{m,n}} \varphi_n(k + sk_m),$$

where $l_{m,n} = k_n/k_m$. Set $B_{m,n} = \prod(1 - k_j/k_{j+1})$ over $m+1 \leqslant j \leqslant n-1$ and $Q_m = \lim_n B_{m,n}$. Recall that for $j < n$ we have $\varphi_n = \varphi_j$ on $(K_{j+1} \cup \ldots \cup K_n)^c$ and observe that the number of j-intervals in $(K_{j+1} \cup \ldots \cup K_n)^c \cap [0, k_n)$ is precisely $k_{j-1}^{-1}|A_{j-1,n} \cap A_{j,n}^c|$. In particular, we can write the above sum as

$$p_m * \varphi_n(k) = k_m k_n^{-1}\big(\varphi_m(k) k_{m-1}^{-1} |A_{m-1,n} \cap A_{m,n}^c|\big) + $$
$$+ k_m k_n^{-1} \sum_{j=m}^{n-1} k_j^{-1} \Big(\sum_{0 \leqslant s < l_{m,j}} a_{j+1,k+sk_m}\Big) |A_{j,n} \cap A_{j+1,n}^c|$$
$$= \varphi_m(k) B_{m-1,n} + \sum_{j=m}^{n-1} l_{m,j}^{-1} \Big(\sum_{0 \leqslant s < l_{m,n}} a_{j+1,k+sk_m}\Big) k_j k_{j+1}^{-1} B_{j,n},$$

by Lemma 3 (i). Therefore

$$\lim_n p_m * \varphi_n(k) = \varphi_m(k) Q_{m-1} + \sum_{j=m}^{\infty} V_{m,j}(k) k_j k_{j+1}^{-1} Q_j,$$

where

$$V_{m,j}(k) = l_{m,j}^{-1} \sum_{0 \leqslant s < l_{m,n}} a_{j+1,k+sk_m},$$

being an average, has modulus bounded by B. Since $\lim_n p_m * \varphi_n = p_m * \psi$ by Lemma 4, $\lim_m Q_m = 1$, and $\lim_m \varphi_m(k) = \varphi(k)$, we conclude

$$\lim_m p_m * \psi(k) = \varphi(k), \qquad k \in Z$$

which proves (i).

Since $\{\varphi_n\}$ and $\{p_n * \psi\}$ converge to ψ in $L^1(\bar{Z})$, the proof of (ii) will follow immediately from $\lim_n \|p_m \circ \varphi - p_m * \varphi_n\|_\infty = 0$, which we now prove. Let $\{f_n\}$ be the averaging kernel on Z defined by $f_n(k) = k_n^{-1}$ for $0 \leqslant k < k_n$ and $f_n(k) = 0$ otherwise. Observe that

$$(1) \qquad \lim_n \|(f_n p_m) * \varphi - p_m \circ \varphi\|_\infty = 0,$$

since φ is ergodic by assumption. Let $k' \in Z$ be arbitrary and choose $0 \leqslant k < k_m$ so that $k \equiv k' \bmod (k_m)$. Since $p_m \circ \varphi$ has period dividing k_m, (1) implies the existence of functions Δ_n on Z satisfying $\|\Delta_n\|_\infty \to 0$ such that

$$(f_n p_m) * \varphi(k') = (f_n p_m) * \varphi(k) + \Delta_n(k').$$

But $\varphi = \varphi_n$ on $(-k_n, k_n)$. Thus, for $m < n$,

$$(f_n p_m) * \varphi(k') = (f_n p_m) * \varphi_n(k) + \Delta_n(k')$$
$$= p_m * \varphi_n(k') + \Delta_n(k'), \text{ for all } k'.$$

Therefore $\lim_n \|p_m \circ \varphi - p_m * \varphi_n\|_\infty = 0$.

