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On Littlewood–Paley functions

by

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Abstract. We define and study a generalization of the Littlewood–Paley function, g_λ^*, by using properties of certain approximations of the identity. L^p estimates are obtained and some examples and applications are given.

Introduction. If f is a reasonable function on $\boldsymbol{R}^n$, the Littlewood–Paley function, $g_\lambda^*(f)$, is defined for $\lambda > 0$ by

$$g_\lambda^*(f, x) = \left\{\int_0^\infty \int_{\boldsymbol{R}^n} \frac{y^{\lambda+1}}{|(|z|+y)|^{n+\lambda}} |\operatorname{grad} P_y * f(x-z)|^2 dz dy\right\}^{1/2}$$

where $P_y(z)$ is the Poisson kernel for the upper half space.

Stein introduced g_λ^* in [9], where he showed that the transformation $f \to g_\lambda^*(f)$ is bounded on $L^p(\boldsymbol{R}^n)$ for $\max\left\{1, \frac{2n}{\lambda+n}\right\} < p < \infty$. In [10], he used g_λ^* to obtain a characterization of the Lebesgue spaces, L_α^p, and in [11], showed how g_λ^* and its variants can be used to obtain Hörmander's version of Mihlin's multiplier theorem.

Subsequently, Segovia and Wheeden, [8], introduced a Littlewood–Paley function, which we call $g_\lambda^{**}(f)$, and which is defined for $\lambda > 0$ by

$$g_\lambda^{**}(f, x) = \left\{\int_0^\infty \int_{\boldsymbol{R}^n} \frac{y^{\lambda+1}}{(\varrho(z)+y)^{n+1+\lambda}} \left|\frac{\partial}{\partial y} \Gamma_y * f(x-z)\right|^2 dz dy\right\}^{1/2}$$

where $\varrho(z) = \sum_{i=1}^{n-1} |z_i| + |z_n|^{1/2}$ and $\Gamma_y(x)$ is the Poisson type kernel for the upper half space associated with the heat equation $\sum_{i=1}^{n-1} u_{x_i x_i} - u_{x_n} + u_{yy} = 0$. They showed that the transformation $f \to g_\lambda^{**}(f)$ is bounded on $L^p(\boldsymbol{R}^n)$ for $\max\left\{1, \frac{2n+2}{\lambda+n+1}\right\} < p < \infty$.

It is the purpose of this paper to show that Littlewood–Paley functions with similar properties can be defined without recourse to Laplace or heat equations.

This article is divided into two chapters. Chapter I contains certain facts, some of which are well known, which are needed in the development and applications of the contents of Chapter II. The promised results are contained in the second chapter.

Some of the material presented here formed part of the author's dissertation directed by Professor Walter Littman at the University of Minnesota.

CHAPTER I

PRELIMINARIES

1. Notation and conventions. $\boldsymbol{R}$ is the real line whose elements are denoted by r, s, and t and $\boldsymbol{R}_+ = \{t \in \boldsymbol{R}\colon t > 0\}$.

$\boldsymbol{R}^n$ is n-dimensional real Euclidean space whose elements are denoted by x, y, z and ξ.

If Ω is a subset of $\boldsymbol{R}^n$, $\bar{\Omega}$ and Ω^c denote the closure and complement of Ω in $\boldsymbol{R}^n$ respectively. If Ω is measurable, $|\Omega|$ denotes the Lebesgue measure of Ω.

If x and z are elements of $\boldsymbol{R}^n$, $\langle x, z\rangle = \sum_{j=1}^{n} x_j z_j$ and $|x| = \sqrt{\langle x, x\rangle}$.

The symbols a and b will always be used to denote linear transformations on $\boldsymbol{R}^n$. $\|a\| = \sup\left\{\frac{|ax|}{|x|}\colon x \in \boldsymbol{R}^n\right\}$ and a^* denotes the adjoint of a. I denotes the identity matrix.

D_j denotes the differential operator $\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$.

All functions considered by us are complex valued and all integrals are over $\boldsymbol{R}^n$, unless denoted otherwise.

If f is a measurable function on $\boldsymbol{R}^n$, then

$$\|f\|_p = \left\{\int |f(x)|^p\,dx\right\}^{1/p}, \quad 1 \leqslant p < \infty,$$

$$\|f\|_\infty = \text{ess.}\sup.|f(x)|,$$

and $L^p(\boldsymbol{R}^n) = L^p$, is the usual Banach space of functions for which $\|f\|_p$ is finite.

p' always denotes the Hölder conjugate of p for $1 \leqslant p \leqslant \infty$, namely $\frac{1}{p} + \frac{1}{p'} = 1$.

If Ω is an open subset of some Euclidean space, $C^\infty(\Omega)$ is the set of infinitely differentiable functions on Ω and $C_0^\infty(\Omega)$ is that subset of $C^\infty(\Omega)$ consisting of functions with compact support. In the case $\Omega = \boldsymbol{R}^n$, we simply write C^∞ and C_0^∞ instead of $C^\infty(\boldsymbol{R}^n)$ and $C_0^\infty(\boldsymbol{R}^n)$.

The subspace of C^∞ consisting of functions which together with all their derivatives tend to zero at infinity faster than any rational function is denoted by $\mathscr{S}$ and is given the usual topology. Its dual, the space of tempered distributions, is denoted by $\mathscr{S}'$ and is also given the usual topology. $\langle T, f\rangle$ denotes the distribution T acting on $f \in \mathscr{S}$.

We say that a distribution T is in $C^\infty(\boldsymbol{R}^n \setminus \{0\})$ (or $\mathscr{S}(\boldsymbol{R}^n \setminus \{0\})$) if the distribution $(1-\varphi)T$ is in C^∞ (or $\mathscr{S}$) for every $\varphi \in C_0^\infty$ which is identically one in a neighborhood of the origin.

The Fourier transform of $f \in \mathscr{S}$ is defined by

$$\mathscr{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2}\int f(x)e^{-i\langle x,\xi\rangle}\,dx,$$

and the inverse Fourier transform of f is defined by

$$\mathscr{F}^{-1}f(x) = \check{f}(x) = (2\pi)^{-n/2}\int f(\xi)e^{i\langle \xi,x\rangle}\,d\xi.$$

Plancherel's formula thus reads:

$$\|\hat{f}\|_2 = \|\check{f}\|_2 = \|f\|_2.$$

The Fourier transform is defined on $\mathscr{S}'$ in the usual manner.

All Fourier transforms and differentiations are to be interpreted in the $\mathscr{S}'$ sense, unless they make sense otherwise.

The convolution of two functions, f and g, defined on $\boldsymbol{R}^n$ is $f*g(x) = \int f(x-z)g(z)\,dz$, whenever this operation makes sense.

The symbol C will be used generically for constants appearing in certain estimates. Sometimes it will be subscripted to denote the parameters it depends on. At other times the parameters C depends on will be clear from the proof of the estimate. It need not be the same at different occurrences.

The end of a proof will always be signalled by ■.

2. Quasi-homogeneous metrics, functions, and distributions. Let a be a linear transformation on $\boldsymbol{R}^n$, and consider the one parameter group $t^a = e^{a\log t}$, $t > 0$. We say that a is *good* if $\|t^a\| \leqslant t$ for $0 < t \leqslant 1$. If there is an $\varepsilon > 0$ such that $\|t^a\| \leqslant t^\varepsilon$ for $0 < t \leqslant 1$, we call a *reasonable*. Observe that if a is reasonable, then there is a positive constant, k, such that ka is good. Also note that if a is reasonable or good, then a^* also has that property.

