Duality and fractional integration in Lipschitz spaces

by

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Abstract. The definition of the spaces \( A(R, X) \) of A. P. Calderón are extended to include the duals of the spaces considered by him. The fractional integration operators on these spaces are constructed. The definitions and theorems are further generalized by considering homogeneity with respect to a one-parameter group of linear transformations. In particular, these results generalize the theory of classical Lipschitz spaces of distributions on \( R^n \) (see \cite{6}).

Introduction. This paper is concerned with the spaces \( A(R, X) \) introduced by A. P. Calderón in \cite{3}. Here \( B \) is a Banach space of tempered distributions on \( R^n \) (\( n \)-dimensional Euclidean space), \( X \) is a Banach lattice of measurable functions on the interval \((0, 1]\) and \( \ast \) is a measure satisfying certain conditions. (The precise definitions are given in Sections 2 and 3.) Then \( u \ast B \) is said to be a member of \( A(R, X) \) if \( \| (u \ast \eta) \|_B \in X \), where \( \ast \) denotes convolution and \( \eta \) is defined for a measurable subset \( E \) of \( R^n \) and for \( t > 0 \) by \( \eta_t(E) = \frac{1}{t} \mathbb{1}_E \).

These spaces generalize and include certain Lipschitz space of smooth functions \((\varepsilon), (\varepsilon), (\varepsilon))\). For example, on \( R^1 \), if we let \( \nu \) be the Dirac measure at \( 0 \) minus the Dirac measure at \(-1\), and let \( X \) be the class of measurable functions \( f \) for which \( f \in L^\infty(0, 1] \) and \( 0 < a < 1 \), then we obtain the space \( A_\varepsilon \) of functions \( f \) for which \( \| f (\cdot) - f (\cdot + t) \|_\infty \leq C |t|^{\varepsilon} \), where \( C \) is a constant independent of \( t \).

A. P. Calderón defined the spaces \( A(R, X) \) for Banach lattices \( X \) of positive type (see Definitions 2.3 and 3.1). In this paper the definitions are extended to include non-positive type. Those of negative type turn out to be the duals of those considered by Calderón. One of the principal contributions of this paper is the construction of the fractional integration operators on the spaces \( A \). These are the analogues of the Riesz and Bessel potential operators.

The definitions and theorems are further generalized by considering homogeneity with respect to a group of linear transformations \( (T_t)_{t \in R} \) on \( R^n \) as studied by de Guzmán in \cite{4}. This point of view allows us to include mixed homogeneity and spaces of functions satisfying different Lipschitz
conditions in different directions. An example of such a family of linear transformations is
given by

\[ T_\lambda(s) = (t_1^\lambda s_1, t_2^\lambda s_2, \ldots, t_n^\lambda s_n) \]

where \( s = (s_1, \ldots, s_n) \in \mathbb{R}^n \) and \( a_1, a_2, \ldots, a_n \) are positive constants. Then the case considered by Calderón amounts to letting

\[ a_1 = a_2 = \ldots = a_n = 1. \]

An example of how spaces of functions satisfying different Lipschitz conditions in different directions are included in this approach is obtained by letting the Banach space \( B \) be \( L^p(\mathbb{R}^n), 1 \leq p < \infty \). The measure \( \nu \) is taken to be equal to twice the Dirac measure at the origin minus the Dirac measures at \( (-1, 0) \) and \( (0, -1) \). For \( \sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2 \), we let

\[ T_\sigma(x_1, x_2) = (\sigma_1 x_1, \sigma_2 x_2) \]

where \( \alpha > 0 \) and \( \beta > 0 \). Let \( X_{\alpha, \beta} \) be the Banach lattice of measurable functions \( g \) on \( (0, 1] \) for which

\[ \|g\| = \left( \int_0^1 (\int_{x_1}^{x_2} |g(t)|^q \, dt)^{1/q} \right)^{1/q}, \quad 1 \leq q < \infty, \]

\[ \sup_{t \in \mathbb{R}} |g(t)|, \quad q = \infty, \]

is finite. If \( f \in L^p \) satisfies the conditions

\[ \|f(x_1 + t, x_2) - f(x_1, x_2)\|_{\alpha, \beta} \]

and

\[ \|f(x_1, x_2 + t) - f(x_1, x_2)\|_{\alpha, \beta} \]

then

\[ \|f(x_1 + t, x_2) - f(x_1, x_2)\|_{\alpha, \beta} \]

and

\[ \|f(x_1, x_2 + t) - f(x_1, x_2)\|_{\alpha, \beta} \]

Thus \( \|f \|_{\alpha, \beta} \) is bounded since

\[ (f \ast g)(x) = 2f(x_1, x_2) - f(x_1 + t, x_2) - f(x_1, x_2 + t) \]

The following is an outline of this paper. Section 1 contains the main tool, which is a representation theorem enabling us to recover a tempered distribution \( \nu \) from \( \nu \ast \nu \). (The exact statement is given by Theorem 1.1.) This theorem, for the case \( \nu = \nu \ast \nu \), was first proved by Calderón and the same proof carries over to the more general setting considered in this paper.

Section 2 contains the definition of a Banach lattice and preliminary lemmas establishing the properties of Banach lattices that are needed in the sequel. In this section is also defined the "type" of a Banach lattice (Definition 3.3). Several examples of Banach lattices are given. We assume that the dual of a Banach lattice is again a Banach lattice.

In Section 3 are defined the spaces \( A(B, X) \), for \( X \) of positive type. Theorem 3.1 is a basic result about a continuous mapping from the direct sum of \( B \) and \( X(B) \) into a \( A \) space. This theorem allows us to prove (see Theorem 3.2 and corollary) that under certain conditions the spaces \( A \) are independent of the measure \( \nu \). In fact, the measure \( \nu \) can be replaced by \( \varphi(x) \, dx \), where \( \varphi \) is a Schwartz test function having certain properties (but otherwise arbitrary) and where \( dx \) denotes Lebesgue measure.

In Section 4 use is made of Theorem 3.1 to construct invertible fractional integration operators mapping a \( A \) space of positive type continuously onto another \( A \) space of positive type. The method of construction is similar to the proof of Theorem 1.1.

The spaces \( A(B, X) \), for \( X \) of non-positive type are defined in Section 5.

This definition is made in terms of the operators constructed in Section 4. The restriction of these operators to spaces of positive type is removed and it is shown (Theorem 5.1) that under certain conditions \( A(B, X) \), is independent of \( \nu \) and that \( \nu \) can be replaced by \( \varphi(x) \, dx \), as was the case for \( X \) of positive type in Section 3.

In the last section it is shown that under certain natural conditions the dual of \( A(B, X) \), is the space \( A(B', X') \), where \( B' \) and \( X' \) are the duals of \( B \) and \( X \) respectively.

The results of this paper form part of the author's Ph. D. thesis [5], which also included a development of the Lipschitz spaces where the Banach lattice is defined on the non-negative integers.

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**Notation.** \( \mathbb{R}^n = \{ x = (x_1, \ldots, x_n) \text{ with } x_1, \ldots, x_n \text{ real} \} \) is \( n \)-dimensional Euclidean space with the norm \( |x| = (\sum_{i=1}^n x_i^2)^{1/2} \), \( d \) is \( n \)-dimensional Lebesgue measure.