To prove (iii) observe that if $\psi \in \mathscr{E}(\bar{Z})$, then $\varphi \in \mathscr{E}(Z)$ by (i) and Corollary 2. The converse follows immediately from (ii). ■

The previous lemma allows us to easily distinguish between $L^\infty(\bar{Z})$ and $\mathscr{E}(\bar{Z})$. For instance, if τ is a bounded nonergodic function on Z, then the φ in the admissable triple $(\{\varphi_n\}, \varphi, \psi)$ corresponding to $a_{ij} = \tau(j - k_{i-2})$ for $0 \leqslant j < k_{i-1}$ and $i \geqslant 3$ is nonergodic. Hence $\psi \notin \mathscr{E}(\bar{Z})$. A simpler example arises from $a_{ij} = (-1)^i$ for $0 \leqslant j < k_{i-1}$ and $i \geqslant 2$.

Now we will construct some nontrivial ergodic functions. To begin, let σ be any bounded function on Z which is zero for $k < 0$ and for $k \in \bigcup_2^\infty K_j$. Inductively define the array $\{a_{ij}\}$ as well as $\{\varphi_n\}$ in the following manner. Let φ_1 be as before and suppose φ_k and a_{ij} have been defined for $1 \leqslant k < n$, $0 \leqslant j < k_{i-1}$, and $2 \leqslant i < n$. Let $0 \leqslant s < k_{n-1}$. Define

$$a_{ns} = \varphi_{n-1}(s) + \sigma(s) = \begin{cases} \varphi_1(s) + \sigma(s) & \text{if } s \notin K_2 \cup \ldots \cup K_{n-1}, \\ \varphi_{n-1}(s) & \text{if } s \in K_2 \cup \ldots \cup K_{n-1}. \end{cases}$$

Define φ_n as before. Since $\{a_{ij}\}$ is bounded, this process gives rise to an admissable triple $(\{\varphi_n\}, \varphi, \psi)$.

In the proof of the next theorem, the following description of the relationships between the φ_n and φ is used. Let $n > 2$ and let $I = [sk_n, (s+1)k_n)$ be an arbitrary n-interval. Either

$$I \subset \Big(\bigcup_{n+1}^{\infty} K_j\Big)^c \quad \text{or} \quad I \subset \Big(\bigcup_{n+1}^{\infty} K_j\Big).$$

In the first case, $\varphi = \varphi_n$ on I. For the second, let $r = r(s, n)$ be the first j such that $I \subset K_j$. Then $I \subset (K_{n+1} \cup \ldots \cup K_{r-1})^c$; hence $\varphi_n = \varphi_{r-1}$ on I. Now choose s_r so that

$$0 \leqslant s_r k_n \leqslant k_{r-1} - k_n$$

and

$$sk_n - s_r k_n \equiv -rk_{r-1} \bmod (k_r).$$

Then, by definition,

$$\varphi_r(sk_n+j) = \varphi_{r-1}(sk_n+j) + \sigma(s_r k_n + j)$$

$$= \varphi_n(sk_n+j) + \sigma(s_r k_n+j), \qquad 0 \leqslant j < k_n.$$

Now suppose $I \subset k_p$ for some $p > r$. Then for the corresponding s_p we have

$$\varphi_p(sk_n+j) = \varphi_{p-1}(sk_n+j) + \sigma(s_p k_n + j), \qquad 0 \leqslant j < k_n.$$

But

$$s_p k_n \equiv pk_{p-1} + sk_n \bmod (k_p) \equiv sk_n \bmod (k_r)$$

$$\equiv -rk_{r-1} + s_r k_n \bmod (k_r);$$

hence $s_p k_n \subset K_r$. Thus $\sigma(s_p k_n + j) = 0$ for $0 \leqslant j < k_n$. Therefore $\varphi_p = \varphi_r$ on I. We conclude

$$\text{either } \varphi = \varphi_n \quad \text{or } \varphi = \varphi_n + \sigma_{(s-s_r)k_n} \text{ on } I.$$

Finally let $m < n$ and recall that $\varphi_m = \varphi_n$ on $(K_{m+1} \cup \ldots \cup K_n)^c$. In particular, for any n-interval I, φ_m differs from φ_n on at most $|A_{m,n}|$ points by at most $\|\sigma\|_\infty$.