Suppose a is reasonable and consider the function $F(x,t) = |t^{-a}x|$, $t > 0$. It is clear that, for fixed $x \neq 0$, $F(x;t)$ is a strictly decreasing continuous function of t, $\lim\limits_{t\to 0} F(x,t) = \infty$ and $\lim\limits_{t\to\infty} F(x,t) = 0$. It follows that $F(x,t) = 1$ has a unique solution which we call $\varrho_a(x)$. Having defined $\varrho_a(x)$ for $x \neq 0$, set $\varrho_a(0) = 0$. The following proposition, whose proof follows immediately from the definitions, gives us some important properties of the function ϱ_a.

PROPOSITION 1.

(i) $\varrho_a \in C^\infty(\boldsymbol{R}^n \setminus \{0\})$.

(ii) $\varrho_a(t^a x) = t\varrho_a(x)$, $t > 0$.

(iii) $\varrho_a(x) = 1$ *if and only if* $|x| = 1$.

(iv) *If* a *is good, then* $\varrho_a(x+z) \leqslant \varrho_a(x) + \varrho_a(z)$.

We call ϱ_a the *quasi-homogeneous "metric" with respect to* a. Note that (iv) of the above proposition implies that if a is good then the function $(x, z) \to \varrho_a(x-z)$ is indeed a metric on $\boldsymbol{R}^n$.

The above considerations suggest the polar change of variables

$$x = r^a x', \tag{1}$$

where $r = \varrho_a(x) > 0$ and $x' = \varrho_a(x)^{-a} x \in S^{n-1} = \{x\colon |x| = 1\}$. Writing x' in terms of the usual polar angles of $\boldsymbol{R}^n$, $\theta_1, \ldots, \theta_{n-1}$, and computing the Jacobian of (1), we get $dx = r^{\operatorname{tr} a - 1} dr\, d\sigma_a$ where $\operatorname{tr} a$ = trace of a and $d\sigma_a = J_a(\theta_1, \ldots, \theta_{n-1}) d\theta_1 \ldots d\theta_{n-1}$ is a measure on S^{n-1}. (For more details concerning the metric and the polar change of variables see [2] or [6].)

Throughout the rest of the section a is always assumed to be reasonable.

The following formulas can be easily verified using the above change of variables:

$$\int_{S^{n-1}} d\sigma_a = \frac{\omega_n}{n} \operatorname{tr} a,$$

$$\int_{\varrho_a(x)<r} dx = \frac{\omega_n}{n} r^{\operatorname{tr} a}$$

where ω_n is the area of S^{n-1}.

If f is a measurable function on $\boldsymbol{R}^n$ such that for some real constants C and m, $|f(x)| \leqslant C\varrho_a(x)^m$, then for any $\varphi \in C_0^\infty$ such that $\varphi = 1$ in a neighborhood of the origin $\varphi(x)f(x)$ is integrable if $m > -\operatorname{tr} a$ and $(1-\varphi(x))f(x)$ is integrable if $m < -\operatorname{tr} a$. Also note that if a is diagonal (i. e. $ax = (a_1 x_1, \ldots, a_n x_n)$) then there is a constant $C > 0$ such that

$$C^{-1}\varrho_a(x) \leqslant \sum_{j=1}^{n} |x_j|^{1/a_j} \leqslant C\varrho_a(x)$$

for all x in $\boldsymbol{R}^n$.

A function f, defined on $\boldsymbol{R}^n$, is said to be *quasi-homogeneous of degree* k, $k \in \boldsymbol{R}$, *with respect to* a if for every $t > 0$ the formula $f(t^a x) = t^k f(x)$ holds for every $x \neq 0$.

The notion of quasi-homogeneity for tempered distributions is analogous to that for functions. For $t > 0$, define the operator π_t^a acting on $\varphi \in \mathscr{S}$ by the formula $\pi_t^a \varphi(x) = \varphi(t^a x)$. For $T \in \mathscr{S}'$ define the operator π_t^a acting on T by the formula

$$\langle \pi_t^a T, \varphi \rangle = \langle T, t^{-\operatorname{tr} a} \pi_{t^{-1}}^a \varphi \rangle.$$

It is clear that $\pi_t^a T$ is in $\mathscr{S}'$. Define T to be quasi-homogeneous of degree k with respect to a if for every $t > 0$, $\pi_t^a T = t^k T$.

A simple computation shows that for $T \in \mathscr{S}'$, $\widehat{\pi_t^a T} = t^{-\operatorname{tr} a} \pi_{t^{-1}}^{a^*} \hat{T}$. Now, if T is quasi-homogeneous of degree k with respect to a, applying the last formula results in $\pi_t^{a^*} \hat{T} = t^{-\operatorname{tr} a - k} \hat{T}$ and hence, $\hat{T}$ is quasi-homogeneous of degree $-\operatorname{tr} a - k$ with respect to a^*.

We conclude this section with

PROPOSITION 2. *If the tempered distribution* T *is locally integrable, in* $C^\infty(\boldsymbol{R}^n \setminus \{0\})$, *and quasi-homogeneous of degree* k *with respect to* a, *where* $-\operatorname{tr} a < k < 0$, *then* $\hat{T}$ *is locally integrable, in* $C^\infty(\boldsymbol{R}^n \setminus \{0\})$, *and quasi-homogeneous of degree* $-\operatorname{tr} a - k$ *with respect to* a^*.

Proof. Let φ be in $C_0^\infty(\boldsymbol{R}_+)$ such that $\int_0^\infty \varphi(t) \frac{dt}{t} = 1$. Write $g(x) = T(x)\varphi(\varrho_a(x))$. Clearly $g \in \mathscr{S}$,

$$T(x) = \int_0^\infty t^{-k} g(t^a x) \frac{dt}{t},$$

$$\hat{T}(\xi) = \int_0^\infty t^{-k-\operatorname{tr} a} \hat{g}(t^{-a^*}\xi) \frac{dt}{t},$$

and the conclusion of the proposition follows. ■

3. Vitali families, maximal functions, and "quasi-homogeneous like" kernels. Let $\{U_s, s > 0\}$ be a family of open subsets of $\boldsymbol{R}^n$ whose closure is compact.

DEFINITION. $\{U_s, s > 0\}$ is a *Vitali family with constant* A if and only if

(i) for $s_1 < s_2$, $U_{s_1} \subset U_{s_2}$ and $\bigcap_{s>0} U_s = \{0\}$,

(ii) $|U_s - U_s| \leqslant A|U_s|$ for all s, where $U_s - U_s$ denotes $\{x\colon x = y - z$ where y and z are both in $U_s\}$,

(iii) $|U_s|$ is a left continuous function of s.

THEOREM 1. *Suppose* Ω *is a measurable set in* $\boldsymbol{R}^n$ *and let* $x \to r(x)$ *be a mapping of* Ω *into* $\boldsymbol{R}_+$ *satisfying*:

(i) $r(x)$ *is bounded and for every* $r_0 > 0$ *the set* $\{x\colon x \in \Omega, r(x) > r_0\}$ *is a bounded subset of* $\boldsymbol{R}^n$.

(ii) *If* $\{x_k\}$ *is a sequence which converges to* x_0 *and* $r(x_k) \uparrow r_0$ *then* $x_0 \in \Omega$ *and* $r(x_0) \geqslant r_0$.