**\( \mathcal{S} = \mathcal{S}(\mathbb{R}^n) \)** will denote the collection of Schwartz test functions, i.e. all those \( C^\infty \) functions \( \varphi \) on \( \mathbb{R}^n \) for which \( \sup_{t \in \mathbb{R}^n} \|D^\alpha \varphi(t)\| < \infty \) for all \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) of non-negative integers. \( \Phi \) will denote the collection of all continuous linear functionals on \( \mathcal{S} \) and is called the space of tempered distributions.

Let \( B \) be a Banach space of tempered distributions on \( \mathbb{R}^n \) and \( X \), a \( B \)-lattice of functions on \( (0, 1] \). Then \( X(B) \) will denote the class of \( B \)-valued functions \( f(t) \) for which \( \|f(t)\|_B \in X \).
1. A REPRESENTATION THEOREM OF A. P. CALDERÓN

The following situation was studied by M. de Guzmán [4]: Let \( \{T_t\} \) for \( t > 0 \) be a group of linear transformations on \( \mathbb{R}^n \) for which \( T_{t_1}T_{t_2} = T_{t_1+t_2}, \) \( T_1 = I, \) the mapping \( t \mapsto T_t \) is continuous, and for each \( x \in \mathbb{R}^n, \) \( \|T_t x\| \rightarrow 0 \) as \( t \downarrow 0 \) and \( \|T_t x\| \rightarrow \infty \) as \( t \rightarrow 0 \) (\( \varepsilon \neq 0 \)). \( T_t \) can be expressed in the form \( T_t = e^{\sum_{k=1}^n a_k t_k P_k} \) where \( P \) is a matrix whose eigenvalues have positive real parts. Furthermore, we assume \( P \) satisfies the conditions of Theorem 3.1.6 of [4], in particular those at the top of page 117. The conditions include the case where \( P \) is diagonal. For each non-zero \( x \in \mathbb{R}^n \) there exists a unique \( t_x > 0 \) such that \( \|T_{t_x} x\| = 1. \) Define \( g(x) = 1/t_x. \) Then \( g(x-y) \) is a metric on \( \mathbb{R}^n, \) \( g(T_t x) = t g(x) \) and \( g(0) = 1 = \|x\|. \) Let \( T_t^* \) be the adjoint of \( T_t. \) Then \( \langle T_t^* \rangle \) has the same properties as \( \{T_t\} \) above and we can define a metric \( g^* \) on \( \mathbb{R}^n \) in the same way as \( g \) above.

Let \( \varphi \) be a vector-valued finite Borel measure on \( \mathbb{R}^n \) (with values in \( \mathbb{R}^m \)) and define \( \varphi(E) = \varphi(T_t E) \) for any \( \varepsilon \)-measurable set \( E. \) If \( \varphi \) is of the form \( \varphi(x) = \Phi(x) \psi(x), \) then \( \varphi(E) = \int E \varphi \psi d t \varphi \) since \( \det T_{t_x} = e^{\sum_{k=1}^n a_k t_k P_k} \)

Theorem 0.1. Let \( \varphi \) be a vector-valued finite Borel measure with the additional property that, for each vector \( \alpha \) of unit length, \( \varphi(T_t \alpha) \) is not identically zero as a function of \( t > 0. \) \( \varphi \) is the Fourier transform of \( \varphi \) where \( \hat{\varphi}(\alpha) = \int \mathcal{M}^{-\|\alpha\|^2} \varphi \psi d t \varphi \) Then there exists a Schwartz test function \( \eta \) such that, for each tempered distribution \( \mu, \)

\[
\mu = \int_{\mathbb{R}^n} \mu * \eta d t \varphi
\]

Also \( \eta \) has the following properties:

(i) \( \eta \) is zero in a neighborhood of the origin and has compact support;

(ii) for each non-zero vector \( \alpha \in \mathbb{R}^n, \) \( \hat{\varphi}(T_t \alpha) \) is not identically zero as a function of \( t > 0. \)

We can also write

\[
u = \mu * \eta + \int_{\mathbb{R}^n} \mu * \eta d t \varphi
\]

where \( \psi \in \mathcal{S} \) (the class of Schwartz test functions).

Proof. We first note that \( \hat{\varphi}(T_t \alpha) = \hat{\varphi}(T_t \alpha) \) Let \( g_N(x) = \int_{\mathbb{R}^n} \hat{\varphi}(T_t \alpha) d t \varphi \)

where \( N > 0. \) Then it is possible to find numbers \( \varepsilon > 0 \) and \( N > 0 \) such that \( \|g_N(x)\| > \varepsilon \) for every vector \( x \) of unit length. For let

\[
E_N = \{x : |x| = 1 \text{ and } g_N(x) \leq \frac{1}{N} \}
\]

Then \( E_{N+1} \subseteq E_N. \) Also \( \bigcap_{N=1}^\infty E_N \) is empty, because if \( \alpha \in \bigcap_{N=1}^\infty E_N \) then

\[
\int_{\mathbb{R}^n} \hat{\varphi}(T_t \alpha) d t \varphi > 0,
\]

from which it would follow that \( \hat{\varphi}(T_t \alpha) = 0, \) contradicting the assumption on \( \varphi. \) Therefore, for every vector \( \alpha \) of unit length, there exists \( N_x > 0 \) such that \( g_{N_x}(x) > \frac{1}{N_x}. \) We now use the continuity of each function \( g_N \) and the compactness of \( \{x : |x| = 1\} \) to deduce the existence of the numbers \( \varepsilon \) and \( N \) above.

Consider now \( \psi \in C^\infty_0 \) with the following property:

\[
\psi(x) = \begin{cases} 
0, \text{ near } 0 \text{ and } \infty, \\
nondecreasingly \text{ for } \frac{1}{N_x} < |x| < N_x, \\
nondecreasingly \text{ otherwise, and } C^\infty_0.
\end{cases}
\]

Next, for \( x \neq 0, \) we let

\[
[N(x)]^{-1} = \int_{\mathbb{R}^n} \hat{\varphi}(T_t \alpha) d t \varphi
\]

Then \( N(x) \) is homogeneous of degree 0, that is \( N(x) = N(x') \) where \( x \) is the unique unit vector for which \( x = T_{t_0} x'. \)

Also \( [N(x)]^{-1} \) is \( C^\infty \) on \( \{x : |x| = 1\} \) and greater than or equal to \( \varepsilon \) there.

Now let

\[
\hat{\eta}(x) = \frac{\overline{\varphi}(x)}{N(x')} \psi(x).
\]

Then \( \hat{\eta} \in C^\infty_0, \) so that \( \eta \in \mathcal{S}. \)
Finally,
\[
\left( \int \frac{u \ast \phi(t)}{t} \, dt \right) = \left( \int \frac{v(t)}{t} \ast \phi(t) \, dt \right) = \left( \int \frac{v(t) \phi(t)}{t} \, dt \right) = \hat{\phi}(0).
\]

We also have \( \phi(x) = \int \frac{v(t) \phi(t)}{t} \, dt \) is a \( C^\infty \) function of \( x \) with compact support. Hence \( u = u \ast \phi + \int \frac{u \ast \phi(t)}{t} \, dt \), where \( \psi \in \mathcal{S}^\prime \) and \( \hat{\psi} = \phi \).