THEOREM 6.

(i) *$\psi \in \mathscr{E}(\bar{Z})$ if and only if $\sigma \in \mathscr{E}(Z)$.*

(ii) *If $\psi \in \mathscr{E}(\bar{Z})$, then $\psi^2 \in \mathscr{E}(\bar{Z})$ if and only if $\sigma^2 \in \mathscr{E}(Z)$.*

Proof. First note $\sigma \in \mathscr{E}(Z)$ precisely when $\mathscr{I}(\sigma) \equiv 0$ (similarly for σ^2) since $\sigma(k) = 0$ for $k < 0$. Define the averaging kernel $\{f_l\}$ on Z by $f_l(k) = k_l^{-1}$ if $-k_l < k \leqslant 0$ and $f_l(k) = 0$ otherwise. Fix $x \in [0, 2\pi)$, let $2 \leqslant m < n$, and let $I = [sk_n, (s+1)k_n)$ be an n-interval. According to the remarks

preceding this theorem, there is a $\delta = 1$ or 0, an $r = r(s, n)$, and an s_r such that

$$(1) \qquad f_n * (\varphi e^{-ix(\cdot)})(sk_n) = k_n^{-1} \sum_I \big(\varphi_n(j) + \delta\sigma(s_r k_n - sk_n + j)\big) e^{-ixj}$$

$$= k_n^{-1} \sum_I \varphi_m(j) e^{-ixj} + \varepsilon(m, n, s) + k_n^{-1} \delta \sum_I \sigma(s_r k_n - sk_n + j) e^{-ixj}$$

$$= f_n * (\varphi_m e^{-ix(\cdot)})(sk_n) + \delta f_n * (\sigma e^{-ix(\cdot)})(s_r k_n) e^{ix(s_r k_n - sk_n)} + \varepsilon(m, n, s),$$

where $|\varepsilon(m, n, s)| \leqslant k_n^{-1} \|\sigma\|_\infty |A_{m,n}| \leqslant \|\sigma\|_\infty \sum_m^\infty k_j/k_{j+1}$. Given $a > 0$ choose $N \geqslant 2$ so that $a > \|\sigma\|_\infty \sum k_j/k_{j+1}$, $N \leqslant j < \infty$. Then $\|\hat{\psi} - \hat{\varphi}_m\|_\infty \leqslant \|\psi - \varphi_m\|_1 \leqslant a$ for all $m \geqslant N$. Fix $m \geqslant N$ and choose $n_1 > m$ so that

$$(2) \qquad \|f_n * (\varphi_m e^{-ix(\cdot)}) - \hat{\varphi}_m(x)\|_\infty < a \quad \text{for all } n \geqslant n_1.$$

Assume first that ψ, hence φ, is ergodic. Then there is an $n_2 \geqslant n_1$ such that

$$(3) \qquad \|f_n * (\varphi e^{-ix(\cdot)}) - \mathscr{I}(\varphi)(x)\|_\infty < a \quad \text{for all } n \geqslant n_2.$$

Since $\mathscr{I}(\varphi) = \hat{\psi}$, we conclude from (1), (2), and (3)

$$(4) \qquad \delta |f_n * (\sigma e^{-ix(\cdot)})(s_r k_n)| < 3a, \qquad n \geqslant n_2,$$

where $\delta, r = r(s, n)$ depend on s, n. We claim (4) remains valid with $\delta = 1$ and s_r replaced by any integer s'. If $s' < 0$ or if $[s'k_n, (s'+1)k_n) \subset \bigcup K_j$ $(j \geqslant n+1)$, this follows from definition of σ. Otherwise choose p so that $s'k_n < k_{p-1}$ and consider the n-interval I determined by $sk_n \equiv k_p - pk_{p-1} + s'k_n$. Then $I \subset K_p$. If $r = r(s, n) < p$, then $s'k_n \equiv s_r k_n - rk_{r-1} \bmod (k_r)$. This implies $[s'k_n, (s'+1)k_n) \subset \bigcup K_j$ $(j \geqslant n+1)$, a contradiction. Therefore, for this I, $p = r(s, n)$, $s_r = s'$, and $\delta = 1$. We conclude

