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Duality and fractional integration in Lipschitz spaces

by

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Abstract. The definition of the spaces $\Lambda(B, X)$ of A. P. Calderón are extended to include the duals of the spaces considered by him. The fractional integration operators on these spaces are constructed. The definitions and theorems are further generalized by considering homogeneity with respect to a one-parameter group of linear transformations. In particular, these results generalize the theory of classical Lipschitz spaces of distributions on $\boldsymbol{R}^n$(see [6]).

Introduction. This paper is concerned with the spaces $\Lambda(B, X)_\nu$ introduced by A.P. Calderón in [3]. Here B is a Banach space of tempered distributions on $\boldsymbol{R}^n$ (n-dimensional Euclidean space), X is a Banach lattice of measurable functions on the interval $(0, 1]$ and ν is a measure satisfying certain conditions. (The precise definitions are given in Sections 2 and 3.) Then $u \in B$ is said to be a member of $\Lambda(B, X)_\nu$ if $\|(u * \nu_t)(\cdot)\|_B \in X$, where $*$ denotes convolution and ν_t is defined for a measurable subset E of $\boldsymbol{R}^n$ and for $t > 0$ by $\nu_t(E) = \nu\left(\frac{1}{t} E\right)$.

These spaces generalize and include certain Lipschitz space of smooth functions ([6], [7], [9]). For example, on $\boldsymbol{R}^1$, if we let ν be the Dirac measure at 0 minus the Dirac measure at -1, and let X be the class of measurable functions f for which $t^{-\alpha}|f(t)| \in L^\infty(0, 1]$, $0 < \alpha < 1$, then we obtain the space Λ_α of functions g for which $\|g(\cdot) - g(\cdot + t)\|_\infty \leqslant Ct^\alpha$, where C is a constant independent of t.

A.P. Calderón defined the spaces $\Lambda(B, X)_\nu$ for Banach lattices X of positive type (see Definitions 2.3 and 3.1). In this paper the definitions are extended to include non-positive type. Those of negative type turn out to be the dual spaces of those considered by Calderón. One of the principal contributions of this paper is the construction of the fractional integration operators on the spaces Λ. These are the analogues of the Riesz and Bessel potential operators.

The definitions and theorems are further generalized by considering homogeneity with respect to a group of linear transformations $\{T_t\}_{t>0}$ on $\boldsymbol{R}^n$ as studied by de Guzmán in [4]. This point of view allows us to include mixed homogeneity and spaces of functions satisfying different Lipschitz conditions in different directions. An example of such a family of linear transformations is given by

$$T_t(x) = (t^{a_1}x_1, t^{a_2}x_2, \ldots, t^{a_n}x_n)$$

where $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$ and $a_1, \ldots, a_n$ are positive constants. Then the case considered by Calderón amounts to letting

$$(*) \qquad a_1 = a_2 = \ldots = a_n = 1.$$

An example of how spaces of functions satisfying different Lipschitz conditions in different directions are included in this approach is obtained by letting the Banach space B be $L^p(\boldsymbol{R}^2)$, $1 \leqslant p \leqslant \infty$. The measure ν is taken to be equal to twice the Dirac measure at the origin minus the Dirac measures at $(-1, 0)$ and $(0, -1)$. For $x = (x_1, x_2) \in \boldsymbol{R}^2$, we let

$$T_t(x_1, x_2) = (tx_1, t^{\alpha/\beta}x_2)$$

where $\alpha > 0$ and $\beta > 0$. Let $X_{\alpha,q}$ be the Banach lattice of measurable functions g on $(0, 1]$ for which

$$\|g\| = \begin{cases} \left[\int\limits_0^1 \left(t^{-\alpha}|g(t)|\right)^q \dfrac{dt}{t}\right]^{1/q}, & 1 \leqslant q < \infty, \\ \operatorname{Sup} t^{-\alpha}|g(t)|, & q = \infty, \end{cases}$$

is finite. If $f \in L^p$ satisfies the conditions

$$\|f(x_1+t, x_2) - f(x_1, x_2)\|_p \in X_{\alpha,q}$$

and

$$\|f(x_1, x_2+t) - f(x_1, x_2)\|_p \in X_{\beta,q},$$

then

$$\|f(x_1+t, x_2) - f(x_1, x_2)\| \in X_{\alpha,q}$$

and

$$\|f(x_1, x_2+t^{\alpha/\beta}) - f(x_1, x_2)\| \in X_{\alpha,q}.$$

Thus $\|f * \nu_t\|_p \in X_{\alpha,q}$ since

$$(f*\nu_t)(x) = 2f(x_1, x_2) - f(x_1+t, x_2) - f(x_1, x_2+t^{\alpha/\beta}).$$

The following is an outline of this paper. Section 1 contains the main tool, which is a representation theorem enabling us to recover a tempered distribution u from $u*\nu_t$. (The exact statement is given by Theorem 1.1.) This theorem, for the case $(*)$ above, was first proved by Calderón and the same proof carries over to the more general setting considered in this paper.

Section 2 contains the definition of a Banach lattice and preliminary lemmas establishing the properties of Banach lattices that are needed in the sequel. In this section is also defined the "type" of a Banach lattice (Definition 3.3). Several examples of Banach lattices are given. We assume that the dual of a Banach lattice is again a Banach lattice.

In Section 3 are defined the spaces $\Lambda(B, X)_\nu$ for X of positive type. Theorem 3.1 is a basic result about a continuous mapping from the direct sum of B and $X(B)$ into a Λ space. This theorem allows us to prove (see Theorem 3.2 and corollary) that under certain conditions the spaces Λ are independent of the measure ν. In fact, the measure ν can be replaced by $\varphi(x)\,dx$, where φ is a Schwartz test function having certain properties (but otherwise arbitrary) and where dx denotes Lebesgue measure.

In Section 4 use is made of Theorem 3.1 to construct invertible fractional integration operators mapping a Λ space of positive type continuously onto another Λ space of positive type. The method of construction is similar to the proof of Theorem 1.1.

The spaces $\Lambda(B, X)_\nu$ for X of non-positive type are defined in Section 5. This definition is made in terms of the operators constructed in Section 4. The restriction of these operators to spaces of positive type is removed and it is shown (Theorem 5.1) that under certain conditions $\Lambda(B, X)_\nu$ is independent of ν and that ν can be replaced by $\varphi(x)\,dx$, as was the case for X of positive type in Section 3.

In the last section it is shown that under certain natural conditions the dual of $\Lambda(B, X)_\nu$ is the space $\Lambda(B', X')_\nu$, where B' and X' are the duals of B and X respectively.

The results of this paper form part of the author's Ph. D. thesis [5], which also included a development of the Lipschitz spaces where the Banach lattice is defined on the non-negative integers.

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Notation. $\boldsymbol{R}^n = \{x = (x_1, \ldots, x_n)$ with $x_1, \ldots, x_n$ real$\}$ is n-dimensional Euclidean space with the norm $|x| = \left(\sum\limits_{i=1}^n x_i^2\right)^{\frac{1}{2}}$. dx is n-dimensional Lebesgue measure.

$\mathscr{S} = \mathscr{S}(\boldsymbol{R}^n)$ will denote the collection of Schwartz test functions, i.e. all those C^∞ functions φ on $\boldsymbol{R}^n$ for which $\operatorname*{Sup}\limits_{x \in \boldsymbol{R}^n} |x^\beta (D^\alpha \varphi)(x)| < \infty$ for all n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ of non-negative integers. $\mathscr{S}'$ will denote the collection of all continuous linear functionals on $\mathscr{S}$ and is called the *space of tempered distributions*.