2. BANACH LATTICES OF MEASURABLE FUNCTIONS

Before we can define the classes of tempered distributions which form the subject matter of this paper, we need to define the concept of Banach lattice. The purpose of this section is to define this concept, to define the "type" of a Banach lattice, to discuss some examples and to establish those properties that will be needed in subsequent sections.

**Definition 2.1.** Let \( X \) be a Banach space of measurable and locally integrable functions on the interval \( (0, 1) \) of the real line. Such a space \( X \) will be called a Banach lattice if whenever \( f \in X \) and \( g \) is a measurable function on \( (0, 1) \) for which \( |g(x)| < |f(x)| \) almost everywhere then \( g \in X \) and \( \|g\|_X \leq \|f\|_X \).

**Definition 2.2.** Let \( \alpha \) be a real number.

(a) A Banach lattice \( X \) on \( (0, 1) \) will be said to satisfy condition \( \Lambda_\alpha \) if the mapping
\[
g \mapsto \int_0^1 g(t) \left( \frac{\phi(t)}{t} \right)^\alpha \, dt
\]

is a bounded linear operator on \( X \).

(b) A Banach lattice \( X \) on \( (0, 1) \) will be said to satisfy condition \( \beta_\alpha \) if the mapping
\[
g \mapsto \int_0^1 g(t) \left( \frac{\phi(t)}{t} \right)^\alpha \, dt
\]

is a bounded linear operator on \( X \).

It is immediate that if \( X \) satisfies condition \( \Lambda_\alpha \) then \( X \) also satisfies condition \( \Lambda_\beta \) for each \( \beta < \alpha \). Similarly, if \( X \) satisfies condition \( \beta_\alpha \) then \( X \) also satisfies condition \( \beta_\beta \) for each \( \beta > \alpha \).

**Definition 2.3.** We shall say that the Banach lattice \( X \) is of type \( \alpha \) if \( X \) satisfies \( \Lambda_\alpha \) for all \( \beta < \alpha \) and satisfies \( \beta_\alpha \) for all \( \beta > \alpha \).

**Definition 2.4.** Let \( X \) be a Banach lattice. By \( \mathcal{B}X \) we shall mean the class of measurable functions \( g \) on \( (0, 1) \) for which \( \mathcal{B}^\alpha g(t) \in X \). Equipped with the norm \( \|\mathcal{B}^\alpha g(t)\|_X \), the space \( \mathcal{B}X \) is a Banach lattice.

It follows immediately from the definitions above that if \( X \) satisfies \( \Lambda_\alpha \) (or \( \beta_\alpha \)) then \( \mathcal{B}X \) satisfies \( \Lambda_{\alpha + \beta} \) (or \( \beta_{\alpha + \beta} \)).

Some examples of the type of Banach lattice discussed above are:

**Example 2.1.** The Orlicz space \( L_\Phi \), see [3], where \( \Phi \) is a Young's function, and a measurable function \( g \) on \( (0, 1) \) is in \( L_\Phi \) if and only if there is a \( \lambda > 0 \) such that
\[
\int_0^1 \Phi \left( \frac{\lambda}{g(t)} \right) \frac{dt}{t} < \Phi(1).
\]

By first establishing a generalization of Hardy's inequality, it can be shown that \( L_\Phi \) is of type \( 0 \). (See [5] for details.) Hence \( \mathcal{B}L_\Phi \) is of type \( \alpha \).

**Example 2.2.** The space \( X_{\alpha, \beta} \) (for \( -\infty < \alpha < \infty \) and \( 1 \leq \beta \)) of all measurable functions \( g \) on \( (0, 1) \) for which, if \( 1 \leq \beta < \infty \),
\[
\|g\|_{X_{\alpha, \beta}} \leq \int_0^1 \left( \mathcal{B}^\alpha g(t) \right)^\beta \, dt
\]
is finite, and for which
\[
\|g\|_{\infty, \beta} = \sup_{t \in (0, 1)} \mathcal{B}^\alpha g(t) \leq \infty \text{ if } \beta = \infty
\]

By noting that if \( 1 \leq \beta < \infty \) then the function \( \Phi(x) = x^\beta \) is a Young's function, and using the observation in the last sentence before this example we have that \( X_{\alpha, \beta} \) is of type \( \alpha \). The fact that \( X_{\infty, \beta} \) is of type \( \alpha \) follows immediately from the definitions.

**Lemma 2.1.** If \( X \) is a Banach lattice of type \( \alpha \), then the function \( \mathcal{B}^{\alpha + \epsilon} X \) for every \( \epsilon > 0 \). Thus if \( \alpha \) is negative, the constant functions are in \( X \).

**Proof.** Let \( \chi \) be the characteristic function of the interval \( [t], 1] \).

Then for any non-negative function \( g \in X \) we have
\[
\chi_e \mathcal{B}^{\alpha + \epsilon} X \leq \int_0^1 g(t) \left( \frac{\chi_e(t)}{t} \right)^{\alpha + \epsilon} \, dt \leq \int_0^1 g(t) \left( \frac{\chi_e(t)}{t} \right)^\alpha \, dt,
\]

where \( e = \max(1, 2^\alpha) \). Thus \( \chi_e X \) and therefore so is the function \( f(t) = \mathcal{B}^{\alpha + \epsilon} X \), since \( f(t) \leq c \chi_e(t) \).

Next, for \( 0 < \beta < \infty \), we note that
\[
\int_0^1 \mathcal{B}^{\alpha + \epsilon} X \left( \frac{\chi_e(t)}{t} \right)^{\alpha + \epsilon} \, dt \leq \int_0^1 g(t) \left( \frac{\chi_e(t)}{t} \right)^\alpha \, dt.
\]
Thus the function \( k(t) \) which is equal to \( t^{1+\delta} \) for \( 0 < t < \frac{1}{4} \) and zero elsewhere is in \( X \).

The result of the lemma now follows from the fact that \( t^{1+\delta} = f(t) + + k(t) \).

**Lemma 2.2.** Suppose \( X \) is of type \( a \) and that \( \beta \) is such that \( a + \beta > 0 \). Then for each function \( g \in X \) we have

\[
\left| \int_0^1 g(t)^\delta \frac{dt}{t^\epsilon} \right| \leq c |g|_X,
\]

where the constant \( c \) is independent of \( g \).

**Proof.** Without loss of generality we may assume that \( g \geq 0 \). After noting that the function \( s^{a+\beta} X \) (see previous lemma) and that \( X \) satisfies the conditions \( A_{a,0} \) and \( B_{a,1} \), the result is a consequence of the following inequality:

\[
s^{a+\beta} \int_0^1 g(t)^\delta \frac{dt}{t^\epsilon} \leq \int_0^1 g(t) \left( \frac{s}{t} \right)^{a+\beta} \frac{dt}{t^\epsilon} + \int_0^1 g(t) \left( \frac{s}{t} \right)^{a+\gamma} \frac{dt}{t}.
\]

We next discuss the duals of Banach lattices.

We assume that the dual, \( X' \), of a Banach lattice \( X \) is again a Banach lattice of measurable and locally integrable functions on \( (0, 1) \). We also assume that the action of \( f \in X' \) on \( g \in X \) is given by \( \frac{1}{g} \int f(t) g(t) \frac{dt}{t} \) and write this \( \langle f, g \rangle \).