$$|f_n * (\sigma e^{-ix(\cdot)})(sk_n)| < 3a$$

for all $n \geqslant n_2$ and all $s \in Z$ which implies

$$\|f_n * (\sigma e^{-ix(\cdot)})\|_\infty < 3a + O(2k_{n_2}/k_n), \qquad n \to \infty.$$

Since $x \in [0, 2\pi)$ and $a > 0$ are arbitrary, $\sigma \in \mathscr{E}(Z)$.

The converse is much easier. The ergodicity of σ together with (1) and (2) implies (3), which means $\varphi \in \mathscr{E}(Z)$. Hence $\psi \in \mathscr{E}(\bar{Z})$ by Lemma 5.

To prove (ii) notice first that $(\{\varphi_n^2\}, \varphi^2, \psi^2)$ is also an admissable triple. In particular we can proceed as in (i). The expressions become

$$(5)\quad f_n*(\varphi^2 e^{-ix(\cdot)})(sk_n) = f_n*(\varphi_m^2 e^{-ix(\cdot)})(sk_n) + \delta f_n*(\sigma^2 e^{-ix(\cdot)})(s_r k_n)\,e^{ix(s_r k_n - sk_n)} + 2\delta f_n*(\sigma\varphi_m e^{-ix(\cdot)})(s_r k_n)\,e^{ix(s_r k_n - sk_n)} + \varepsilon'(m, n, s),$$

where $|\varepsilon'(m, n, s)| < (\|\varphi\|_\infty + \|\sigma\|_\infty)^2 \sum_m^\infty k_j/k_{j+1}$. Since we are assuming $\psi \in \mathscr{E}(\bar{Z})$, part (i) tells us that $\sigma \in \mathscr{E}(Z)$. In particular the first and third terms of (5) converge uniformly in s to $(\varphi_m^2)\hat{}\,(x)$ and 0 respectively as $n \to \infty$. Now argue precisely as in the proof of (i) to conclude (ii). ■

Suppose now that $\sigma \in C(Z)$ satisfies

$$(\mathrm{P})\quad \begin{cases} (\mathrm{a}) & \sigma(k) = 0 \quad \text{if } k \in (-\infty, 0) \cup \big(\bigcup_2^\infty K_j\big), \\ (\mathrm{b}) & \sigma \in \mathscr{E}(Z), \\ (\mathrm{c}) & \sigma^2 \notin \mathscr{E}(Z), \end{cases}$$

and let $(\{\varphi_n\}, \varphi, \psi)$ be the admissable triple determined (up to φ_1) by σ. Then Theorem 6 implies $\psi \in \mathscr{E}(\bar{Z})$ but $\psi^2 \notin \mathscr{E}(\bar{Z})$. It follows that $\psi \in \mathscr{E}(\bar{Z}) \cap W(\bar{Z})^c$ since $W(Z)$ and hence $W(\bar{Z})$ is closed under products. In particular $\varphi \in \mathscr{E}(Z) \cap W(Z)^c$. The referee has pointed out that φ is in fact a bounded Besicovitch almost periodic function ([1], pages 73,95) which is ergodic but not Weyl almost periodic. That is, there is a sequence $\{p_m\} \subset P(Z)$ such that the sequence

$$r_m = \limsup_n \frac{1}{2n+1} \sum_{k=-n}^{n} |\varphi(k) - p_m(k)|$$

converges to zero. In this case set $p_m = \varphi_m$ and observe that $|r_m| = O(\sum_m^\infty k_j/k_{j+1})$ as $m \to \infty$. This estimate, which holds for any admissable triple, is based on Lemma 3 and the fact that $k_{j-1}/(k_j - jk_{j-1}) \to 0$.