If $\{U_s, s>0\}$ is a Vitali family with constant A, then there exists a sequence $\{x_k\}\subset \Omega$ such that

1) $\{x_k+U_{r(x_k)}\}$ *is disjoint,*

2) $\Omega\subset \bigcup_{k=1}^{\infty}\{x_k+(U_{r(x_k)}-U_{r(x_k)})\}$,

3) $|\Omega|\leqslant A\sum_{k=1}^{\infty}|U_{r(x_k)}|$.

THEOREM 2. *Let $\{U_s, s>0\}$ be a Vitali family with constant A. For $f\in L^1$, define*

$$Mf(x) = \sup_{s>0}\frac{1}{|U_s|}\int_{U_s}|f(x-z)|\,dz.$$

Then

(i) $|\{x\colon Mf(x)>1\}|\leqslant A\|f\|_1$,

(ii) $\|Mf\|_p\leqslant C\|f\|_p$ *for* $1<p\leqslant\infty$, *where C is a constant depending only on A and p.*

For proofs of the above theorems see Rivière [6].

In the statements of the next two theorems, we take a to be reasonable.

THEOREM 3. *Suppose $H\in L^1$ and $|H(x)|\leqslant h\big(\varrho_a(x)\big)$ where $h(t)$ is a decreasing function on $\boldsymbol{R}_+$ and $\int h\big(\varrho_a(x)\big)dx\leqslant$ constant.*

Consider the transformation $f\to M_{H_b}f$ where

$$M_{H_b}f(x) = \sup_{t>0}\Big|\int t^{-\operatorname{tr} b}H(t^{-b}z)f(x-z)\,dz\Big|.$$

If b is reasonable and commutes with a, then

$$\|M_{H_b}f\|_p\leqslant C\|f\|_p \quad \text{for } 1<p\leqslant\infty,$$

where C is a constant which depends only on H, p, and n.

Proof. Write

$$\Big|\int t^{-\operatorname{tr} b}H(t^{-b}z)f(x-z)\,dz\Big|$$

$$\leqslant \sum_{k=-\infty}^{\infty}\int_{2^{k-1}\leqslant\varrho_a(t^{-b}z)<2^k} t^{-\operatorname{tr} b}|H(t^{-b}z)|\,|f(x-z)|\,dz$$

$$\leqslant \sum_{k=-\infty}^{\infty}h(2^{k-1})t^{-\operatorname{tr} b}\int_{\varrho_a(t^{-b}z)<2^k}|f(x-z)|\,dz.$$

Taking the sup over $t>0$, we get

$$M_{H_b}f(x)\leqslant\sum_{k=-\infty}^{\infty}h(2^{k-1})2^{k\operatorname{tr} a}M_kf(x)$$

where

$$M_kf(x) = \sup_{t>0}\frac{1}{2^{k\operatorname{tr} a}t^{\operatorname{tr} b}}\int_{|2^{-ka}t^{-b}z|<1}|f(x-z)|\,dz.$$

Let $\{U_t^k, t>0\}$ be the family of open subsets of $\boldsymbol{R}^n$ defined by $U_t^k = \{z\colon |2^{-ka}t^{-b}|<1\}$. Clearly $\{U_t^k, t>0\}$ is a Vitali family with constant 2^n. Observing that $|U_t^k| = \frac{\omega_n}{n}2^{k\operatorname{tr} a}t^{\operatorname{tr} b}$ and applying Theorem 2, it follows that $\|M_kf\|_p\leqslant C\|f\|_p$ for $1<p\leqslant\infty$, where C is a constant depending only on n and p. Since this is true for each $k=0,\pm1,\pm2,\ldots$, we conclude that

$$\|M_{H_b}f\|_p\leqslant\sum_{k=-\infty}^{\infty}h(2^{k-1})2^{k\operatorname{tr} a}C\|f\|_p\leqslant\Big[C\int h\big(\varrho_a(x)\big)dx\Big]\|f\|_p. \ \blacksquare$$

The next theorem is a quasi-homogeneous version of a classical result generally referred to as Sobolev's imbedding theorem. For a proof in the case $a=I$, see Stein [11].

THEOREM 4. *Suppose H is locally integrable on $\boldsymbol{R}^n$ and $|H(x)|\leqslant C\varrho_a(x)^{\alpha-\operatorname{tr} a}$, where $0<\alpha<\operatorname{tr} a$. If p and q satisfy $1<p<\frac{\operatorname{tr} a}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{\operatorname{tr} a}$, then for $f\in L^p$ the transformation $f\to H*f$ is well defined and $\|H*f\|_q\leqslant C\|f\|_p$, where C depends on H, p, and n.*

4. Distributions whose Fourier transforms are in L^1.

PROPOSITION 3. *Let $\beta_1,\ldots,\beta_n$ be positive integers such that $\sum_{j=1}^{n}\frac{1}{\beta_j}<2$. Suppose that $f\in\mathscr{S}'$ satisfies*

(i) $\|f\|_2\leqslant B$,

(ii) $\|D_j^{\beta_j}f\|_2\leqslant B$, $j=1,\ldots,n$;

then $\hat f\in L^1$ and $\|\hat f\|_1\leqslant CB$, where C is a constant which depends only on $\beta_1,\ldots,\beta_n$.

Proof. From (i) it is clear that $\hat f$ is a function in L^2. To compute the L^1 norm of $\hat f$, let b be the linear transformation defined by $bx=\Big(\frac{1}{\beta_1}x_1,\ldots,\frac{1}{\beta_n}x_n\Big)$ and write

$$\int|\hat f(x)|\,dx\leqslant\Big\{\int\big|\big(1+\varrho_b(x)\big)\hat f(x)\big|^2dx\Big\}^{1/2}\cdot\Big\{\int\big(1+\varrho_b(x)\big)^{-2}dx\Big\}^{1/2} \tag{2}$$

where $\varrho_b(x)$ is the quasi-homogeneous "metric" with respect to b. Recalling that $\varrho_b(x) \leqslant C\sum_{j=1}^{n}|x_j|^{\beta_j}$ and applying Plancherel's formula we have

$$\Big\{\int \big|(1+\varrho_b(x))\hat{f}(x)\big|^2dx\Big\}^{1/2} \leqslant C\Big(\|f\|_2+\sum_{j=1}^{n}\|D_j^{\beta_j}f\|_2\Big).$$

Using the polar change of variables we have

$$\Big\{\int (1+\varrho_b(x))^{-2}dx\Big\}^{1/2} = C'\Big\{\int_0^\infty (1+r)^{-2}r^{\operatorname{tr}b-1}dr\Big\}^{1/2} = C$$

since $\operatorname{tr}b = \sum_{j=1}^{n}\frac{1}{\beta_j} < 2$. Slipping the last two estimates into (2) gives us the desired result. ■

THEOREM 5. *Suppose a is reasonable and let $\beta_1, \ldots, \beta_n$ be positive integers such that $\sum_{j=1}^{n}\frac{1}{\beta_j} < 2$. If f is in L^∞ such that*

(i)
$$\int_{\frac12<\varrho_a(\xi)<2} |D_j^{\gamma_j}f_k(\xi)|^2d\xi \leqslant B_k^2$$

for all $k = 0, \pm1, \pm2, \ldots$, and integers γ_j, $0\leqslant\gamma_j\leqslant\beta_j$, $j = 1, \ldots, n$, where $f_k(\xi) = f(2^{ka}\xi)$ and the B_k's are positive numbers with $\sum_{-\infty}^{\infty}B_k < \infty$, then $\hat{f}$ is in L^1 and the L^1 norm of $\hat{f}$ is bounded by a constant which depends only on $\beta_1, \ldots, \beta_n$ and $\sum_{-\infty}^{\infty}B_k$.