Let \( g \in X' \) and \( f \in X \). Then \( L_f \) defined on \( X' \) by \( L_f(g) = \int f(t) g(t) \frac{dt}{t} \) is a bounded linear functional on \( X' \) whose norm is equal to the norm of \( f \) in the space \( X \). Conversely, every bounded linear functional on \( X' \) is of the above form. Thus we have

**Lemma 2.3.** The dual of the Banach lattice \( X \) is \( X' \).

It is known [8], that if the Young's function \( \Phi \) is such that \( \Phi(\alpha g(t)) dt = \infty \) for all \( \alpha > 0 \) coincides with \( L_\alpha \) then the dual of \( L_\alpha \) is \( L_\alpha \) where \( \Phi \) is the complementary Young's function and the action of \( f \in L_\alpha \) on \( g \in L_\alpha \) is given by \( \int f(t) g(t) \frac{dt}{t^\epsilon} \). Thus the dual of \( X_{a,0} \) is \( X_{a,0} \),

where \( 1 \leq q < \infty \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \).

Finally, we need the following

**Lemma 2.4.** If the Banach lattice \( X \) satisfies the condition \( A_\alpha \) (or \( B_\alpha \)) then \( X' \) satisfies the condition \( B_{a,0} \) (or \( A_{a,0} \)). Thus if \( X \) is of type \( a \), then \( X' \) is of type \(-a\).

The proof is a straightforward application of the definitions.

### 3. The spaces \( A(B, X) \) for \( X \) of positive type

**Definition 3.1.** Let \( \nu \) be a vector-valued finite Borel measure on \( \mathbb{R}^n \) having one of the following properties: (i) \( \nu \) has compact support, or (ii) \( \nu \) is of the form \( \nu = \rho \eta \) where \( \rho \) does not necessarily have compact support, but for each non-negative integer \( I \) there exists a constant \( D_I \) such that, for all \( \alpha \in \mathbb{R}^n \),

\[
|\nu(\alpha)| \leq D_I \frac{|\alpha|}{(1 + |\alpha|)^{1+I}}.
\]

Let \( B \) be a Banach space of tempered distribution in which \( \mathcal{S} \) is continuously embedded and for which \( \|\nu \|_{\mathcal{S}'} \leq C \|\nu\|_B \) for every \( \nu \in B \). (If \( \nu \in \mathcal{S} \), we define \( \langle \nu, u \rangle \) by \( \langle \nu, u \rangle = \langle \nu', u \rangle \).

Let \( X \) be a Banach lattice of type \( a \), where \( a \) is positive, as defined in the previous section. Then A.P. Calderón defined the space \( A(B, X) \) (for the case \( T_\nu = f\nu \)) to be

\[
\{ u \in B : \|u * \nu\|_{\mathcal{S}'} \}
\]

(\( \nu \) was defined in Section 1.) Equipped with the norm \( \|u\|_{A(B, X)} = \|u\|_B + + \|u * \nu\|_{\mathcal{S}'} \), \( A(B, X) \) is a Banach space, whose embedding in \( B \) is continuous. (See A. P. Calderón [3], page 126.)

We shall often write \( \Lambda \) instead of \( A(B, X) \), and denote the norm of \( u \in \Lambda \) by \( \|u\|_{A(B, X)} \), when it is clear which Banach space \( B \), which Banach lattice \( X \) of type \( a \) and which measure \( \nu \) are involved.

The case where \( X = X_{a,0} \) and \( B = L_p(\mathbb{R}^n) \), where \( 1 < p < \infty \), and \( T_\nu = \nu \) was studied by M.H. Taibleson in [6] and [7].

The following two theorems are due to A.P. Calderón for the case \( T_\nu = f\nu \).

**Theorem 3.1.** Let the measure \( \nu \), and the spaces \( X \) and \( B \) be as above. Suppose also that all the moments of \( \nu \) up to and including \( k \) are zero, where \( 0 < a < b \), i.e.

\[
\int \Phi(\alpha g(t)) dt = 0
\]

for all non-negative integers \( \beta = (b_1, b_2, \ldots, b_k) \) for which \( |\beta| = b_1 + + + b_k < k \).

Let \( u \in B \) and \( f \in X(B) \). Then the mapping

\[
(u, \nu) \rightarrow u * \nu = \int_0^1 \frac{f(t) \nu(t) \frac{dt}{t}}{1-t^\epsilon}
\]

is continuous for \( f(t) \rightarrow f(t) \) as \( t \rightarrow 0 \).
where \( \gamma \) and \( \eta \) are test functions, is defined when \( 0 < \alpha + \beta < k + 1 \) and maps \( B \otimes X(B) \) continuously into \( A(B, E X) = A_{\alpha+\beta} \).

Remark. For some different mapping theorems, see [2].

**Lemma 3.1.** Let \( \gamma \) be the same as in Theorem 3.1. Let \( \eta \) be a \( C^m \) function for which \( \|D^\beta \eta(x)\| \leq \frac{D_3}{(1 + |x|)^{m+2}} \) for all \( x \in \mathbb{R}^n \) and all \( n \)-tuples \( \beta \) of non-negative integers for which \( |\beta| \leq k + 1 \), and where \( D_3 \) is a constant. Then \( \|\eta_{\gamma} \eta_{\gamma}\|_1 \leq C \min \left( 1, \frac{s}{t} \right)^m \) for any \( m \in \{0, k+1\} \) and where \( 0 < s \leq t < 1 \).

**Proof.** Clearly \( \|\eta_{\gamma} \eta_{\gamma}\|_1 \leq \|\eta_{\gamma}\|_1 \cdot \|\eta_{\gamma}\|_1 = \|\eta_{\gamma}\|_1 \cdot \|\eta_{\gamma}\|_1 = C \). Now let \( m \) be one of the integers \( 1, 2, \ldots, k+1 \), and let \( 0 < s \leq t < 1 \). We want to show that

\[
\|\eta_{\gamma} \eta_{\gamma}\|_1 \leq C \frac{s}{t}^m.
\]

By Taylor's theorem we have

\[
\eta(x-y) = \sum_{|\alpha| \leq m} \frac{\partial^\alpha \eta}{\alpha!} (x) y^\alpha + R(x, y)
\]

where \( |R(x, y)| \leq C(x) \cdot |y|^m \) and \( C(x) \in C(\mathbb{R}^n) \). Therefore

\[
\|\eta_{\gamma} \eta_{\gamma}\|_1 = \int_{\mathbb{R}^n} R(T_{1\alpha}, T_{1\gamma}) t^{-\alpha} dv(T_{1\alpha} y)
\]

because the moments of \( \gamma \) of orders less than \( m \) are zero.

(3.3) \[ \cdot \cdot \cdot \leq t^{-\alpha} C(T_{1\alpha} y) \int_{\mathbb{R}^n} |T_{1\alpha} y|^{m} dv(T_{1\alpha} y). \]

We want to show that the integral in this last expression is less than or equal to a constant times \( \frac{s}{t}^m \cdot (3.3) \) would then follow immediately.