Let B be a Banach space of tempered distributions on $\boldsymbol{R}^n$ and X a B-lattice of functions on $(0, 1]$. Then $X(B)$ will denote the class of B-valued functions $f(t)$ for which $\|f(t)\|_B \in X$.

1. A REPRESENTATION THEOREM OF A. P. CALDERÓN

The following situation was studied by M. de Guzmán [4]: Let $\{T_t\}$ for $t>0$ be a group of linear transformations on $\boldsymbol{R}^n$ for which $T_s \cdot T_t = T_{st}$, $T_1 = I$, the mapping $t \to T_t$ is continuous, and for each $x \in \boldsymbol{R}^n$, $\|T_t x\| \to 0$ as $t \to 0$ and $\|T_t x\| \to \infty$ as $t \to \infty$ $(x \neq 0)$. T_t can be expressed in the form $T_t = e^{P \log t}$ where P is a matrix whose eigenvalues have positive real parts. Furthermore, we assume P satisfies the conditions of Theorem 3.1.6 of [4], in particular those at the top of page 117. The conditions include the case where P is diagonal. For each non-zero $x \in \boldsymbol{R}^n$ there exists a unique $t_x > 0$ such that $\|T_{t_x} x\| = 1$. Define $\varrho(x) = 1/t_x$. Then $\varrho(x-y)$ is a metric on $\boldsymbol{R}^n$, $\varrho(T_t x) = t\varrho(x)$ and $\varrho(x) = 1 \Leftrightarrow \|x\| = 1$. Let T_t^* be the adjoint of T_t. Then $\{T_t^*\}$ has the same properties as $\{T_t\}$ above and we can define a metric ϱ^* on $\boldsymbol{R}^n$ in the same way as ϱ above.

Let ν be a vector-valued finite Borel measure on $\boldsymbol{R}^n$ (with values in $\boldsymbol{R}^n$) and define $\nu_t(E) = \nu(T_{1/t}E)$ for any ν-measurable set E. If ν is of the form $d\nu(x) = \varphi(x)\,dx$, then $\varphi_t(x) = t^{-\operatorname{tr} P}\varphi(T_{1/t}x)$ since $|\det T_{1/t}| = t^{-\operatorname{tr} P}$.

We first prove a theorem of A.P. Calderón developed in [3] and a seminar at the University of Chicago (for the case $T_t x = tx$).

THEOREM 1.1. *Let ν be a vector-valued finite Borel measure with the additional property that, for each vector x of unit length, $\hat{\nu}(T_t^* x)$ is not identically zero as a function of $t>0$. ($\hat{\nu}$ is the Fourier transform of ν where $\hat{\nu}(x) = \int_{\boldsymbol{R}^n} e^{-2\pi i x \cdot y}\,d\nu(y)$.) Then there is a Schwartz test function η such that, for each tempered distribution u,*

$$u = \int_0^\infty u * \nu_t * \eta_t \frac{dt}{t}.$$

Also η has the following properties:

(i) *$\hat{\eta}$ is zero in a neighbourhood of the origin and has compact support;*

(ii) *for each non-zero vector $x \in \boldsymbol{R}^n$, $\hat{\eta}(T_t^* x)$ is not identically zero as a function of $t>0$.*

We can also write

$$u = u*\psi + \int_0^1 u*\nu_t*\eta_t \frac{dt}{t}$$

where $\psi \in \mathscr{S}$ (the class of Schwartz test functions).

Proof. We first note that $\hat{\nu}_t(x) = \hat{\nu}(T_t^* x)$. Let $g_N(x) = \int_{1/N}^{N} |\hat{\nu}(T_t^* x)|^2 \frac{dt}{t}$ where $N>0$. Then it is possible to find numbers $\varepsilon > 0$ and $N>0$ such that $|g_N(x)| > \varepsilon$ for every vector x of unit length. For let

$$E_N = \left\{x\colon |x| = 1 \text{ and } |g_N(x)| \leqslant \frac{1}{N}\right\}.$$

Then $E_{N+1} \subseteq E_N$. Also $\bigcap_{N=1}^{\infty} E_N$ is empty, because if $x \in \bigcap_{N=1}^{\infty} E_N$ then $\int_0^\infty |\hat{\nu}(T_t^* x)|^2 \frac{dt}{t} = 0$, from which it would follow that $\hat{\nu}(T_t^* x) \equiv 0$, contradicting the assumption on ν. Therefore, for every vector x of unit length, there exists $N_x > 0$ such that $g_{N_x}(x) > \frac{1}{N_x}$. We now use the continuity of each function g_N and the compactness of $\{x\colon |x| = 1\}$ to deduce the existence of the numbers ε and N above.

Consider now $\varphi \in C_0^\infty$ with the following property:

$$\varphi(x) = \begin{cases} 0, \text{ near } 0 \text{ and } \infty, \\ \text{identically } 1 \text{ for } \frac{1}{N} \leqslant \varrho^*(x) \leqslant N, \\ \text{non-negative otherwise, and } C_0^\infty. \end{cases}$$

Next, for $x \neq 0$, we let

$$[N(x)]^{-1} = \int_0^\infty |\hat{\nu}(T_t^* x)|^2 \varphi(T_t^* x) \frac{dt}{t}.$$

Then $N(x)$ is homogeneous of degree 0, that is $N(x) = N(x')$ where x is the unique unit vector for which

$$x = T^*_{\varrho^*(x)} x'.$$

Also $[N(x)]^{-1}$ is C^∞ on $\{x\colon |x| = 1\}$ and greater than or equal to ε there.

Now let

$$\hat{\eta}(x) = \overline{\hat{\nu}}(x) N(x') \varphi(x).$$

Then $\hat{\eta} \in C_0^\infty$, so that $\eta \in \mathscr{S}$.

Finally,

$$\Big(\int_0^\infty u*\nu_t*\eta_t\frac{dt}{t}\Big)^\wedge = \hat u\int_0^\infty \hat\nu(tx)\hat\eta(tx)\frac{dt}{t} = \hat u\int_0^\infty |\hat\nu(tx)|^2\psi(tx)N(x')\frac{dt}{t} = \hat u.$$

We also have $\varphi(x) = \int_1^\infty |\hat\nu(tx)|^2\psi(tx)N(x')\frac{dt}{t}$ is a C^∞ function of x with compact support. Hence $u = u*\psi+\int_0^1 u*\nu_t*\eta_t\frac{dt}{t}$, where $\psi\in\mathscr{S}$ and $\hat\psi = \varphi$.

2. BANACH LATTICES OF MEASURABLE FUNCTIONS

Before we can define the classes of tempered distributions which form the subject matter of this paper, we need to define the concept of Banach lattice. The purpose of this section is to define this concept, to define the "type" of a Banach lattice, to discuss some examples and to establish those properties that will be needed in subsequent sections.

DEFINITION 2.1. Let X be a Banach space of measurable and locally integrable functions on the interval $(0,1]$ of the real line. Such a space X will be called a *Banach lattice* if whenever $f\in X$ and g is a measurable function on $(0,1]$ for which $|g(x)|\leqslant|f(x)|$ almost everywhere then $g\in X$ and $\|g\|_X\leqslant\|f\|_X$.

DEFINITION 2.2. Let α be a real number.