Proof. Let ψ be a positive function in $C_0^\infty(\boldsymbol{R}_+)$ with support in $[\frac12, 2]$ and such that $\psi(t) > 0$ for $\frac{1}{\sqrt2}\leqslant t\leqslant\sqrt2$. Set $\Phi(t) = \psi(t)/\sum_{-\infty}^{\infty}\psi(2^{-k}t)$ and $\varphi(\xi) = \Phi(\varrho_a(\xi))$. Observe that $\varphi\in C_0^\infty$, $\varphi(2^{-ka}\xi)$ has support in $\{\xi\colon 2^{k-1}\leqslant\varrho_a(\xi)\leqslant 2^{k+1}\}$ and $\sum_{-\infty}^{\infty}\varphi(2^{-ka}\xi) = 1$ for $\xi\neq 0$. Write

(3)
$$f(\xi) = \sum_{-\infty}^{\infty}\varphi(2^{-ka}\xi)f(\xi) = \sum_{-\infty}^{\infty}f^k(\xi).$$

It is clear that the $\hat{f^k}$'s are in L^2. To obtain an estimate on their L^1 norms, using the same method as in proof of Proposition 3, write

(4)
$$\int|\hat{f^k}(x)|dx \leqslant \Big\{\int\big|(1+\varrho_b(2^{ka^*}x))\hat{f^k}(x)\big|^2dx\Big\}^{1/2}\Big\{\int(1+\varrho_b(2^{ka^*}x))^{-2}dx\Big\}^{1/2}$$
$$\leqslant\Big(\|f^k\|_2+\Big\{\int|\varrho_b(2^{ka^*}x)\hat{f^k}(x)|^2dx\Big\}^{1/2}\Big)\cdot C2^{\frac{-k\operatorname{tr}a}{2}}.$$

Now, using Plancherel's formula and (i), we have

$$\|\hat{f^k}\|_2 = \|f^k\|_2 \leqslant \Big\{\int_{\frac12<\varrho(2^{-ka}\xi)\leqslant2}|f(\xi)|^2d\xi\Big\}^{1/2}\leqslant 2^{\frac{k\operatorname{tr}a}{2}}B_k$$

and

$$\Big\{\int|\varrho_b(2^{ka^*}x)\hat{f^k}(x)|^2dx\Big\}^{1/2} = 2^{\frac{-k\operatorname{tr}a}{2}}\Big\{\int|\varrho_b(x)\hat{f^k}(2^{-ka^*}x)|^2dx\Big\}^{1/2}$$
$$\leqslant C''2^{\frac{-k\operatorname{tr}a}{2}}\sum_{j=1}^{n}\Big\{\int\big|2^{k\operatorname{tr}a}D_j^{\beta_j}\big(f_k(\xi)\varphi(\xi)\big)\big|^2d\xi\Big\}^{1/2}$$
$$\leqslant C'2^{\frac{k\operatorname{tr}a}{2}}\sum_{j=1}^{n}\sum_{\gamma_j=0}^{\beta_j}\Big\{\int_{\frac12<\varrho(\xi)<2}|D_j^{\gamma_j}f_k(\xi)|^2d\xi\Big\}^{1/2}$$
$$\leqslant C2^{\frac{k\operatorname{tr}a}{2}}B_k.$$

Slipping the last two estimates into (4), we conclude that

(5)
$$\|\hat{f^k}\|_1\leqslant CB_k,\qquad k = 0, \pm1, \pm2, \ldots$$

Since $\sum_{-\infty}^{\infty}B_k < \infty$, it follows from (5) that there is an F in L^1 such that $\lim_{m\to\infty}\|F-\sum_{k=-m}^{m}\hat{f^k}\|_1 = 0$ and whose L^1 norm is bounded by $C\sum_{-\infty}^{\infty}B_k$. Observe that (3) implies that $\lim_{m\to\infty}\sum_{k=-m}^{m}f^k = f$ in $\mathscr{S}'$, and using the fact that the Fourier transform is continuous on $\mathscr{S}'$, we conclude that $\hat{f} = F$. ■

The following corollary of Theorem 5 can be used for most applications.

COROLLARY. *Suppose a is diagonal (i.e. $ax = (a_1x_1, \ldots, a_nx_n)$ and $a_j > 0$, $j = 1, \ldots, n$) and let $\beta_1, \ldots, \beta_n$ be as in Theorem 5.*

If f is in L^∞ and sufficiently smooth with

$$\sup_{2^{k-1}<\varrho_a(\xi)<2^{k+1}}|2^{ka_j\gamma_j}D_j^{\gamma_j}f(\xi)|\leqslant B_k$$

for $k = 0, \pm1, \pm2, \ldots$, and integers γ_j, $0\leqslant\gamma_j\leqslant\beta_j$, $j = 1, \ldots, n$, where the B_k's are positive numbers with $\sum_{k=-\infty}^{\infty}B_k < \infty$, then $\hat{f}$ is in L^1 and L^1 norm of f depends only on $\beta_1, \ldots, \beta_n$ and $\sum_{-\infty}^{\infty}B_k$.

5. Vector valued singular integrals. In this section we assume that a is a good linear transformation on $\boldsymbol{R}^n$. In this case, the function $(x, z)\to\varrho_a(x-z)$ is a metric on $\boldsymbol{R}^n$.

If $\mathscr{H}$ is a Hilbert space, then $|u|_{\mathscr{H}}$ denotes the norm of the element u of $\mathscr{H}$ and $L^p(\mathscr{H})$, $1\leqslant p\leqslant\infty$, denotes the space of strongly measurable

$\mathscr{H}$ valued functions defined on $\boldsymbol{R}^n$ such that $|f(x)|_{\mathscr{H}}$ belongs to L^p with norm $\|f\|_{L^p(\mathscr{H})} = L^p$ norm of $|f(x)|_{\mathscr{H}}$. If $\mathscr{H}_1$ and $\mathscr{H}_2$ are two Hilbert spaces, then $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ denotes the space of bounded linear operators from $\mathscr{H}_1$ to $\mathscr{H}_2$ and if L is in $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ then $|L|_{\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)} = \sup\{|Lu|_{\mathscr{H}_2}: u \in \mathscr{H}_1, |u|_{\mathscr{H}_1} \leqslant 1\}$.

The following theorem is a generalization of the Calderón–Zygmund inequality and its proof can be found in Rivière [6].