To prove the statement about the integral in (3.3) we need the following facts from M. de Guzman's work [4]:

(3.4) **There exist numbers** \( 1 \leq p \leq q \) **such that**, for all \( x \in \mathbb{R}^n \),

\[
\|\gamma(x)\|^p \leq |x| \cdot \|\gamma(x)\|^p \quad \text{if} \quad |x| \leq 1
\]

and

\[
\|\gamma(x)\|^p \leq |x| \cdot \|\gamma(x)\|^p \quad \text{if} \quad |x| \geq 1
\]

(3.5) **\( \gamma(x) \) always lies between \( 1 \) and \( |x| \). Thus**

\[
|x| \leq \gamma(x) \leq 1 \text{ if } |x| \leq 1 \quad \text{and} \quad 1 \leq \gamma(x) \leq |x| \text{ if } |x| \geq 1
\]

(3.6) \[ q(T_{1\alpha} y) = q(x) \quad \text{for all } t > 0 \text{ and all } x \in \mathbb{R}^n. \]

We consider separately the two possible forms of \( \gamma \). Firstly, suppose \( \gamma \) has compact support. By a change of variable we have that the integral in (3.3) is equal to

(3.7) \[ \int_{\mathbb{R}^n} |T_{1\alpha} y|^{m} dv(T_{1\alpha} y). \]

By integrating separately over the sets \( \{ x : |T_{1\alpha} y| \leq 1 \} \) and \( \{ x : |T_{1\alpha} y| > 1 \} \) and applying (3.4) and (3.6), we obtain that the integral (3.7) is less than or equal to

\[ \left( \frac{s}{t} \right)^m \int_{|t| \leq 1} \int_{|t| > 1} \int_{\mathbb{R}^n} q(x)^m dv(x) \]

which is less than or equal to

\[ \left( \frac{s}{t} \right)^m \left( \int_{|t| \leq 1} \int_{|t| > 1} \int_{\mathbb{R}^n} q(x)^m dv(x) \right) \]

by using (3.5).

Secondly, suppose \( \gamma \) is of the form \( d \varphi = \varphi(x) dx \) with \( \varphi \) satisfying condition (ii) of Definition 3.1. We first notice that

\[ T_{1\alpha} y = T_{1\alpha} (T_{1\gamma} y), \]

so that by using (3.4) and (3.6) we have

\[ |T_{1\alpha} y| \leq \frac{s}{t} \cdot |T_{1\alpha} y|, \]

where

\[ r = \frac{s}{t} \quad \text{if} \quad |T_{1\alpha} y| \leq 1 \quad \text{and} \quad r = 1 \quad \text{if} \quad |T_{1\alpha} y| > 1. \]

Thus, after a change of variable, the integral in (3.3) will be seen to be less than

\[ \left( \frac{s}{t} \right)^m \int_{|t| \leq 1} \int_{|t| > 1} \int_{\mathbb{R}^n} |x|^{m} \cdot |\varphi(x)| dx \]

by the condition on \( \varphi \).

**Proof of Theorem 3.1.** We need to show that \( \nu \in E, \nu \in \nu_{\gamma} (E X) \) and

\[ \|\nu\|_{\alpha+\beta} \leq c(\|\nu\|_{\alpha} + \|\nu\|_{X(\alpha)}). \]
We first show that the integral in (3.1) is defined and that its norm in the space $B$ is bounded by the norm of $F$ in the space $X(B)$.

Using Lemma 2.2, we have

$$\left\| \int_0^t F \psi u \ dt \right\|_B \leq c \left\| F \right\|_{t^{-\beta}}^\beta dt \leq c \left\| F \right\|_{X(B)}.$$

Next, $u \ast \psi \in B$ and $\left\| u \ast \psi \right\|_B \leq \left\| u \right\|_B \left\| \psi \right\|_1 = c \left\| u \right\|_B$.

Finally, using Lemma 3.1, we have

$$s^{-\beta} \left\| u \ast \psi \right\|_B \leq s^{-\beta} \left\| u \right\|_B \left\| \psi \right\|_1 + s^{-\beta} \int_0^t \left\| F \psi \right\|_{t^{-\beta}}^\beta dt \leq c \left\| u \right\|_B \left\| \psi \right\|_1 + \left\| F \right\|_{X(B)} \int_0^t \frac{1}{t^{1-\beta}} dt,$$

It follows that $\left\| u \ast \psi \right\|_B \leq c \left\| u \right\|_B \left\| \psi \right\|_1 + \left\| F \right\|_{X(B)} \int_0^t \frac{1}{t^{1-\beta}} dt.$

**Corollary 3.1**. Let the measure $\nu$ of Theorem 3.1 have the additional property that, for each non-zero vector $x$, $\hat{\nu}(T_x^\beta x)$ is not identically zero as a function of $t > 0$. Then the $\nu$ and $\eta$ of Theorem 3.1 can be so chosen that, for $\beta = 0$,

$$S(u, \nu) = u \ast \nu + \int_0^t F \psi u \ dt \leq \int_0^\infty u \ast \nu \ dt,$$

is a projection of $B \oplus X(B)$ onto $A(B, X)$, $A_{\nu} = A_{\nu} \oplus X_B$.

Proof. If we choose the $\nu$ and $\eta$ the same as in Theorem 1.1, then, for $u \in A_{\nu}$, we have $u = S(u, \nu)$. 

**Theorem 3.2**. Let $\nu$ and $\mu$ be two measures satisfying the conditions of Theorem 3.1. Suppose in addition that, for each non-zero vector $x$, $\hat{\nu}(T_x^\beta x)$ and $\hat{\mu}(T_x^\beta x)$ are not identically zero as functions of $t > 0$. Let the Banach space $B$ and the Banach lattice $X$ be as above. Then $A(B, X)$, and $A(B, X)_\beta$ are equal algebraically and topologically.

Proof. Consider the mapping $\Sigma$ of the above corollary and let $u \in A_{\nu}$. Then $S(u, \nu) = u$ and, by Theorem 3.1, $u \in A_{\nu}$ and $\left\| u \right\|_B \leq c \left\| u \right\|_B$. Thus we have shown that $A_{\nu}$ is continuously embedded in $A_{\nu}$. Similarly, $A_{\mu}$ is continuously embedded in $A_{\mu}$.

**Corollary 3.2**. Let $\nu$ be a test function such that, for each non-zero vector $x$, $\hat{\nu}(T_x^\beta x)$ is not identically zero as a function of $t > 0$ and for which $\hat{\nu}$ is zero in a neighborhood of the origin. (Such a test function was constructed in Theorem 1.1.) Let the measure $\nu$ satisfy the conditions of Theorem 3.2. Then $A(B, X) = A(B, X)_\beta$, algebraically and topologically.

**Remark 3.1**. Thus, from now on we may assume that the measure $\nu$ is of the form $d\nu(x) = \varphi(x) dx$ with $\varphi$ as in Corollary 3.2.

**Proof**. The fact that $\hat{\varphi}$ is zero in a neighborhood of the origin implies that all the moments of $\varphi$ are zero. The result now follows from Theorem 3.2.

**4. Fractional Integration Operators**

For $u \in A_{\lambda} = A(B, X)$, let us write

$$T_{\lambda} u = u \ast \psi + \int_0^t u \ast \eta \ dt$$

where $\psi$ and $\eta \in \mathcal{F}$. We showed in Theorem 3.1 that $T_{\lambda}$ maps $A$ continuously into $A_{\lambda + \beta} = A(B, r^{\beta} X)$, provided $\alpha + \beta > 0$.