(a) A Banach lattice X on $(0,1]$ will be said to satisfy *condition* A_α if the mapping

$$g\to\int_0^s g(t)\Big(\frac{s}{t}\Big)^\alpha\frac{dt}{t}\ \text{ is a bounded linear operator on } X.$$

(b) A Banach lattice X on $(0,1]$ will be said to satisfy *condition* B_α if the mapping

$$g\to\int_s^1 g(t)\Big(\frac{s}{t}\Big)^\alpha\frac{dt}{t}\ \text{ is a bounded linear operator on } X.$$

It is immediate that if X satisfies condition A_α then X also satisfies condition A_β for each $\beta\leqslant\alpha$. Similarly, if X satisfies condition B_α then X also satisfies condition B_β for each $\beta\geqslant\alpha$.

DEFINITION 2.3. We shall say that the Banach lattice X is of *type* α if X satisfies A_β for all $\beta<\alpha$ and satisfies B_β for all $\beta>\alpha$.

DEFINITION 2.4. Let X be a Banch lattice. By $t^\beta X$ we shall mean the class of measurable functions g on $(0,1]$ for which $t^{-\beta}g(t)\in X$. Equipped with the norm $\|t^{-\beta}g(t)\|_X$, the space $t^\beta X$ is a Banach lattice.

It follows immediately from the definitions above that if X satisfies A_α (or B_α) then $t^\beta X$ satisfies $\mathrm{A}_{\alpha+\beta}$ (or $\mathrm{B}_{\alpha+\beta}$).

Some examples of the type of Banach lattice discussed above are:

EXAMPLE 2.1. The Orlicz space L_Φ, see [8], where Φ is a Young's function, and a measurable function g on $(0,1]$ is in L_Φ if and only if there is a $\lambda>0$ such that $\int_0^1\Phi\Big(\frac{1}{\lambda}|g(t)|\Big)\frac{dt}{t}$ is finite. The norm is defined by

$$\|g\|_\Phi = \inf\Big\{\lambda>0:\int_0^1\Phi\Big(\frac{1}{\lambda}|g(t)|\Big)\frac{dt}{t}\leqslant\Phi(1)\Big\}.$$

By first establishing a generalization of Hardy's inequality, it can be shown that L_Φ is of type 0. (See [5] for details.) Hence $t^\alpha L_\Phi$ is of type α.

EXAMPLE 2.2. The space $X_{\alpha,q}$ (for $-\infty<\alpha<\infty$ and $1\leqslant q$), of all measurable functions g on $(0,1]$ for which, if $1\leqslant q<\infty$,

$$\|g\|_{\alpha,q} = \Big\{\int_0^1[t^{-\alpha}|g(t)|]^q\frac{dt}{t}\Big\}^{\frac1q}\quad\text{is finite},$$

and for which

$$\|g\|_{\alpha,\infty} = \operatorname*{Ess.\,Sup}_{0<t\leqslant1} t^{-\alpha}|g(t)|\quad\text{is finite if } q=\infty.$$

By noting that if $1\leqslant q<\infty$, then the function $\Phi(s) = s^q$ is a Young's function, and using the observation in the last sentence before this example we have that $X_{\alpha,q}$ is of type α. The fact that $X_{\alpha,\infty}$ is of type α follows immediately from the definitions.

LEMMA 2.1. *If X is a Banach lattice of type α, then the function $t^{\alpha+\varepsilon}\in X$ for every $\varepsilon>0$. Thus if α is negative, the constant functions are in X.*

Proof. Let χ be the characteristic function of the interval $[\frac12,1]$. Then for any non-negative function $g\in X$ we have

$$\chi(s)\int_0^{1/2} g(t)\Big(\frac1t\Big)^{\alpha-\varepsilon}\frac{dt}{t}\leqslant c\int_0^s g(t)\Big(\frac{s}{t}\Big)^{\alpha-\varepsilon}\frac{dt}{t}$$

where $c = \max\{1,2^{\alpha-\varepsilon}\}$. Thus $\chi\in X$ and therefore so is the function $f(t) = t^{\alpha+\varepsilon}\chi(t)$, since $f(t)\leqslant c\chi(t)$.

Next, for $0<s<x$, we notice that

$$s^{\alpha+\varepsilon}\int_{1/2}^1 g(t)\Big(\frac1t\Big)^{\alpha+\varepsilon}\frac{dt}{t}\leqslant\int_s^1 g(t)\Big(\frac{s}{t}\Big)^{\alpha+\varepsilon}\frac{dt}{t}.$$

Thus the function $k(t)$ which is equal to $t^{a+\varepsilon}$ for $0 < t < \frac{1}{2}$ and zero elsewhere is in X.

The result of the lemma now follows from the fact that $t^{a+\varepsilon} = f(t)+ +k(t)$.

LEMMA 2.2. *Suppose X is of type a* and *that β is such that $a+\beta > 0$. Then for each function $g \in X$ we have*

$$\left|\int_0^1 g(t)t^\beta \frac{dt}{t}\right| \leqslant c\|g\|_X,$$

where the constant c is independent of g.

Proof. Without loss of generality we may assume that $g \geqslant 0$. After noting that the function $s^{a+1} \in X$ (see previous lemma) and that X satisfies the conditions $A_{-\beta}$ and B_{a+1}, the result is a consequence of the following inequality:

$$s^{a+1}\int_0^1 g(t)t^\beta \frac{dt}{t} \leqslant \int_0^s g(t)\left(\frac{s}{t}\right)^{-\beta}\frac{dt}{t} + \int_s^1 g(t)\left(\frac{s}{t}\right)^{a+1}\frac{dt}{t}.$$

We next discuss the duals of Banach lattices.

We assume that the dual, X', of a Banach lattice X is again a Banach lattice of measurable and locally integrable functions on $(0,1]$. We also assume that the action of $f \in X'$ on $g \in X$ is given by $\int_0^1 f(t)g(t)\frac{dt}{t}$, and write this $\langle f, g\rangle$.

Let $g \in t^a X$ and $f \in t^{-a}X'$. Then L_f defined on $t^a X$ by $L_f(g) = \int_0^1 f(t)g(t)\frac{dt}{t}$ is a bounded linear functional on $t^a X$ whose norm is equal to the norm of f in the space $t^a X$. Conversely, every bounded linear functional on $t^a X$ is of the above form. Thus we have

LEMMA 2.3. *The dual of the Banach lattice $t^a X$ is $t^{-a}X'$.*

It is known, see [8], that if the Young's function Φ is such that $E_\Phi = \left\{g: \int_0^1 \Phi(a|g(t)|)\frac{dt}{t} < \infty \text{ for all } a > 0\right\}$ coincides with L_Φ, then the dual of L_Φ is L_Ψ where Ψ is the complementary Young's function and the action of $f \in L_\Psi$ on $g \in L_\Phi$ is given by $\int_0^1 f(t)g(t)\frac{dt}{t}$. Thus the dual of $X_{a,q}$ is $X_{-a,q'}$, where $1 \leqslant q < \infty$ and where $\frac{1}{q}+\frac{1}{q'} = 1$.

Finally, we need the following

LEMMA 2.4. *If the Banach lattice X satisfies the condition A_a (or B_a) then X' satisfies the condition B_{-a} (or A_{-a}). Thus if X is of type a, then X' is of type $-a$.*

The proof is a straightforward application of the definitions.