THEOREM 6. *Let $\mathscr{K}(x)$ be a function on $\boldsymbol{R}^n$ with values in $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ such that $\mathscr{K}$ is measurable and integrable on compact subsets of $\boldsymbol{R}^n\setminus\{0\}$. Suppose $\mathscr{K}$ has the following properties:*

(i) $\Big|\int\limits_{\varepsilon<\varrho_a(x)<\delta} \mathscr{K}(x)\,dx\Big|_{\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)} \leqslant C_1$, *where C_1 is independent of ε and δ, $0<\varepsilon<\delta<\infty$, and for each $u \in \mathscr{H}_1$, $\lim\limits_{\varepsilon\to 0}\int\limits_{\varepsilon<\varrho_a(x)<1} \mathscr{K}(x)\,dx\,u$ exists.*

(ii) For $u \in \mathscr{H}_1$, $\int\limits_{\varepsilon<\varrho_a(x)<2\varepsilon} |\mathscr{K}(x)u|_{\mathscr{H}_2}\,dx \leqslant C_2|u|_{\mathscr{H}_1}$, *where C_2 is independent of u and ε, $\varepsilon>0$.*

(iii) For $u \in \mathscr{H}_1$, $\int\limits_{\varrho_a(x)>2\varrho_a(z)} |[\mathscr{K}(x-z)-\mathscr{K}(x)]u|_{\mathscr{H}_2}\,dx \leqslant C_3|u|_{\mathscr{H}_1}$, *where C_3 is independent of u and z.*

Also assume that $\mathscr{K}^*(x)$ enjoys the same properties as $\mathscr{K}(x)$.

Under these conditions, the transformation $f \to \mathscr{K}f$ given by $\mathscr{K}f(x) = \lim\limits_{\varepsilon\to 0}\int\limits_{\varepsilon<\varrho_a(z)<\frac{1}{\varepsilon}} \mathscr{K}(z)f(x-z)\,dz$ is well defined on $L^p(\mathscr{H}_1)$ to $L^p(\mathscr{H}_2)$ for $1<p<\infty$ and $\|\mathscr{K}f\|_{L^p(\mathscr{H}_2)} \leqslant C\|f\|_{L^p(\mathscr{H}_1)}$ where C depends only on C_1, C_2, C_3 and p.

CHAPTER II

LITTLEWOOD–PALEY FUNCTIONS

1. g_{K_a}. Suppose that K is in L^1 and a is good; then for $f \in L^p$, $1 \leqslant p < \infty$, the Littlewood–Paley function $g_{K_a}(f)$ is defined by the formula

$$g_{K_a}(f, x) = \Big\{\int\limits_0^\infty |K_{t^a} * f(x)|^2 \frac{dt}{t}\Big\}^{1/2}, \quad \text{where } K_{t^a}(x) = t^{-\operatorname{tr} a}K(t^{-a}x). \tag{6}$$

The following theorem gives some conditions on K which imply that the semi-linear transformation, $f \to g_{K_a}(f)$, maps L^p boundedly into L^p.

THEOREM 7. *Suppose K has the following properties:*

(i) $|K(x)| \leqslant h(\varrho_a(x))$ *where $h(t)$ is a decreasing function on $\boldsymbol{R}_+$ and $h(t) \leqslant C_1 t^{\delta-\operatorname{tr} a}(1+t)^{-2\delta}$ for some $\delta>0$.*

(ii) $\int K(x)\,dx = 0$.

(iii) $\int |K(x-z)-K(x)|\,dx \leqslant C_2\varrho_a(z)^\varepsilon$ *for some $\varepsilon > 0$.*

Under these conditions it follows that

$$\|g_{K_a}(f)\|_p \leqslant C\|f\|_p, \quad 1<p<\infty,$$

where C depends only on C_1, C_2, δ, ε, and p.

Proof. This result is an easy consequence of Theorem 5. To see this, let $\mathscr{H}_1$ be the complex numbers and let

$$\mathscr{H}_2 = \Big\{\varphi\colon \varphi \text{ measurable on } \boldsymbol{R}_+ \text{ with } \Big\{\int\limits_0^\infty |\varphi(t)|^2\frac{dt}{t}\Big\}^{1/2} < \infty\Big\}.$$

Let $\mathscr{K}(x)$ be the linear operator transforming the complex number ζ into the element $K_{t^a}(x)\zeta$ of $\mathscr{H}_2$. Now, if $\mathscr{K}(x)$ satisfies the hypothesis of Theorem 5, observe that $g_{K_a}(f, x) = |\mathscr{K}f(x)|_{\mathscr{H}_2}$ and hence

$$\|g_{K_a}(f)\|_p = \|\mathscr{K}f\|_{L^p(\mathscr{H}_2)} \leqslant C\|f\|_{L^p(\mathscr{H}_1)} = C\|f\|_p.$$

The calculations showing that $\mathscr{K}(x)$ and $\mathscr{K}^*(x)$ satisfy (i), (ii), and (iii) are analogous to those in Benedek, Calderón, and Panzone [1], so we omit them. ■

Observe that if K satisfies the hypothesis of Theorem 6, then $\int\limits_0^\infty |\hat{K}_{t^a}(\xi)|^2\frac{dt}{t} < \infty$ and recall that $\hat{K}_{t^a}(\xi) = \hat{K}(t^{a^*}\xi)$. Now, if $\hat{K}$ is a function which depends only on $\varrho_{a^*}(\xi)$ (i.e. $\hat{K}(\xi) = \varkappa(\varrho_{a^*}(\xi))$ where $\varkappa$ is a function on $\boldsymbol{R}_+$) then $\int\limits_0^\infty |\hat{K}_{t^a}(\xi)|^2\frac{dt}{t} = \int\limits_0^\infty |\varkappa(t)|^2\frac{dt}{t} = C =$ constant independent of ξ. In this case, for f_1 and f_2 in $\mathscr{S}$, we have

$$\int\int\limits_0^\infty K_{t^a}*f_1(x)\overline{K_{t^a}*f_2(x)}\frac{dt}{t}\,dx = \int\limits_0^\infty\int |\hat{K}_{t^a}(\xi)|^2\hat{f}_1(\xi)\overline{\hat{f}_2(\xi)}\,d\xi\frac{dt}{t}$$
$$= C\int \hat{f}_1(\xi)\overline{\hat{f}_2(\xi)}\,d\xi = C\int f_1(x)\overline{f_2(x)}\,dx.$$

Applying Hölder's inequality to the last formula results in

$$C\Big|\int f_1(x)\overline{f_2(x)}\,dx\Big| \leqslant \|g_{K_a}(f_1)\|_p\|g_{K_a}(f_2)\|_{p'}.$$

The last inequality together with Riesz representation and a simple limiting argument give us the following:

COROLLARY. *Suppose K satisfies the hypothesis of Theorem 6 and $\hat{K}(\xi)$ depends only on $\varrho_{a^*}(\xi)$, then if K is not identically zero*

$$\|f\|_p \leqslant C\|g_{K_a}(f)\|_p, \quad 1<p<\infty,$$

where C is a constant depending only on K and p.

2. g_{K_a,H_b}. Suppose H is a non-negative function in L^1 and b is reasonable, if K and a satisfy the hypothesis of Theorem 7 and f is in L^p, $1 \leqslant p < \infty$, the Littlewood–Paley function $g_{K_a,H_b}(f)$ is defined by the formula

$$g_{K_a,H_b}(f, x) = \Big\{\int_0^\infty \int H_{t^b}(x-z)|K_{t^a}*f(z)|^2 dz \frac{dt}{t}\Big\}^{1/2} \tag{7}$$

where $H_{t^b}(x) = t^{-\mathrm{tr}\, b} H(t^{-b}x)$. Observe that if f is in L^2 then

$$\|g_{K_a,H_b}(f)\|_2 \leqslant C\|f\|_2 \tag{8}$$

where $C = \Big\{\|H\|_1 \Big\|\int_0^\infty |\hat{K}_{t^b}(\xi)|^2 \frac{dt}{t}\Big\|_\infty\Big\}^{1/2}$. The following theorems show that with somewhat better H, the semi-linear operator $f \to g_{K_a,H_b}(f)$ maps L^p boundedly into L^p for other values of p also.