The existence of $\sigma \in C(Z)$ satisfying (P) follows essentially from a result of Kahane ([5], Théorème 2). However this implication requires some notation and a few comments. To begin, choose $g \in C(\boldsymbol{R})$ supported on $[0, 1]$ so that $\hat{g}(t) \neq 0$ for $t \in [0, 2\pi]$. ($g = 1$ at $\frac{1}{2}$, 0 off $[0, 1]$, and linear on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$ will work.) For $\tau \in C(Z)$ define $T_g(\tau) \in C(\boldsymbol{R})$ by

$$T_g(\tau)(r) = \sum_{-\infty}^{\infty} g(r-k)\,\tau(k).$$

By using an averaging kernel, it is easy to see that $T_g(\tau) \in \mathscr{E}(\boldsymbol{R})$ if and only if $\tau \in \mathscr{E}(Z)$. In fact if $\tau \in \mathscr{E}(Z)$, then $\mathscr{F}(T_g(\tau)) = \hat{g}\mathscr{F}(\tau)$ where $\mathscr{F}(\tau)$ is considered as a periodic function on $\boldsymbol{R}$. Define $T_g(\tau)_n = T_g(\tau)$ on $[n, n+1)$ and $T_g(\tau)_n = 0$ elsewhere. The theorem of Kahane implies there exists a ± 1 valued function Δ on Z such that $f \equiv \sum \Delta(n) T_g(\tau)_n \in \mathscr{E}(\boldsymbol{R})$ and $\mathscr{F}(f) \equiv 0$ (Δ depends on τ). For our purposes let τ be the characteristic function of $[0, \infty) \cap (\bigcup_2^\infty K_j)^c$ and observe $\sum \Delta(n) T_g(\tau)_n = T_g(\Delta\tau)$. Set $\sigma = \Delta\tau$. Then σ satisfies (a) and (b) of (P). Since $\sigma^2 = \tau$ and since

$$k_n^{-1} \sum_{0 \le k < k_n} \tau(k) = 1 - k_n^{-1}|\Delta_{1,n}| = \prod_{j=1}^{n-1} (1 - k_j/k_{j+1})$$

is bounded away from zero, σ also satisfies (c).

Lemma 5, Theorem 6, and the remarks concerning Weyl and Besicovitch almost periodic functions remain valid when Z is replaced by $\boldsymbol{R}$. To show this we need: if $\{f_n\}$ and $\{h_n\}$ are averaging kernels on Z and $\boldsymbol{R}$ respectively and if $x \in \boldsymbol{R}$, then

$$(*)\qquad \lim_n \|\hat{g} f_n*(\tau e^{-ix(\cdot)})\|_\infty = \lim_n \big\|h_n*\big(T_g(\tau)\,e^{-ix(\cdot)}\big)\big\|_\infty.$$