3. THE SPACES $\Lambda(B, X)$ FOR X OF POSITIVE TYPE

DEFINITION 3.1 Let ν be a vector-valued finite Borel measure on $\boldsymbol{R}^n$ having one of the following properties: (i) ν has compact support, or (ii) ν is of the form $d\nu = \varphi(x)dx$ where φ does not necessarily have compact support, but for each non-negative integer l there exists a constant D_l such that, for all $x \in \boldsymbol{R}^n$,

$$|\varphi(x)| \leqslant \frac{D_l}{(1+|x|)^l}.$$

Let B be a Banach space of tempered distribution in which $\mathscr{S}$ is continuously embedded and for which $\|\tau_x u\|_B \leqslant C\|u\|_B$ for every $u \in B$. (If $\psi \in \mathscr{S}$, we define $\tau_x u$ by $\langle \tau_x u, \psi\rangle = \langle u, \psi(\cdot - x)\rangle$.)

Let X be a Banach lattice of type a, where a is positive, as defined in the previous section. Then A.P. Calderón defined the space $\Lambda(B, X)$ (for the case $T_t x = tx$) to be

$$\{u \in B: \|u * \nu_t\|_B \in X\}.$$

(ν_t was defined in Section 1.) Equipped with the norm $\|u\|_\Lambda = \|u\|_B + +\|u*\nu_t\|_{X(B)}$, $\Lambda(B, X)_\nu$ is a Banach space, whose embedding in B is continuous. (See A. P. Calderón [3], page 126.)

We shall often write Λ_a instead of $\Lambda(B, X)_\nu$ and denote the norm of $u \in \Lambda_a$ by $\|u\|_a$ when it is clear which Banach space B, which Banach lattice X of type a and which measure ν are involved.

The case where $X = X_{aq}$, $B = L_p(\boldsymbol{R}^n)$, where $1 \leqslant p \leqslant \infty$, and $T_t x = tx$ was studied by M.H. Taibleson in [6] and [7].

The following two theorems are due to A.P. Calderón for the case $T_t x = tx$.

THEOREM 3.1. *Let the measure ν, and the spaces X and B be as above. Suppose also that all the moments of ν up to and including k are zero, where $0 < a < k+1$, i.e. $\int_{\boldsymbol{R}^n} x^\beta d\nu(x) = 0$ for all multiindices of non-negative integers $\beta = (b_1, b_2, \ldots, b_n)$ for which $|\beta| = b_1+\ldots+b_n \leqslant k$.*

Let $u \in B$ and $F \in X(B)$. Then the mapping

$$(u, F) \to v = u*\varphi + \int_0^1 F*\eta_t \frac{dt}{t^{1-\beta}} \tag{3.1}$$

where ψ and η are test functions, is defined when $0 < \alpha+\beta < k+1$ and maps $B \oplus X(B)$ continuously into $\Lambda(B, t^\beta X)_\nu = \Lambda_{\alpha+\beta}$.

Remark. For some different mapping theorems, see [2].

LEMMA 3.1. *Let ν be the same as in Theorem 3.1. Let η be a C^∞ function for which $|D^\beta \eta(x)| \leqslant \frac{D_\beta}{(1+|x|)^{n+1}}$ for all $x \in \mathbf{R}^n$ and all n-tuples β of non-negative integers for which $|\beta| \leqslant k+1$, and where D_β is a constant. Then $\|\eta_t * \nu_s\|_1 \leqslant C \min\left(1, \left(\frac{s}{t}\right)^m\right)$ for any $m \in (0, k+1]$ and where $0 < s, t \leqslant 1$.*

Proof. Clearly $\|\eta_t * \nu_s\|_1 \leqslant \|\eta_t\|_1 \cdot \|\nu_s\|_1 = \|\eta\|_1 \cdot \|\nu\|_1 = C$. Now let m be one of the integers $1, 2, \ldots, k+1$, and let $0 < s \leqslant t \leqslant 1$. We want to show that

$$\|\eta_t * \nu_s\|_1 \leqslant C\left(\frac{s}{t}\right)^m. \tag{3.2}$$

By Taylor's theorem we have

$$\eta(x-y) = \sum_{|\alpha| \leqslant m-1} D^\alpha \eta(x) \frac{(-y)^\alpha}{\alpha!} + R(x, y)$$

where $|R(x, y)| \leqslant C(x) \cdot |y|^m$ and $C(x) \in L'(\mathbf{R}^n)$. Therefore

$$(\eta_t * \nu_s)(x) = \int_{\mathbf{R}^n} R(T_{1/t}x, T_{1/t}y) t^{-\operatorname{tr} P} d\nu(T_{1/s}y)$$

because the moments of ν of orders less than m are zero.

$$\therefore |(\eta_t * \nu_s)(x)| \leqslant t^{-\operatorname{tr} P} C(T_{1/t}x) \int_{\mathbf{R}^n} |T_{1/t}y|^m d\nu(T_{1/s}y). \tag{3.3}$$

We want to show that the integral in this last expression is less than or equal to a constant times $\left(\frac{s}{t}\right)^m$. (3.2) would then follow immediately. To prove the statement about the integral in (3.3) we need the following facts from M. de Guzmán's work [4]:

(3.4) *There exist numbers $1 \leqslant p \leqslant q$ such that, for all $x \in \mathbf{R}^n$,*

$$[\varrho(x)]^q \leqslant |x| \leqslant [\varrho(x)]^p \quad \text{if} \quad |x| \leqslant 1$$

and

$$[\varrho(x)]^p \leqslant |x| \leqslant [\varrho(x)]^q \quad \text{if} \quad |x| \geqslant 1.$$

(3.5) *$\varrho(x)$ always lies between 1 and $|x|$. Thus*

$$|x| \leqslant \varrho(x) \leqslant 1 \text{ if } |x| \leqslant 1 \quad \text{and} \quad 1 \leqslant \varrho(x) \leqslant |x| \text{ if } |x| \geqslant 1,$$

$$\varrho(T_t x) = t\varrho(x) \quad \text{for all } t > 0 \text{ and all } x \in \mathbf{R}^n. \tag{3.6}$$

We consider separately the two possible forms of ν. Firstly, suppose ν has compact support. By a change of variable we have that the integral in (3.3) is equal to

$$\int_{\mathbf{R}^n} |T_{s/t}z|^m d\nu(z). \tag{3.7}$$

By integrating separately over the sets $\{z\colon |T_{s/t}z| \leqslant 1\}$ and $\{z\colon |T_{s/t}z| > 1\}$ and applying (3.4) and (3.6), we obtain that the integral (3.7) is less than or equal to

$$\left(\frac{s}{t}\right)^{pm} \int_{|T_{s/t}z| \leqslant 1} \varrho(z)^{pm} d\nu(z) + \left(\frac{s}{t}\right)^{qm} \int_{|T_{s/t}z| > 1} \varrho(z)^{qm} d\nu(z)$$

which is less than or equal to

$$\left(\frac{s}{t}\right)^m \left\{ \int_{\varrho(z) \leqslant 1} [|z|^{pm/q} + |z|^m] d\nu(z) + \int_{\varrho(z) > 1} [|z|^m + |z|^{qm/p}] d\nu(z) \right\} = C\left(\frac{s}{t}\right)^m,$$

by using (3.5).

Secondly, suppose ν is of the form $d\nu = \varphi(x)dx$ with φ satisfying condition (ii) of Definition 3.1. We first notice that

$$T_{1/t}y = T_{s/t}(T_{1/s}y),$$

so that by using (3.4) and (3.6) we have

$$|T_{1/t}y| \leqslant \frac{s}{t} |T_{1/s}y|^r,$$

where

$$r = \frac{p}{q} \text{ if } |T_{1/s}y| \leqslant 1 \quad \text{and} \quad r = 1 \text{ if } |T_{1/s}y| > 1.$$

Thus, after a change of variable, the integral in (3.3) will be seen to be less than

$$\left(\frac{s}{t}\right)^m \left\{ \int_{|x| \leqslant 1} |x|^{mp/q} |\varphi(x)| dx + \int_{|x| > 1} |x|^{mq/p} |\varphi(x)| dx \right\} = C\left(\frac{s}{t}\right)^m$$

by the condition on φ.