Consider the transformation $f \to M_{H_b} f$ defined by the formula $M_{H_b}f(x) = \sup_{t>0} |H_{t^b}*f(x)|$. Clearly this operation maps L^∞ boundedly into L^∞ and recall that Theorem 3 gives conditions on H which insure that it maps L^p boundedly into L^p for p's less than ∞.

THEOREM 8. *Suppose $f \to M_{H_b} f$ maps L^q boundedly into L^q for $1 < q \leqslant \infty$, then there is a constant, C_p, independent of f, such that*

$$\|g_{K_a,H_b}(f)\|_p \leqslant C_p\|f\|_p, \quad \text{for } 2 \leqslant p < \infty.$$

Proof. Since we already have the result for $p = 2$, assume that $2 < p < \infty$. Let φ be any function in L^q, where $q = \left(\frac{p}{2}\right)'$, and write

$$\Big|\int \varphi(x)\{g_{K_a,H_b}(f, x)\}^2 dx\Big| \leqslant \int\int_0^\infty \Big\{\int H_{t^b}(x-z)|\varphi(x)|dx\Big\}|K_{t^a}*f(z)|^2 \frac{dt}{t} dz$$
$$\leqslant \int M_{H_b}\tilde{\varphi}(-z)\{g_{K_a}(f, z)\}^2 dz$$

where $\tilde{\varphi}(x) = |\varphi(-x)|$. Applying the hypothesis and Hölder's inequality to the right-hand side of the last estimate we get

$$\Big|\int \varphi(x)\{g_{K_a,H_b}(f, x)\}^2 dx\Big| \leqslant C\|\varphi\|_q \|g_{K_a}(f)\|_p^2.$$

The desired result now follows from Riesz representation and Theorem 7. ■

THEOREM 9. *Suppose $H(x) \leqslant \varphi(\varrho_a(x))$ where $\varphi(t)$ is a decreasing function on $\boldsymbol{R}_+$, $\varphi(\varrho_a(x))$ is in L^1, and $\varphi(t) \leqslant Ct^{-\mathrm{tr}\, a - \lambda}$ for $\lambda > 0$. If $\hat{K}(\xi) \leqslant C(1+\varrho_{a^*}(\xi))^{-\gamma}$, $\gamma > \frac{1}{2}\min\{\mathrm{tr}\, a, \lambda\}$, then there is a constant, C_p, independent of f such that $\|g_{K_a,H_a}(f)\|_p \leqslant C_p\|f\|_p$ for $\max\Big\{1, \frac{2\,\mathrm{tr}\, a}{\lambda + \mathrm{tr}\, a}\Big\} < p < \infty$.*

This theorem is a simple corollary of the following lemma and the Marcinkiewicz interpolation theorem. The proof presented here is a modification of the calculation used by Fefferman, [3], in obtaining the corresponding estimate for g_λ^*.

LEMMA 1. *Suppose H and K satisfy the hypothesis of Theorem 9 and λ satisfies $0 < \lambda < \mathrm{tr}\, a$. Then, for $p = \frac{2\,\mathrm{tr}\, a}{\lambda + \mathrm{tr}\, a}$ and any f in L^p,*

$$|\{x\colon g_{K_a,H_a}(f, x) > s\}| \leqslant Cs^{-p}\|f\|_p^p$$

for each $s > 0$, where C is a constant independent of f and s.

Proof. Let f be a function in L^p, $p = \frac{2\,\mathrm{tr}\, a}{\lambda + \mathrm{tr}\, a}$. To prove the lemma it suffices to show that

$$|\{x\colon g_{K_a,H_a}(f, x) > 1\}| \leqslant C\|f\|_p^p \tag{9}$$

where C is a constant independent of f.

Let $B(x, s) = \{z\colon \varrho_a(x-z) < s\}$ and observe that $\{B(0, s), s > 0\}$ is a Vitali family with constant 2^n. Now set

$$F(x) = \sup_{s>0}\Big\{\frac{1}{|B(x, s)|}\int_{B(x,s)} |f(z)|^p dz\Big\} \quad \text{and} \quad \Omega = \{x\colon F(x) > 1\}.$$

Since $|f(x)|^p$ is in L^1, it follows from Theorem 2 that $|\Omega| = 2^n\|f\|_p^p$.

Let $x \to r(x)$ be a mapping of Ω into $\boldsymbol{R}_+$ defined by $r(x) = \frac{1}{8}\inf_{z\in\Omega^c} \varrho_a(x-z)$. Since Ω is open and of finite measure, $r(x)$ satisfies the hypothesis of Theorem 1 and hence there exists a sequence in Ω such that $\{B(x_j, r(x_j))\}$ is disjoint and $\bigcup_{j=1}^\infty B(x_j, 2r(x_j)) \supset \Omega$. Let $\{V_j\}$ be a disjoint family of measurable sets such that $B(x_j, r(x_j)) \subset V_j \subset B(x_j, 2r(x_j))$, $j = 1, 2, \ldots$, $\bigcup_{j=1}^\infty V_j = \Omega$. Define the functions f' and f'' by

$$f'(x) = \begin{cases} \dfrac{1}{|V_j|}\displaystyle\int_{V_j} f(z)dz, & x \in V_j, \\ f(x), & x \in \Omega^c \end{cases}$$

and $f''(x) = f(x) - f'(x)$, and observe that the estimates

$$|\{x\colon g_{K_a,H_a}(f', x) > \tfrac{1}{2}\}| \leqslant C\|f\|_p^p \tag{10}$$

and

$$|\{x\colon g_{K_a,H_a}(f'', x) > \tfrac{1}{2}\}| \leqslant C\|f\|_p^p \tag{11}$$

imply (9).

To see (10), note that, by definition, $|f'(x)| \leqslant C'$ for almost all x, where C' is independent of f. This together with the fact that $\|f'\|_p \leqslant \|f\|_p$ imply that f' is in L^2 and $\|f'\|_2^2 \leqslant C\|f\|_p^p$. Hence, applying (8), we have

$$|\{x:\ g_{K_a,H_a}(f',x) > \tfrac{1}{2}\}| \leqslant 4\|g_{K_a,H_a}(f',x)\|_2^2 \leqslant C\|f\|_p^p,$$

which proves (10).