**Theorem 4.1.** The test functions $\psi$ and $\eta$ in (4.1) can be so chosen that the fractional integration operator $T_{\lambda}$ maps $A_{\lambda}$ continuously onto $A_{\lambda + \beta}$. Furthermore we can construct a fractional integration operator $T_{\alpha, \lambda}$ mapping $A_{\lambda + \beta}$ continuously onto $A_{\lambda}$ such that $T_{\lambda}$ and $T_{\alpha, \lambda}$ are inverse maps.

We also note that if $T_{\lambda}$ and $T_{\alpha}$ are constructed as in Theorem 4.1 such that $\alpha + \beta > 0, \alpha + \beta > 0$, and $\alpha + \beta + \gamma > 0$, then $T_{\lambda + \beta} T_{\gamma} = T_{\lambda + \beta + \gamma}$.

**Proof**. We first show how to construct test functions $\Phi$ and $\Psi$ such that the mapping

$$u \rightarrow S_{\alpha, \lambda} u = \int_0^\infty u \ast \Psi \ dt$$

maps $A_{\lambda}$ continuously into $A_{\lambda + \beta}$, and such that the mapping

$$\omega \rightarrow S_{\alpha, \lambda} \omega = \int_0^\infty \omega \ast \Phi \ dt$$

maps $A_{\lambda + \beta}$ continuously into $A_{\lambda}$ such that $S_{\lambda}$ and $S_{\alpha, \lambda}$ are inverses of each other.

Let the positive numbers $N$ and $\varepsilon$ and the test function $\nu$ be the same as in the proof of Theorem 1.1.

Now define $N_{\gamma}(x)$ for $x \neq 0$ by

$$[N_{\gamma}(x)]^{-1} = \int_0^\infty |\gamma(x)|^\beta T_{\gamma}^\beta x \ dt,$$

$N_{\gamma}(x)$ is homogeneous of degree $\beta$ in the sense that, for $x \neq 0$, $N_{\gamma}(x) = (|\gamma(x)|^\beta) N_{\gamma}(x')$

where $x = T_{\gamma}^\beta x'$ and $|x'| = 1$. 

Also for every unit vector \( \hat{z} \in \mathbb{R}^n \) we have
\[
|N_{\hat{z}}(\hat{z})|^{-1} \geq \varepsilon \min (N^{-1}, N^0)
\]
and \( N^0 \) is \( C^\infty \) on the unit sphere.

Similarly, let
\[
|N_{\hat{z}}(\hat{z})|^{-1} = \int_0^1 |v_t(x)|^2 \varphi(T_{\hat{z}} x) \frac{dt}{t^{1+\beta}}.
\]

Then \( N_{\hat{z}}(\hat{z}) \) is homogeneous of degree \( -\beta \) and has otherwise the same two properties as \( N_{\hat{z}} \) above.

Also let
\[
\tilde{\varphi}(z) = \tilde{\varphi}(x) \cdot \varphi(x) \cdot N_{\hat{z}}(x')
\]
and
\[
\tilde{\varphi}(z) = \tilde{\varphi}(x) \cdot \varphi(x) \cdot N_{\hat{z}}(x'),
\]
where \( x = T_{\hat{z}} x' \) and \( |x'| = 1 \). Then \( \tilde{\varphi} \) and \( \tilde{\varphi} \) are \( C^\infty \) functions and have compact support. Therefore \( \tilde{\varphi} \) and \( \tilde{\varphi} \) are test functions. We also notice that \( \tilde{\varphi} \) and \( \tilde{\varphi} \) are zero in a neighborhood of the origin.

Now for \( u \in A_{\alpha} \) let \( S_{\alpha} u = \int_0^1 u \cdot \tilde{\varphi}(T_{\hat{z}} x) \frac{dt}{t^{1+\beta}} \) and for \( \alpha \in A_{\alpha+\beta} \) let \( S_{\alpha} \psi = \int_0^1 \psi \cdot \tilde{\varphi}(T_{\hat{z}} x) \frac{dt}{t^{1+\beta}} \).

Firstly we show that these integrals are defined.

We have shown in Theorem 3.1 that
\[
\int_0^1 u \cdot \tilde{\varphi}(T_{\hat{z}} x) \frac{dt}{t^{1+\beta}} \quad \text{and} \quad \int_0^1 \psi \cdot \tilde{\varphi}(T_{\hat{z}} x) \frac{dt}{t^{1+\beta}}
\]
are defined. Also
\[
\left( \int_0^1 u \cdot \tilde{\varphi}(T_{\hat{z}} x) \frac{dt}{t^{1+\beta}} \right) (x) = \int_0^1 u \cdot \tilde{\varphi}(T_{\hat{z}} x) \cdot \tilde{\varphi}(x) \cdot N_{\hat{z}}(x') \frac{dt}{t^{1+\beta}}.
\]

This last expression is \( C^\infty \) and has compact support.

\( v = \int_0^\infty u \cdot \tilde{\varphi}(T_{\hat{z}} x) \frac{dt}{t^{1+\beta}} \) is a test function, so that \( \int_0^\infty u \cdot \tilde{\varphi}(T_{\hat{z}} x) \frac{dt}{t^{1+\beta}} \) exists and
\[
S_{\alpha} u = u \cdot \psi + \int_0^\infty u \cdot \tilde{\varphi}(T_{\hat{z}} x) \frac{dt}{t^{1+\beta}}.
\]

Similarly \( \int_0^\infty \psi \cdot \tilde{\varphi} \frac{dt}{t^{1+\beta}} \) exists and
\[
S_{\alpha} \psi = \psi \cdot \psi + \int_0^\infty \psi \cdot \tilde{\varphi} \frac{dt}{t^{1+\beta}} \quad \text{where} \quad \psi \cdot \psi.
\]

In Theorem 3.1 we showed that \( S_{\alpha} \) maps \( A_{\alpha} \) continuously into \( A_{\alpha+\beta} \) and that \( S_{\alpha} \) maps \( A_{\alpha+\beta} \) continuously into \( A_{\alpha} \). It remains to show that \( S_{\alpha} \) and \( S_{\alpha} \) are inverses of each other. We have, for \( u \in A_{\alpha} \),
\[
[S_{\alpha+\beta}(S_{\alpha} u)] = \tilde{\varphi} \cdot \left( \int_0^\infty \tilde{\varphi}(x) \cdot \tilde{\varphi}(x) \frac{dt}{t^{1+\beta}} \right) \left( \int_0^\infty \tilde{\varphi}(x) \cdot \tilde{\varphi}(x) \frac{dt}{t^{1+\beta}} \right) = \tilde{\varphi}.
\]

Similarly \( S_{\alpha+\beta} S_{\alpha} = \psi \cdot \psi \) for \( \psi \in A_{\alpha+\beta} \).

5. The Spaces \( A(B, X) \) of Non-Positive Type

**Definition 5.1.** Let \( B \) be a Banach space of tempered distributions on \( \mathbb{R}^n \) satisfying \( \| \tau v \|_{B} \leq C \| v \|_{B} \) for each \( v \in B \), and in which \( \psi \) is continuously embeded. Let \( X \) be a Banach lattice of type \( \alpha \). Let \( \nu \) be a finite vector-valued Borel measure on \( \mathbb{R}^n \) of the form \( \nu(x) = \varphi(x) dx \) where \( \varphi \) has the following properties:

(i) \( \varphi \) is a \( C^\infty \) function;

(ii) \( \varphi \) has moments of all orders up to and including \( k \) equal to zero, where \( |a| < k + 1 \);

(iii) \( |D^k \varphi(x)| \leq \frac{D_{\varphi}}{(1 + |x|)^{k+1}} \) for all \( x \) and all \( n \)-tuples \( \beta \) of non-negative integers for which \( |\beta|_{=k+1} \) and where \( D_{\varphi} \) is a constant. We also assume \( |\varphi(x)| \leq \frac{D_{\varphi}}{(1 + |x|)^{m}} \) for all non-negative integers \( m \);

(iv) for each unit vector \( \hat{z} \), \( \tilde{\varphi}(\hat{z}) \) is not identically zero as a function of \( t > 0 \).