Its proof is straightforward if nf_n and nh_n are the characteristic functions of $[0, n) \subset Z$ and $[0, n) \subset \boldsymbol{R}$, respectively. The general case then follows immediately from the theorem in the introduction. Before continuing, let us assume $(g^2)\hat{}\,(t) \neq 0$ for $t \in [0, 2\pi)$ (the g mentioned earlier works) and observe that $\big(T_g(\tau)\big)^2 = T_{g^2}(\tau^2)$. Given ψ in the admissable triple $(\{\varphi_n\}, \varphi, \psi)$, define $T_g(\psi) \in L^\infty(\boldsymbol{R})$ by $T_g(\psi) = \lim T_g(\varphi_n)$ in $L^1(\bar{\boldsymbol{R}})$. We call $\big(\{T_g(\varphi_n)\}, T_g(\varphi), T_g(\psi)\big)$ an admissable triple on $\boldsymbol{R}$. Now observe that $(*)$ together with Proposition 1 and Corollary 2 is all that is needed to extend Lemma 5 and Theorem 6 to $\boldsymbol{R}$. The fact that $T_g(\psi)$ no longer necessarily has precompact spectrum in $\boldsymbol{R}$ offers no problem. This assumption was only used in Proposition 1 to guarantee the existence of certain cluster points. Here $T_g(\varphi)$ is such a point. Finally, we note that if $T_g(\psi)\hat{}$ is the transform of a general Ryll-Nardzewski or Weyl almost periodic χ on $\boldsymbol{R}$([5], [1]) with $\int_t^{t+1} |\chi|\,dx$ uniformly bounded (hence any Weyl a-p function), then $T_g(\varphi) \in \mathscr{E}(\boldsymbol{R})$ or $W(\boldsymbol{R})$, respectively. To see this let $\{f_a\} \subset L^1(\boldsymbol{R})$ be a bounded approximate identity with support f_a compact. Then $f_a * \chi \in C(\boldsymbol{R})$ and is in the same class as χ. Hence $f_a * T_g(\varphi) \in \mathscr{E}(\boldsymbol{R})$ or $W(\boldsymbol{R})$. But $T_g(\varphi)$ is the uniform limit of $\{f_a * T_g(\varphi)\}$.

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Integration of evolution equations in a locally convex space

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Abstract. Let $H = H(\boldsymbol{R}^m)$ be the space of all real-valued functions in $C^\infty(\boldsymbol{R}^m)$ having every partial derivative in $L_2(\boldsymbol{R}^m)$ and topologized by the seminorms defined as follows:

$$p_i(\varphi) = \Big(\sum_{|\nu|=0}^{i} \int_{\boldsymbol{R}^m} |D^{(\nu)}\varphi(\underset{\sim}{t})|^2 d\underset{\sim}{t}\Big)^{1/2}, \quad \varphi \in H,\ i = 0, 1, 2, \ldots$$

Let A be an elliptic differential operator with coefficients possessing bounded derivatives of all orders. This paper solves the Cauchy problem for the system:

$$\frac{\partial u(\xi, \underset{\sim}{t})}{\partial \xi} = (Au)(\xi, t), \quad \xi > 0,\ \underset{\sim}{t} \in \boldsymbol{R}^m,$$

$$u(0, \underset{\sim}{t}) = f(\underset{\sim}{t}), \quad f \in H,\ \underset{\sim}{t} \in \boldsymbol{R}^m.$$

1. Introduction. The present paper is a follow-up to [2], and its knowledge is assumed here. Let Ω be an open subset of a Euclidean space. For convenience we shall denote by $C^\infty = C^\infty(\Omega)$ the space of all infinite times continuously differentiable real-valued functions on Ω and by $C_0^\infty(\Omega)$ the space of functions in $C^\infty(\Omega)$ having compact support in Ω.

Now let A be the partial differential operator of $2n$th order in m-dimensional Euclidean space $\boldsymbol{R}^m$ given by

$$A = -(-1)^n \sum_{|\varrho|,|\nu|=0}^{n} D^{(\varrho)} a_{\varrho;\nu}(\underset{\sim}{t}) D^{(\nu)}, \tag{1.1}$$

where the coefficients $a_{\varrho;\nu}$ belong to $C^\infty(\boldsymbol{R}^m)$ with bounded partial derivatives of all orders. We assume further that $a_{\varrho;\nu}(\underset{\sim}{t}) = a_{\nu;\varrho}(\underset{\sim}{t})$ for $|\varrho| = |\nu| = n$ and there is a constant $\varepsilon_0 > 0$ such that

$$\sum_{|\varrho|=|\nu|=n} a_{\varrho;\nu}(\underset{\sim}{t}) t_1^{\varrho_1} \ldots t_m^{\varrho_m} \cdot t_1^{\nu_1} \ldots t_m^{\nu_m} \geqslant \varepsilon_0 \Big(\sum_{j=1}^{m} t_i^2\Big)^n \tag{1.2}$$

for each $(t_1, \ldots, t_m) \in \boldsymbol{R}^m$; so that A is an elliptic differential operator.