Let $B_j = B(x_j,\ 4r(x_j))$, $f_j(x) = f''(x)\chi_{V_j}(x)$, and write $K_{t^a}*f''(x) = \sum_{j=1}^{\infty} K_{t^a}*f_j(x) = A_1(x)+A_2(x)$, where

$$A_1(x,t) = \sum_{j=1}^{\infty}[K_{t^a}*f_j(x)]\chi_{B_j^c}(x), \qquad A_2(x,t) = \sum_{j=1}^{\infty}[K_{t^a}*f_j(x)]\chi_{B_j}(x),$$

and χ_E always denotes the characteristic function of the set E. Now, if

$$g_1(x) = \left\{\int_0^{\infty}\int H_{t^a}(x-z)|A_1(z,t)|^2 dz\frac{dt}{t}\right\}^{1/2}$$

and

$$g_2(x) = \left\{\int_0^{\infty}\int H_{t^a}(x-z)|A_2(z,t)|^2 dz\frac{dt}{t}\right\}^{1/2}$$

then

$$g_{K_a,H_a}(f'',x) \leqslant g_1(x)+g_2(x)$$

and it is clear that (11) will follow if we can show

(12) $$|\{x:\ g_1(x) > \tfrac{1}{4}\}| \leqslant C\|f\|_p^p$$

and

(13) $$|\{x:\ g_2(x) > \tfrac{1}{4}\}| \leqslant C\|f\|_p^p.$$

To see (12), write

(14) $$|\{x:\ g_1(x) > \tfrac{1}{4}\}| \leqslant 16\|g_1\|_2^2 = 16\int_0^{\infty}\int |A_1(x,t)|^2 dz\frac{dt}{t}.$$

To obtain an estimate on $|A_1(z,t)|$, recall that $|K(x)| \leqslant h(\varrho_a(x))$, where $h(s)$ is a decreasing function on $\boldsymbol{R}_+$ and $\int h(\varrho_a(x))dx < \infty$, and observe that

(15) $$\chi_{B_j^c}(z)\sup_{y\in V_j}|K_{t^a}(z-y)| \leqslant \frac{1}{|V_j|}\int_{V_j} t^{-\operatorname{tr}a}h\left(\frac{1}{2}\varrho_a\big(t^{-a}(z-y)\big)\right)dy.$$

Also note that

(16) $$\int_{V_j}|f''(y)|dy \leqslant C|V_j|,$$

where C is a constant depending only on a. Using (15) and (16), we have

(17) $$\begin{aligned}|A_1(z,t)| &\leqslant \sum_{j=1}^{\infty}\chi_{B_j^c}(z)|K_{t^a}*f_j(z)| \\ &\leqslant \sum_{j=1}^{\infty}\chi_{B_j^c}(z)\sup_{y\in V_j}|K_{t^a}(z-y)|\int_{V_j}|f''(y)|dy \\ &\leqslant C\int h(\varrho_a(x))dx.\end{aligned}$$

To obtain an estimate on $\int_0^{\infty}\int|A_1(z,t)|^2dz\frac{dt}{t}$, use (17) and write

(18) $$\int_0^{\infty}\int|A_1(z,t)|^2dz\frac{dt}{t} \leqslant C\sum_{j=1}^{\infty}\int_0^{\infty}\int|K_{t^a}*f_j(z)\chi_{B_j^c}(z)|dz\frac{dt}{t}.$$

Since K satisfies the hypothesis of Theorem 7, it is easy to see that

$$\int_0^{\infty}\int|K_{t^a}(z-y)-K_{t^a}(z-x_j)|\chi_{B_j^c}(z)dz\frac{dt}{t} \leqslant C \quad \text{for any } y\in V_j,$$

where x_j is the center B_j. Hence, using the fact that $\int_{V_j}f''(y)dy = 0$ and (16) we have

(19) $$\begin{aligned}&\int_0^{\infty}\int|K_{t^a}*f_j(z)\chi_{B_j^c}(z)|dz \\ &= \int_0^{\infty}\int\left|\int_{V_j}[K_{t^a}(z-y)-K_{t^a}(z-x_j)]f''(y)dy\right|\chi_{B_j^c}(z)dz\frac{dt}{t} \\ &\leqslant \int_{V_j}|f(y)|\int_0^{\infty}\int|K_{t^a}(z-y)-K_{t^a}(z-x_j)|\chi_{B_j^c}(z)dz\frac{dt}{t}dy \\ &\leqslant C|V_j|.\end{aligned}$$

Slipping (19) into (18) results in

(20) $$\int_0^{\infty}\int|A_1(z,t)|^2dz\frac{dt}{t} \leqslant C'\sum_{j=1}^{\infty}|V_j| = C'|\Omega| \leqslant C\|f\|_p^p.$$

Estimate (12) now follows from (14) and (20).

To complete the proof of the lemma, it remains to show (13). Since $|\Omega| < C\|f\|_p^p$, (13) will follow from

(21) $$|\{x\in\Omega^c:\ g_2(x) > \tfrac{1}{4}\}| \leqslant C\|f\|_p^p.$$

Recall that $g_2(x) = \left\{\int_0^\infty \int H_{t^a}(x-z)|A_2(z,t)|^2 dz \frac{dt}{t}\right\}^{1/2}$ and observe that $A_2(z,t) = \sum_{j=1}^\infty [K_{t^a} * f_j(z)]\chi_{B_j}(z)$ is identically zero for $z \in \Omega^c$, since $B_j \subset \Omega$ for all j. It follows that

$$[g_2(x)]^2 = \sum_{m=1}^\infty \int_0^\infty \int \chi_{V_m}(z) H_{t^a}(x-z)|A_2(z,t)|^2 dz \frac{dt}{t}.$$

Now, if x_m is the center of B_m and x is in Ω^c, then for any z in V_m, $\varrho_a(x-z) \geqslant \frac{3}{4}\varrho(x-x_m)$ and hence, since $H(x) \leqslant C\varrho_a(x)^{-\mathrm{tr}\,a-\lambda}$, $H_{t^a}(x-z) \leqslant Ct^\lambda \varrho_a(x-x_m)^{-\mathrm{tr}\,a-\lambda}$. Using this fact together with the last formula for $g_2(x)$ gives us

(22) $$[g_2(x)]^2 \leqslant \sum_{m=1}^\infty \varrho_a(x-x_m)^{-\mathrm{tr}\,a-\lambda} \int_0^\infty \int t^\lambda \chi_{V_m}(x)|A_2(z,t)|^2 dz \frac{dt}{t},$$

for all $x \in \Omega^c$.

To obtain an estimate on $\int_0^\infty \int t^\lambda \chi_{V_m}(z)|A_2(z,t)|^2 dz \frac{dt}{t}$, first observe that the integrand can be non-zero only if z is contained in $V_m \cap B_j$ for some j. Now recall that $B(x_j, r(x_j)) \subset V_j \subset B(x_j, 2r(x_j))$ and $B_j = B(x_j, 4r(x_j))$ where $r(x_j) = \frac{1}{8}\inf_{x\in\Omega^c} \varrho_a(x_j - x)$. It is easy to see that if $B_j \cap V_m \neq \emptyset$ then $B_j \subset B'_m$ where $B'_m = B(x_m, 22r(x_m))$ and if N denotes the number of B_j's whose intersection with V_m is not empty then $N \leqslant (44)^{\mathrm{tr}\,a}$. Hence if $\sum_{j\sim m}$ denotes the sum of those j's where $B_j \cap V_m \neq \emptyset$, then $\chi_{V_m}(z)|A_2(z,t)|^2 \leqslant N \sum_{j\sim m} |K_{t^a} * f_j(z)|^2$. Using the last inequality, we have

(23) $$\int_0^\infty \int t^\lambda \chi_{V_m}(z)|A_2(z,t)|^2 dz \frac{dt}{t} \leqslant N \sum_{j\sim m} \int_0^\infty \int t^\lambda |K_{t^a} * f_j(z)|^2 dz \frac{dt}{t}.$$

Now, using Plancherel, Fubini, and the fact that $\hat{K}(\xi) \leqslant C(1+\varrho_{a^*}(\xi))^{-\gamma}$ for $\gamma > \lambda/2$, write