Proof of Theorem 3.1. We need to show that $v \in B$, $v * \nu_s \in (s^\beta X)(B)$ and

$$\|v\|_{\alpha+\beta} \leqslant c\{\|u\|_B + \|F\|_{X(B)}\}.$$

We first show that the integral in (3.1) is defined and that its norm in the space B is bounded by the norm of F in the space $X(B)$.

Using Lemma 2.2, we have

$$\Big\| \int_0^1 F*\eta_t \frac{dt}{t^{1-\beta}} \Big\|_B \leqslant c \int_0^1 \|F\|_B t^\beta \frac{dt}{t} \leqslant c\|F\|_{X(B)}.$$

Next, $u*\psi \in B$ and $\|u*\psi\|_B \leqslant \|u\|_B \cdot \|\psi\|_1 = c\|u\|_B$.

Finally, using Lemma 3.1, we have

$$s^{-\beta}\|v*\nu_s\|_B \leqslant s^{-\beta}\|u\|_B \cdot \|\psi*\nu_s\|_1 + s^{-\beta} \int_0^1 \|F\|_B \cdot \|\eta_t*\nu_s\|_1 \frac{dt}{t^{1-\beta}}$$

$$\leqslant c\|u\|_B s^{k+1-\beta} + c\int_0^s \|F\|_B \Big(\frac{s}{t}\Big)^{-\beta} \frac{dt}{t} + c\int_s^1 \|F\|_\beta \Big(\frac{s}{t}\Big)^{k+1-\beta} \frac{dt}{t}.$$

It follows that $\big\|s^{-\beta}\|v*\nu_s\|_B\big\|_X \leqslant c\{\|u\|_B + \|F\|_{X(B)}\}$.

COROLLARY 3.1. *Let the measure ν of Theorem 3.1 have the additional property that, for each non-zero vector x, $\hat{\nu}(T_t^* x)$ is not identically zero as a function of $t > 0$. Then the ψ and η of Theorem 3.1 can be so chosen that, for $\beta = 0$,*

$$S(u, F) = u*\psi + \int_0^1 F*\eta_t \frac{dt}{t}$$

is a projection of $B \oplus X(B)$ onto $\Lambda(B, X)_\nu = \Lambda_a$.

Proof. If we choose the ψ and η the same as in Theorem 1.1, then, for $u \in \Lambda_a$, we have $u = S(u, u_t)$.

THEOREM 3.2. *Let ν and μ be two measures satisfying the conditions of Theorem 3.1. Suppose in addition that, for each non-zero vector x, $\hat{\nu}(T_t^* x)$ and $\hat{\mu}(T_t^* x)$ are not identically zero as functions of $t > 0$. Let the Banach space B and the Banach lattice X be as above. Then $\Lambda(B, X)_\nu$ and $\Lambda(B, X)_\mu$ are equal algebraically and topologically.*

Proof. Consider the mapping S of the above corollary and let $u \in \Lambda_\nu$. Then $S(u, u_t) = u$, and, by Theorem 3.1, $u \in \Lambda_\mu$ and $\|u\|_\mu \leqslant c\|u\|_\nu$. Thus we have shown that Λ_ν is continuously embedded in Λ_μ. Similarly, Λ_μ is continuously embedded in Λ_ν.

COROLLARY 3.2. *Let φ be a test function such that, for each non-zero vector x, $\hat{\varphi}(T_t^* x)$ is not identically zero as a function of $t > 0$ and for which $\hat{\varphi}$ is zero in a neighbourhood of the origin. (Such a test function was constructed in Theorem 1.1.) Let the measure ν satisfy the conditions of Theorem 3.2. Then $\Lambda(B, X)_\nu = \Lambda(B, X)_\varphi$, algebraically and topologically.*

Remark 3.1. Thus, from now on we may assume that the measure ν is of the form $d\nu(x) = \varphi(x)dx$ with φ as in Corollary 3.2.

Proof. The fact that $\hat{\varphi}$ is zero in a neighbourhood of the origin implies that *all* the moments of φ are zero. The result now follows from Theorem 3.2.

4. FRACTIONAL INTEGRATION OPERATORS

For $u \in \Lambda_a = \Lambda(B, X)_\nu$ let us write

$$T_\beta u = u*\psi + \int_0^1 u_t * \eta_t \frac{dt}{t^{1-\beta}} \tag{4.1}$$

where ψ and $\eta \in \mathscr{S}$. We showed in Theorem 3.1. that T_β maps Λ continuously into $\Lambda_{a+\beta} = \Lambda(B, t^\beta X)_\nu$ provided $a+\beta > 0$.

THEOREM 4.1. *The test functions ψ and η in (4.1) can be so chosen that the fractional integration operator T_β maps Λ_a continuously onto $\Lambda_{a+\beta}$. Furthermore we can construct a fractional integration operator $T_{-\beta}$ mapping $\Lambda_{a+\beta}$ continuously onto Λ_a such that T_β and $T_{-\beta}$ are inverse maps.*

We also note that if T_β and T_γ are constructed as in Theorem 4.1 such that $a+\beta > 0$, $a+\gamma > 0$ and $a+\beta+\gamma > 0$, then $T_\beta \circ T_\gamma = T_\gamma \circ T_\beta = T_{\gamma+\beta}$.

Proof. We first show how to construct test functions Φ and Ψ such that the mapping

$$u \to S_\beta u = \int_0^\infty u_t * \Psi_t \frac{dt}{t^{1-\beta}}$$

maps Λ_a continuously into $\Lambda_{a+\beta}$, and such that the mapping

$$\omega \to S_{-\beta}\omega = \int_0^\infty \omega_t * \Phi_t \frac{dt}{t^{1+\beta}}$$

maps $\Lambda_{a+\beta}$ continuously into Λ_a and such that S_β and $S_{-\beta}$ are inverses of each other.

Let the positive numbers N and ε and the test function ψ be the same as in the proof of Theorem 1.1.

Now define $N_\beta(x)$ for $x \neq 0$ by

$$[N_\beta(x)]^{-1} = \int_0^\infty |\hat{\nu}_t(x)|^2 \psi(T_t^* x) \frac{dt}{t^{1-\beta}}.$$

$N_\beta(x)$ is homogeneous of degree β in the sense that, for $x \neq 0$,

$$N_\beta(x) = [\varrho^*(x)]^\beta N_\beta(x')$$

where $x = T^*_{\varrho^*(x)} x'$ and $|x'| = 1$.

Also for every unit vector $x \in \boldsymbol{R}^n$ we have

$$[N_\beta(x)]^{-1} \geqslant \varepsilon \min(N^{-\beta}, N^\beta)$$

and N^β is C^∞ on the unit sphere.

Similarly, let

$$[N_{-\beta}(x)]^{-1} = \int_0^\infty |\hat{\nu}_t(x)|^2 \psi(T_t^* x) \frac{dt}{t^{1+\beta}}.$$

Then $N_{-\beta}(x)$ is homogeneous of degree $-\beta$ and has otherwise the same two properties as N_β above.