Let \( \delta = |a| - \alpha \) if \( \alpha \neq 0 \), and \( \delta = \frac{|a|}{2} \) if \( \alpha = 0 \). Then we define
\[
A(B, X)_\nu = \{ u : \varphi \in B \} \quad \text{and} \quad u \cdot \nu \in X(B)
\]
where the fractional integration operator \( T_{\hat{z}} \) is defined as in Theorem 4.1. Equipped with the norm
\[
\| T_{\hat{z}} u \|_{X} + \| u \cdot \nu \|_{X(B)}
\]
\( A(B, X) \) becomes a Banach space.
For $a > 0$ this is the same definition as was given in Section 3.

**Lemma 5.1.** Let $X$ be a Banach lattice of type $a$ and let $A_a = A(B, X)$, where $B$ and $v$ are the same as in Definition 5.1. Then for $v \in \mathcal{D}$ and $u \in A_a$, $u * v \in B$ and $\|u * v\|_B \leq c\|v\|_B$.

Proof. Let $\delta$ be as before, let $T_\delta u = v$ and let $T_\delta$ be given by

$$T_\delta \omega = \omega * \Phi + \frac{1}{\delta} \int_0^\infty \omega * \Psi_t \frac{dt}{t^{1/\delta}},$$

where $\Phi$ and $\Psi_t \in \mathcal{D}$ and the moments of all orders of $\Psi$ are zero. Then

$$\|u * v\|_B \leq \|u * \Phi * v\|_B + \frac{1}{\delta} \int_0^\infty \|\omega * \Psi_t * v\|_B \frac{dt}{t^{1/\delta}} \leq c\|u\|_B + \frac{1}{\delta} \int_0^\infty \|\Psi_t * v\|_B \frac{dt}{t^{1/\delta}},$$

after an application of Lemma 3.1.

**Lemma 5.2.** Let $B$, $v$ and $X$ be the same as in Definition 5.1. Let $A_v = \Lambda(B, X)$, and let $\beta$ be chosen so that $|a + \beta| < k + 1$. Then the mapping $U_v$ defined for $u \in A_v$ and $G \in \mathcal{X}(B)$ by

$$U_v(u, G) = u * v + \frac{1}{\delta} \int_0^\infty G * v_t \frac{dt}{t^{1/\delta}},$$

where $v$ and $v_t$ are test functions, maps $A_v \otimes \mathcal{X}(B)$ continuously into $A_{a+\beta} = \Lambda(B, \mathcal{X}(B))$.

Proof. Let

$$\varepsilon = \begin{cases} 0, & \text{if } a + \beta > 0, \\ \frac{1}{\delta}, & \text{if } a + \beta = 0, \\ -2(a + \beta), & \text{if } a + \beta < 0, \end{cases}$$

and let $T_\varepsilon$, defined as in Theorem 4.1, be given by

$$T_\varepsilon \omega = \omega * \xi + \frac{1}{\delta} \int_0^\infty \omega * \xi_t \frac{dt}{t^{1/\delta}},$$

where $\xi$ and $\xi_t$ are test functions and $\xi$ is zero in a neighbourhood of the origin.

Then, choosing $\gamma$ such that $-(a + \beta) < \gamma < \varepsilon$ and using Lemmas 5.1, 3.1 and 2.2, we have

$$\|T_\gamma (U_v(u, G))\|_{A_v} \leq c\|u * v\|_B + \frac{1}{\delta} \int_0^\infty \|u_t * v\|_B \frac{dt}{t^{1/\delta}} + \int \|G\|_B \|v_t\|_B \frac{dt}{t^{1/\delta}} +$$

$$+ \frac{1}{\delta} \int \|G_t\|_B \|v\|_B \frac{dt}{t^{1/\delta}} + \frac{1}{\delta} \int \|G_t\|_B \|v\|_B \frac{dt}{t^{1/\delta}} +$$

$$\leq c\|u\|_B + \frac{1}{\delta} \int \|u_t\|_B \frac{dt}{t^{1/\delta}} + c\|L_{D\lambda} u\|_B \frac{dt}{t^{1/\delta}} +$$

$$+ c\|L_{D\lambda} u\|_B \frac{dt}{t^{1/\delta}} + \frac{1}{\delta} \int \|G\|_B \|v\|_B \frac{dt}{t^{1/\delta}} +$$

$$\leq c\|u\|_B + c\|u\|_B \frac{dt}{t^{1/\delta}} + \frac{1}{\delta} \int \|G\|_B \|v\|_B \frac{dt}{t^{1/\delta}} +$$

Next let $\delta$ be as in Definition 5.1, let $T_\delta u = v$ and suppose $T_\delta \omega$ is given as in (5.1). Then for $0 < s \leq 1$, we have using Lemma 5.1,

$$\|s^{-s}\|U_v(u, G) * v\|_B \leq s^{-s}\|u\|_B \|\Phi * v\|_B + s^{-s}\|v\|_B \left( \int_0^\infty \|v_t\|_B \frac{dt}{t^{1/\delta}} \right)^{s-1} \frac{dt}{t^{1/\delta}}$$

$$\leq c\|u\|_B s^{a+1-s} + \frac{1}{\delta} \int \|G\|_B \|v\|_B \frac{dt}{t^{1/\delta}} +$$

Thus, using Lemma 2.1 and the fact that $X$ satisfies conditions $A_{a-1-s}$ and $D_{b+1-s}$, we have

$$\|s^{-s}\|U_v(u, G) * v\|_B \leq c\|v\|_B + c\|G\|_B \frac{dt}{t^{1/\delta}},$$

This completes the proof of Lemma 5.2.

By constructing the test functions defining $T_\delta$ and $T_{-\delta}$ as in Theorem 4.1, i.e. such that $T_\delta$ and $T_{-\delta}$ are inverses, and using Lemma 5.2 and Theorem 4.1 and 3.2 and its corollary, we have the following

**Theorem 5.1.** Let the Banach lattice $X$ be of type $a$, where $|a| < k + 1$ and the Banach space $B$ of tempered distributions be as described in Definition 5.1. Let $v$ and $\mu$ be two measures satisfying the conditions (i)-(iv) of that definition, and let $\omega$ be a test function whose Fourier transform satisfies condition (iv) of Definition 5.1 and is identically zero in a neighbourhood of...
the origin. Then
\[ A(B; X)_b = A(B; X)_b = A(B; X)_p \]
algebraically and topologically.

Remark 5.1. If \( X \) is of any type \( a \in (-\infty, \infty) \) we may from now on assume that the measure \( \nu \) is of the form \( d\nu(x) = \varphi(x)dx \), where \( \varphi \) is a test function whose Fourier transform is identically zero in a neighbourhood of the origin and such that for each non-zero vector \( \alpha \), \( \varphi_\alpha(x) \) is not identically zero as a function of \( t > 0 \).