Also let

$$\hat{\Psi}(x) = \overline{\hat{\nu}(x)} \cdot \psi(x) \cdot N_\beta(x')$$

and

$$\hat{\Phi}(x) = \overline{\hat{\nu}(x)} \cdot \psi(x) \cdot N_{-\beta}(x'),$$

where $x = T^*_{\varrho^*(x)} x'$ and $|x'| = 1$. Then $\hat{\Psi}$ and $\hat{\Phi}$ are C^∞ functions and have compact support. Therefore Ψ and Φ are test functions. We also notice that $\hat{\Psi}$ and $\hat{\Phi}$ are zero in a neighbourhood of the origin.

Now for $u \in \Lambda_a$ let $S_\beta u = \int_0^\infty u_t * \Psi_t \frac{dt}{t^{1-\beta}}$ and for $\omega \in \Lambda_{a+\beta}$ let $S_{-\beta}\omega = \int_0^\infty \omega_t * \Phi_t \frac{dt}{t^{1+\beta}}$.

Firstly we show that these integrals are defined.

We have shown in Theorem 3.1 that

$$\int_0^1 u_t * \Psi_t \frac{dt}{t^{1-\beta}} \quad \text{and} \quad \int_0^1 \omega_t * \Phi_t \frac{dt}{t^{1+\beta}}$$

are defined. Also

$$\left(\int_1^\infty \nu_t * \Psi_t \frac{dt}{t^{1-\beta}}\right)^\wedge (x) = \int_1^\infty \hat{\nu}_t(x) \hat{\Psi}_t(x) \frac{dt}{t^{1-\beta}} = \int_1^\infty |\hat{\nu}_t(x)|^2 \psi_t(x) N_\beta(x') \frac{dt}{t^{1-\beta}}.$$

This last expression is C^∞ and has compact support.

$\therefore\ \psi = \int_1^\infty \nu_t * \Psi_t \frac{dt}{t^{1-\beta}}$ is a test function, so that $\int_1^\infty u_t * \Psi_t \frac{dt}{t^{1-\beta}}$ exists and

$$S_\beta u = u * \psi + \int_0^1 u_t * \Psi_t \frac{dt}{t^{1-\beta}}.$$

Similarly $\int_0^\infty \omega_t * \Phi_t \frac{dt}{t^{1+\beta}}$ exists and

$$S_{-\beta} u = \omega * \varphi + \int_0^1 \omega_t * \Phi_t \frac{dt}{t^{1+\beta}} \quad \text{where} \quad \varphi \in \mathscr{S}.$$

In Theorem 3.1 we showed that S_β maps Λ_a continuously into $\Lambda_{a+\beta}$ and that $S_{-\beta}$ maps $\Lambda_{a+\beta}$ continuously into Λ_a. It remains to show that S_β and $S_{-\beta}$ are inverses of each other. We have, for $u \in \Lambda_a$,

$$[S_{-\beta}(S_\beta u)]^\wedge = \hat{u} \cdot \left(\int_0^\infty \hat{\nu}_s(x) \cdot \hat{\Psi}_s(x) \frac{ds}{s^{1-\beta}}\right) \cdot \left(\int_0^\infty \hat{\nu}_t(x) \cdot \hat{\Phi}_t(x) \frac{dt}{t^{1+\beta}}\right) = \hat{u}.$$

Similarly $S_\beta \circ S_{-\beta} \omega = \omega$ for $\omega \in \Lambda_{a+\beta}$.

5. THE SPACES $\Lambda(B, X)$ OF NON-POSITIVE TYPE

DEFINITION 5.1. Let B be a Banach space of tempered distributions on $\boldsymbol{R}^n$ satisfying $\|\tau_x u\|_B \leqslant c\|u\|_B$ for each $u \in B$, and in which $\mathscr{S}$ is continuously embedded. Let X be a Banach lattice of type a. Let ν be a finite vector-valued Borel measure on $\boldsymbol{R}^n$ of the form $d\nu(x) = \varphi(x)dx$ where φ has the following properties:

(i) φ is a C^∞ function;

(ii) φ has moments of all orders up to and including k equal to zero, where $|a| < k+1$;

(iii) $|D^\beta \varphi(x)| \leqslant \frac{D_{|\beta|}}{(1+|x|)^{n+1}}$ for all x and all n-tuples β of non-negative integers for which $|\beta| \leqslant k+1$ and where $D_{|\beta|}$ is a constant. We also assume $|\varphi(x)| \leqslant \frac{D_m}{(1+|x|)^m}$ for all non-negative integers m;

(iv) for each unit vector x, $\hat{\nu}_t(x)$ is not identically zero as a function of $t > 0$.

Let $\delta = |a| - a$ if $a \neq 0$, and $\delta = \frac{1}{2}$ if $a = 0$. Then we define

$$\Lambda(B, X)_\nu = \{u \in \mathscr{S}' \colon T_\delta u \in B \text{ and } u * \nu_t \in X(B)\},$$

where the fractional integration operator T_δ is defined as in Theorem 4.1. Equipped with the norm

$$\|T_\delta u\|_B + \|u * \nu_t\|_{X(B)},$$

$\Lambda(B, X)_\nu$ becomes a Banach space.

For $\alpha > 0$ this is the same definition as was given in Section 3.

LEMMA 5.1. *Let X be a Banach lattice of type α and let $\Lambda_\alpha = \Lambda(B, X)_\nu$ where B and ν are the same as in Definition 5.1. Then for $\psi \in \mathscr{S}$ and $u \in \Lambda_\alpha$, $u * \psi \in B$ and $\|u * \psi\|_B \leqslant c\|u\|_\alpha$.*

Proof. Let δ be as above, let $T_\delta u = \omega$ and let $T_{-\delta}$ be given by

$$T_{-\delta}\omega = \omega * \Phi + \int_0^1 \omega_t * \Psi_t \frac{dt}{t^{1+\delta}}, \tag{5.1}$$

where Φ and $\Psi \in \mathscr{S}$ and the moments of all orders of Ψ are zero. Then

$$\|u * \psi\|_B \leqslant \|\omega * \Phi * \psi\|_B + \int_0^1 \|\omega * \nu_t * \Psi_t * \psi\|_B \frac{dt}{t^{1+\delta}}$$

$$\leqslant c\|\omega\|_B \left\{1 + \int_0^1 \|\Psi_t * \psi\|_1 t^{-\delta} \frac{dt}{t}\right\} \leqslant c\|\omega\|_B \leqslant c\|u\|_\alpha,$$

after an application of Lemma 3.1.

LEMMA 5.2. *Let B, ν and X be the same as in Definition 5.1. Let $\Lambda_\alpha = \Lambda(B, X)_\nu$ and let β be chosen so that $|\alpha + \beta| < k+1$. Then the mapping U_β defined for $u \in \Lambda_\alpha$ and $G \in X(B)$ by*

$$U_\beta(u, G) = u * \psi + \int_0^1 G * \eta_t \frac{dt}{t^{1-\beta}},$$

where ψ and η are test functions, maps $\Lambda_\alpha \oplus X(B)$ continuously into $\Lambda_{\alpha+\beta} = \Lambda(B, t^\beta X)_\nu$.

Proof. Let

$$\varepsilon = \begin{cases} 0, & \text{if } \alpha+\beta > 0, \\ \frac{1}{2}, & \text{if } \alpha+\beta = 0, \\ -2(\alpha+\beta), & \text{if } \alpha+\beta < 0, \end{cases}$$

and let T_ε, defined as in Theorem 4.1, be given by

$$T_\varepsilon \omega = \omega * \xi + \int_0^1 \omega * \nu_t * \zeta_t \frac{dt}{t^{1-\varepsilon}},$$

where ξ and ζ are test functions and $\hat{\zeta}$ is zero in a neighbourhood of the origin.

Then, choosing γ such that $-(\alpha+\beta) < \gamma < \varepsilon$ and using Lemmas 5.1, 3.1 and 2.2, we have

$$\big\|T_\varepsilon\big(U_\beta(u, G)\big)\big\|_B \leqslant \|u * \psi * \xi\|_B + \int_0^1 \|u_t\|_B \cdot \|\psi * \zeta_t\|_1 \frac{dt}{t^{1-\varepsilon}} + \int_0^1 \|G\|_B \cdot \|\eta_t * \xi\|_1 \frac{dt}{t^{1-\beta}} +$$

$$+ \int_0^1 \int_0^1 \|G\|_B \cdot \|\nu_s\|_1 \cdot \|\eta_t * \zeta_s\|_1 \frac{dt}{t^{1-\beta}} \frac{dt}{s^{1-\varepsilon}}$$

$$\leqslant c\|u\|_\alpha + c\int_0^1 \|u_t\|_B t^{\varepsilon+k+1} \frac{dt}{t} + c\int_0^1 \|G\|_B \cdot t^{\beta+\varepsilon} \frac{dt}{t} +$$

$$+ c\int_0^1 \int_0^1 s^{\varepsilon-\gamma} \frac{ds}{s} \cdot \|G\|_B t^{\gamma+\beta} \frac{dt}{t}$$

$$\leqslant c\{\|u\|_\alpha + \|u_t\|_{X(B)} + \|G\|_{X(B)}\} \leqslant c\{\|u\|_\alpha + \|G\|_{X(B)}\}.$$

Next let δ be as in Definition 5.1, let $T_\delta u = \omega$ and suppose $T_{-\delta}\omega$ is given as in (5.1). Then for $0 < s \leqslant 1$, we have using Lemma 3.1,

$$s^{-\beta}\|U_\beta(u, G) * \nu_s\|_B \leqslant s^{-\beta}\|\omega\|_B \cdot \|\psi * \Phi * \nu_s\|_1 + s^{-\beta}\|\omega\|_B \int_0^1 \|\nu_s * \nu_t\|_1 \cdot \|\Psi_t * \psi\|_1 \frac{dt}{t^{1+\delta}} +$$

$$+ \int_0^1 \|G\|_B \cdot \|\eta_t * \nu_s\|_1 \left(\frac{s}{t}\right)^{-\beta} \frac{dt}{t}$$

$$\leqslant c\|u\|_\alpha s^{k+1-\beta} + c\int_0^s \|G\|_B \left(\frac{s}{t}\right)^{-k-1-\beta} \frac{dt}{t} +$$

$$+ c\int_s^1 \|G\|_B \left(\frac{s}{t}\right)^{k+1-\beta} \frac{dt}{t}.$$

Thus, using Lemma 2.1 and the fact that X satisfies conditions $A_{-k-1-\beta}$ and $B_{k+1-\beta}$, we have

$$\big\|s^{-\beta}\|U_\beta(u, G) * \nu_s\|_B\big\|_X \leqslant c\{\|u\|_\alpha + \|G\|_{X(B)}\}.$$

This completes the proof of Lemma 5.2.

By constructing the test functions defining T_δ and $T_{-\delta}$ as in Theorem 4.1, i.e. such that T_δ and $T_{-\delta}$ are inverses, and using Lemma 5.2 and Theorem 4.1 and 3.2 and its corollary, we have the following

THEOREM 5.1. *Let the Banach lattice X be of type α, where $|\alpha| < k+1$ and let the Banach space B of tempered distributions be as described in definition 5.1. Let ν and μ be two measures satisfying the conditions (i)–(iv) of that definition, and let φ be a test function whose Fourier transform satisfies condition (iv) of Definition 5.1 and is identically zero in a neighbourhood of* *the origin. Then*

$$\Lambda(B, X)_\nu = \Lambda(B, X)_\mu = \Lambda(B, X)_\varphi$$

algebraically and topologically.

Remark 5.1. If X is of any type $\alpha \in (-\infty, \infty)$ we may from now on assume that the measure ν is of the form $d\nu(x) = \varphi(x)\,dx$, where φ is a test function whose Fourier transform is identically zero in a neighbourhood of the origin and such that for each non-zero vector x, $\hat{\varphi}_t(x)$ is not identically zero as a function of $t > 0$.

Finally, by constructing the fractional integration operators T_β and $T_{-\beta}$ as in Theorem 4.1 and again using Lemma 5.2, we have

THEOREM 5.2. *Let B be a Banach space of tempered distributions as in Definition 5.1, let X be a Banach lattice of type α and let $\Lambda_\alpha = \Lambda(B, X)$. Then T_β maps Λ_α continuously onto $\Lambda_{\alpha+\beta} = \Lambda(B, t^\beta X)$, $T_{-\beta}$ maps $\Lambda_{\alpha+\beta}$ continuously onto Λ_α, and T_β and $T_{-\beta}$ are inverses.*

6. DUALITY

Let X be a Banach lattice of measurable functions as described in Section 2. Let B be a Banach space of tempered distributions on $\mathbf{R}^n$ for which $\|\tau_x u\|_B \leqslant c\|u\|_B$ for each $u \in B$ and in which $\mathscr{S}$ is continuously embedded. This latter condition ensures that B', the dual of B, is again a Banach space of tempered distributions. We assume that the dual of the space $X(B)$ is the space $X'(B')$ and that the action of $F \in X'(B')$ on $G \in X(B)$ is of the form

$$\langle F, G\rangle = \int_0^1 \langle F(\cdot, t), G(\cdot, t)\rangle \frac{dt}{t}.$$

An example of this situation is obtained by taking X to be $X_{\alpha,q}$ $(1 < q < \infty)$ of Section 2 and B to be $L_p(\mathbf{R}^n)$ for $1 \leqslant p < \infty$. Then using Lemma 2.3 and the results of A. Benedek and R. Panzone [1] on L^p spaces, we have that $X(B)' = X'(B')$.

The main result of this section is

THEOREM 6.1. *Let B be a Banach space of tempered distributions as described above, and let X be a Banach lattice. Then the dual of $\Lambda_1 = \Lambda(B, X)$ is $\Lambda_2 = \Lambda(B', X')$.*

We first prove the following lemma to show that it is sufficient to assume that X is of positive type.

LEMMA 6.1. *Let X be a Banach lattice of type α. Then T_β maps $\Lambda'_\alpha = \Lambda(B, X)'$ continuously onto $\Lambda'_{\alpha-\beta} = \Lambda(B, t^{-\beta}X)'$.*

Proof. Let $v \in \Lambda'_\alpha$ and $u \in \Lambda_{\alpha-\beta}$. Then

$$|\langle T_\beta v, u\rangle| = |\langle v, T_\beta u\rangle|, \quad \text{where } T_\beta u \in \Lambda_\alpha.$$

Thus $|\langle T_\beta v, u\rangle| \leqslant \|v\|'_\alpha \cdot \|T_\beta u\|_\alpha \leqslant c\|v\|'_\alpha \cdot \|u\|_{\alpha-\beta}$. Therefore $\|T_\beta v\|'_{\alpha-\beta} \leqslant c\|v\|'_\alpha$.