Finally, by constructing the fractional integration operators \( T_p \) and \( T_{-p} \) as in Theorem 4.1 and again using Lemma 5.2, we have

**Theorem 5.2.** Let \( B \) be a Banach space of tempered distributions as in Definition 5.1, let \( X \) be a Banach lattice of type \( a \) and let \( A_a = A(B; X) \). Then \( T_p \) maps \( A_a \) continuously onto \( A_{a+p} = A(B; \varphi^\beta X) \), \( T_{-p} \) maps \( A_{a+p} \) continuously onto \( A_{a-1} \), and \( T_p \) and \( T_{-p} \) are inverses.

### 6. Duality

Let \( X \) be a Banach lattice of measurable functions as described in Section 2. Let \( B \) be a Banach space of tempered distributions on \( R^n \) for which \( |v|_p < \infty \) for each \( u \in B \) and in which \( \mathcal{S} \) is continuously embedded. This latter condition ensures that \( B' \), the dual of \( B \), is again a Banach space of tempered distributions. We assume that the dual of the space \( X(B) \) is the space \( X'(B') \) and that the action of \( F \times X'(B') \) on \( g \in X(B) \) is of the form
\[
\langle F, G \rangle = \int \frac{dt}{t} \langle F(t), G(t) \rangle dt.
\]

An example of this situation is obtained by taking \( X \) to be \( L_q(1 < q < \infty) \) of Section 2 and \( B \) to be \( L_p(R^n) \) for \( 1 < p < \infty \). Then using Lemma 2.3 and the results of A. Benedek and R. Panzone [1] on \( L^p \) spaces, we have that \( X'(B') = X(B) \).

The main result of this section is

**Theorem 6.1.** Let \( B \) be a Banach space of tempered distributions as described above, and let \( X \) be a Banach lattice. Then the dual of \( A_a = A(B; X) \) is \( A_a = A(B'; X') \).

We first prove the following lemma to show that it is sufficient to assume that \( X \) is of positive type.

**Lemma 6.1.** \( X \) is a Banach lattice of type \( a \). Then \( T_p \) maps \( A_a \) continuously onto \( A_{a+p} \) continuously onto \( A_{a+p} \).

**Proof.** Let \( v \in A_a \) and \( u \in A_{a+p} \). Then
\[
\langle T_p v, u \rangle = \langle v, T_p u \rangle, \quad \text{where} \quad T_p u \in A_a.
\]

Thus \( \|T_p v\|_{a+p} \leq \|v\|_{a+p} \leq \|v\|_{a} \|w\|_{a+p} \). Therefore \( \|T_p v\|_{a+p} \leq \|v\|_{a} \).

We need one more

**Lemma 6.2.** Let \( B \) be a Banach space of tempered distributions as described at the beginning of this section. Let the Banach lattice \( X \) be of type \( \gamma \) positive. Then \( B \) is continuously embedded in \( A(B; X) \), where \( \varphi \) is as described in Theorem 5.1.

**Proof.** Let \( u \in B \) and let \( T_{-p} \) be given by
\[
T_{-p} u = u \ast \varphi + \int \frac{u \ast \varphi}{1 + \frac{t}{t}} dt
\]
where \( \varphi \) and \( \Psi \in \mathcal{S} \). Then
\[
\|T_{-p} u\|_B \leq c \|u\|_{B} + c \int \frac{\|u\|_{B} t^{-\gamma} dt}{t} = c \|u\|_{B}.
\]

Using the representation theorem of A. P. Calderón (Theorem 1.1), we may write
\[
u = u \ast \psi + \int \frac{u \ast \varphi_\gamma}{t} dt
\]
where \( \psi \in \mathcal{S} \) and \( \gamma \) has moments of all orders equal to zero. Then, choosing \( 0 < \alpha < |\gamma| \) and applying Lemma 3.1, we have for \( 0 < s \leq 1 \),
\[
\|u \ast \varphi_\gamma\|_{B} \leq c \|u\|_{B} + c \int \frac{\|u\|_{B} |\varphi_\gamma| dt}{t} \leq c \|u\|_{B} + c \int \frac{\|u\|_{B} t^{\alpha} dt}{t} = c \|u\|_{B}.
\]

An application of Lemma 2.1 shows that
\[
\|u \ast v\|_{X(\mathcal{S})} \leq c \|u\|_{B}.
\]

**Proof of Theorem 6.1.** Let the measure \( \nu \) be of the form \( d\nu(x) = \varphi(x)dx \), where \( \varphi \) is as described in Theorem 5.1, viz. \( \varphi \in \mathcal{S} \), and the Fourier transform of \( \varphi \) is zero in a neighbourhood of the origin and, for each non-zero vector \( \alpha \), \( \varphi_\alpha(x) \) is not identically zero as a function of \( t > 0 \).

Let \( \psi \) and \( \eta \) be the test functions of Theorem 1.1. Thus, for each \( u \in \mathcal{S} \),
\[
\psi = u \ast \psi + \int \frac{u \ast \varphi_\gamma}{t} dt
\]
where \( \psi \) has the same properties as \( \varphi \) above.

We shall show that, if \( X \) is of positive type, then the dual of \( A_a = A(B; X) \) is \( A_a = A(B'; X') \).
Let $u \in A_1$ and $v \in A_1$. Then the functional $L_u$ defined on $A_1$ by

$$L_u(w) = \langle v \ast w, u \rangle + \int_0^1 \langle v \ast w_t, u \ast w_t \rangle \frac{dt}{t}$$

makes sense, since $v \ast w_t B$ by Lemma 3.1, and

$$|L_u(w)| \leq \|v \ast w\|_B \|w\|_B + \int_0^1 \|v_t\|_B \|u \ast w_t\|_B \frac{dt}{t} \leq c \|v\|_B \|w\|_B + \int_0^1 \|v_t\|_B \|u \ast w_t\|_B \frac{dt}{t} \leq c \|v\|_B \|w\|_B.$$

Therefore the mapping $v \mapsto L_u$ is a linear functional on $A_1$.

In Corollary 3.1 we showed that the mapping $S: B \otimes X(B) \to A_1$ defined for $(u, F) \in B \otimes X(B)$ by

$$S(u, F) = u \ast v + \int_0^1 F_t \ast v_t \frac{dt}{t}$$

is continuous and onto $A_1$. In fact $S(u, v \ast w_t) = u$ for each $u \in A_1$.

Now define $\tilde{L}$ on $B \otimes X(B)$ by $\tilde{L}(u, F) = L(u, F)$. Then $\tilde{L}$ is a bounded linear operator on $B \otimes X(B)$. Therefore there exist $w \in B$ and $G \in X(B)$ such that

$$L(u, F) = \langle w, u \rangle + \int_0^1 \langle G_t, F_t \rangle \frac{dt}{t},$$

and

$$\|w\|_B \leq \|\tilde{L}\| \quad \text{and} \quad \|G\|_{X(B)} \leq \|\tilde{L}\|.$$

Thus if $u \in A_1$, then

$$L(u) = \tilde{L}(u, u \ast w_t) = \langle w, u \rangle + \int_0^1 \langle G_t, u \ast w_t \rangle \frac{dt}{t}.$$