We need one more

LEMMA 6.2. *Let B be a Banach space of tempered distributions as described at the beginning of this section. Let the Banach lattice X be of type γ negative. Then B is continuously embedded in $\Lambda(B, X)_\varphi$, where φ is as described in Theorem 5.1.*

Proof. Let $u \in B$ and let $T_{-2\gamma}$ be given by

$$T_{-2\gamma}u = u * \Phi + \int_0^1 u * \varphi_t * \Psi_t \frac{dt}{t^{1+2\gamma}}$$

where Φ and $\Psi \in \mathscr{S}$. Then

$$\|T_{-2\gamma}u\|_B \leqslant c\|u\|_B + c\int_0^1 \|u\|_B \cdot t^{-2\gamma}\frac{dt}{t} = c\|u\|_B.$$

Using the representation theorem of A. P. Calderón (Theorem 1.1), we may write

$$u = u * \psi + \int_0^1 u * \varphi_t * \eta_t \frac{dt}{t}$$

where ψ and $\eta \in \mathscr{S}$ and η has moments of all orders equal to zero. Then, choosing $0 < \delta < |\gamma|$ and applying Lemma 3.1, we have for $0 < s \leqslant 1$,

$$\|u * \varphi_s\|_B \leqslant c\|u\|_B + c\int_0^1 \|u\|_B \cdot \|\varphi_t * \varphi_s\|_1 \frac{dt}{t}$$

$$\leqslant c\|u\|_B + c\int_0^1 \|u\|_B \cdot \left(\frac{t}{s}\right)^\delta \frac{dt}{t} \leqslant cs^{-\delta}\|u\|_B.$$

An application of Lemma 2.1 shows that

$$\|u * \nu_s\|_{X(B)} \leqslant c\|u\|_B.$$

Proof of Theorem 6.1. Let the measure ν be of the form $d\nu(x) = \varphi(x)\,dx$, where φ is as described in Theorem 5.1, viz. $\varphi \in \mathscr{S}$, and the Fourier transform of φ is zero in a neighbourhood of the origin and, for each non-zero vector x, $\hat{\varphi}_t(x)$ is not identically zero as a function of $t > 0$.

Let ψ and η be the test functions of Theorem 1.1. Thus, for each $u \in \mathscr{S}'$, $u = u * \psi + \int_0^1 u * \varphi_t * \eta_t \frac{dt}{t}$, where η has the same two properties as φ above.

We shall show that, if X is of positive type, then the dual of $\Lambda_1 = \Lambda(B, X)_\eta$ is $\Lambda_2 = \Lambda(B', X')_\varphi$.

Let $u \in \Lambda_1$ and $v \in \Lambda_2$. Then the functional L_v defined on Λ_1 by

$$L_v(u) = \langle v*\psi, u\rangle + \int_0^1 \langle v*\varphi_t, u*\eta_t\rangle \frac{dt}{t}$$

makes sense, since $v*\psi \in B'$ by Lemma 5.1, and

$$|L_v(u)| \leqslant \|v*\psi\|_{B'}\cdot\|u\|_B + \int_0^1 \|v_t\|_{B'}\cdot\|u*\eta_t\|_B \frac{dt}{t}$$
$$\leqslant c\|v\|_2\cdot\|u\|_1 + \|v_t\|_{X'(B')}\cdot\|u*\eta_t\|_{X(B)} \leqslant c\|v\|_2\cdot\|u\|_1.$$

Therefore the mapping $v \to L_v$ is continuous from Λ_2 into Λ_1', and $\|L_v\| \leqslant c\|v\|_2$.

Conversely, suppose L is a linear functional on Λ_1.

In Corollary 3.1 we showed that the mapping $S\colon B\oplus X(B)\to\Lambda_1$ defined for $(u, F) \in B\oplus X(B)$ by

$$S(u, F) = u*\psi + \int_0^1 F*\varphi_t \frac{dt}{t}$$

is continuous and onto Λ_1. In fact $S(u, u*\eta_t) = u$ for each $u \in \Lambda_1$.

Now define $\bar{L}$ on $B\oplus X(B)$ by $\bar{L}(u, F) = L\circ S(u, F)$. Then $\bar{L}$ is a bounded linear operator on $B\oplus X(B)$. Therefore there exist $w \in B'$ and $G \in X'(B')$ such that

$$\bar{L}(u, F) = \langle w, u\rangle + \int_0^1 \langle G, F\rangle \frac{dt}{t},$$

and

(6.1) $$\|w\|_{B'} \leqslant \|\bar{L}\| \quad\text{and}\quad \|G\|_{X'(B')} \leqslant \|\bar{L}\|.$$

Thus if $u \in \Lambda_1$, then

$$L(u) = \bar{L}(u, u*\eta_t) = \langle w, u\rangle + \int_0^1 \langle G, u*\eta_t\rangle \frac{dt}{t}.$$

Now $w = w*\psi + \int_0^1 w*\varphi_t*\eta_t \frac{dt}{t}$, and

$$\int_0^1 \langle G, u*\eta_t\rangle \frac{dt}{t} = \int_0^1 \langle G*\eta_t, u\rangle \frac{dt}{t}$$
$$= \int_0^1 \left\langle G*\eta_t, u*\psi + \int_0^1 u*\eta_s*\varphi_s \frac{ds}{s}\right\rangle \frac{dt}{t}$$
$$= \langle W*\psi, u\rangle + \int_0^1 \langle W*\varphi_s, u*\eta_s\rangle \frac{ds}{s},$$

where $W = \int_0^1 G*\eta_t \frac{dt}{t}$. After noting that X' is of type $-\alpha$ (Lemma 2.4) we appeal to Lemma 5.2 to deduce that $W \in \Lambda_2$ and

(6.2) $$\|W\|_2 \leqslant \|G\|_{X'(B')}.$$

If we now write $v = w + W$, then we have shown that

$$L(u) = \langle v*\psi, u\rangle + \int_0^1 \langle v*\varphi_s, u*\eta_s\rangle \frac{ds}{s},$$

and

$$\|v\|_2 \leqslant \|w\|_2 + \|W\|_2$$
$$\leqslant c\|w\|_{B'} + \|G\|_{X'(B')} \quad \text{(by lemma 6.2 and (6.2))}$$
$$\leqslant c\|\bar{L}\| \quad \text{(by (6.1))}.$$

Remark. It should be pointed out that the proof of Theorem 6.1 differs from that used by M.H. Taibleson in [7] for the cases he considers there. While he used the theories of semi-groups and the interpolation of linear operators, the present paper does not appeal to these theories.

References

[1] A. Benedek and R. Panzone, *The spaces L^P, with mixed norm*, Duke Math. J. 28 (1961), pp. 301–324.

[2] J. Boman and H. S. Shapiro, *Comparison theorems for a generalized modulus of continuity*, Bull. AMS (1969), pp. 1266–1268.

[3] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), pp. 113–190.

[4] M. de Guzmán, *Singular integral operators with generalized homogeneity*, Rev. Acad. Ci. Madrid (1970).

[5] N. J. H. Heideman, *Duality and fractional integration in the spaces $\Lambda(B, X)$ of A. P. Calderón*, Ph. D. thesis, Washington University 1970.

[6] M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space I, Principal properties*, J. Math. Mech. 13 (1964), pp. 407–480.

[7] — *On the theory of Lipschitz spaces of distributions on Euclidean n-space II, translation invariant operators, duality and interpolation*, J. Math. Mech. 14 (1965), pp. 821–840.

[8] G. L. Weiss, *On certain classes of function spaces and on the interpolation of sublinear operators*, Ph. D. thesis, University of Chicago 1956.

[9] A. Zygmund, *Trigonometric series*, London 1959.

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