(24) $$\int_0^\infty \int t^\lambda |K_{t^a} * f_j(z)|^2 dz \frac{dt}{t} = \int |\hat{f}_j(\xi)|^2 \int_0^\infty t^\lambda |\hat{K}(t^a\xi)|^2 \frac{dt}{t} d\xi$$
$$\leqslant C \int |[\varrho_{a^*}(\xi)]^{-\lambda/2} \hat{f}_j(\xi)|^2 d\xi$$
$$= C\|R^{\lambda/2} * f_j\|_2^2,$$

where $R^{\lambda/2}$ is the inverse Fourier transform of $\varrho_{a^*}(\xi)^{-\lambda/2}$. Applying Proposition 2 and Theorem 4, we see that the transformation $f \to R^{\lambda/2} * f$ maps L^p, $p = \frac{2\mathrm{tr}\,a}{\lambda + \mathrm{tr}\,a}$, boundedly into L^2. In view of this fact, (24) becomes

(25) $$\int_0^\infty \int t^\lambda |K_{t^a} * f_j(z)|^2 dz \frac{dt}{t} \leqslant C\|f_j\|_p^2.$$

Now observe that

(26) $$\sum_{j\sim m} \|f_j\|_p^2 = \sum_{j\sim m} \left\{\int_{V_j} |f''(\xi)|^p dz\right\}^{2/p} \leqslant \sum_{j\sim m} C|V_j|^{2/p}$$
$$\leqslant CN|B'_m|^{2/p} \leqslant C'|V_m|^{2/p}.$$

Putting estimates (25) and (26) into (23) and then slipping the result into (22) gives us

(27) $$[g_2(x)]^2 \leqslant C \sum_{m=1}^\infty |V_m|^{2/p} \varrho_a(x-x_m)^{-\mathrm{tr}\,a-\lambda}, \quad \text{for} \quad x \in \Omega^c.$$

Now that we have (27), simply write

$$|\{x \in \Omega^c \colon g_2(x) > \tfrac{1}{4}\}| \leqslant 16 \int_{\Omega^c} [g_2(x)]^2 dx$$
$$\leqslant C' \sum_{m=1}^\infty |V_m|^{2/p} \int_{V_m^c} \varrho_a(x-x_m)^{-\mathrm{tr}\,a-\lambda} dx$$
$$\leqslant C'' \sum_{m=1}^\infty |V_m| = C\|f\|_p^p$$

which is the desired result. ∎

Note that the assumption that a is good in Theorems 7, 8, and 9 was used only to take advantage of the fact that in this case the function $(x,z) \to \varrho_a(x-z)$ is a metric. A simple change of variables shows that it suffices to take a to be reasonable.

3. Examples. Suppose a is reasonable and φ is a function in $C_0^\infty(\boldsymbol{R}_+)$ with support in $[1/2, 2]$, write $\Phi(x) = \int \varphi(\varrho_{a^*}(\xi)) e^{i\langle x,\xi\rangle} d\xi$ and $\Phi'(x) = \int |\varphi(\varrho_{a^*}(\xi))|^2 e^{i\langle x,\xi\rangle} d\xi$. Clearly Φ and Φ' both satisfy the hypothesis of Theorem 7 and its corollary and hence, if φ is not identically zero, we have

(28) $$C_p^{-1}\|f\|_p \leqslant \|g^{\Phi'a}(f)\|_p \leqslant C_p\|f\|_p$$

for $1 < p < \infty$ and an analogous inequality for $g_{\Phi_a}(f)$. Now let m be the least integer greater than $n/2$ and write $\Psi(x) = (|x|^m+1)^{-2}$. It is clear from Theorems 3 and 8 that

$$\|g_{\Phi_a, \Psi_a}(f)\|_p \leqslant C_p \|f\|_p \tag{29}$$

for $2 \leqslant p < \infty$.

PROPOSITION 4. *Suppose h is in L^∞ with $\|h\|_\infty \leqslant B$ and*

(i) $$\int_{\frac{1}{2} \leqslant \varrho_{a^*}(\xi) \leqslant 2} |D_j^{\gamma_j} h_t(\xi)|^2 d\xi \leqslant B^2$$

for all $t > 0$ and integers γ_j, $0 \leqslant \gamma_j \leqslant m$, $j = 1, \ldots, n$, where $h_t(\xi) = h(t^{a^*}\xi)$ and B is a positive constant, then for any f in $\mathcal{S}$,

$$g_{\Phi'_a}(T_h f, x) \leqslant C g_{\Phi_a, \Psi_a}(f, x)$$

where $T_h f$ is defined by $\widehat{T_h f}(\xi) = h(\xi)\hat{f}(\xi)$ and C is a constant depending only on φ and B.

Proof. Observe that

$$\Phi'_{t^a} * T_h f(x) = \int \Phi_h(z, t)\, \Phi_{t^a} * f(x-z)\, dz \tag{30}$$

where $\Phi_h(z, t) = \int h(\xi)\varphi\big(\varrho_{a^*}(t^{a^*}\xi)\big) e^{i\langle z, \xi\rangle} d\xi$.

Now break up the integration in (30) into an integral over the set $\{z\colon |t^{-a}z| < 1\}$ and its complement and write

$$\begin{aligned} |\Phi'_{t^a} * T_h f(x)|^2 \leqslant 2 \Big\{ & t^{\mathrm{tr}a} \int_{|t^{-a}z|<1} |\Phi_h(z, t)\, \Phi_{t^a} * f(x-z)|^2 dz + \\ & + C \int \big| |t^{-a}z|^m \Phi_h(z, t)\big|^2 dz \int_{|t^{-a}z| \geqslant 1} |t^{-a}z|^{-2m} |\Phi_{t^a} * f(x-z)|^2 dz \Big\}. \end{aligned} \tag{31}$$

Note that

$$|\Phi_h(z, t)| \leqslant \|h\|_\infty \int |\varphi\big(\varrho_{a^*}(t^{a^*}\xi)\big)|\, d\xi \leqslant CBt^{-\mathrm{tr}a}$$

and

$$\begin{aligned} \int \big| |t^{-a}z|^m \Phi_h(z, t)\big|^2 dz &= t^{\mathrm{tr}a} \int \big| |z|^m \Phi_h(t^a z, t)\big|^2 dz \\ &\leqslant C'' t^{\mathrm{tr}a} \int \Big| \sum_{j=1}^n |z_j|^m \Phi_h(t^a z, t)\Big|^2 dz \\ &\leqslant C' t^{\mathrm{tr}a} \sum_{j=1}^n \int \big| t^{-\mathrm{tr}a} D_j^m \big[h(t^{-a^*}\xi)\, \Phi\big(\varrho_a^*(\xi)\big)\big]\big|^2 d\xi \\ &\leqslant CB^2 t^{-\mathrm{tr}a}. \end{aligned}$$

Slipping the last two estimates into (31) gives us

$$\int_0^\infty |\Phi'_{t^a} * T_h f(x)|^2 \frac{dt}{t} \leqslant C \int_0^\infty \int t^{-\mathrm{tr}a} (|t^{-a}z|^{-2m} + 1) |\Phi_{t^a} * f(x-z)|^2 dz \frac{dt}{t},$$

which is the desired result. ■

As an immediate corollary, we see that if h satisfies the hypothesis of Proposition 2 then the transformation $f \to T_h f$ maps L^p boundedly into L^p for $1 < p < \infty$.

Let P^j, $j = 0, 1, \ldots, n$, be the L^1 functions whose Fourier transform is given by $\widehat{P^0}(\xi) = |\xi| e^{-|\xi|}$, $\widehat{P^j}(\xi) = \xi_j e^{-|\xi|}$, $j = 1, \ldots, n$, and for $\lambda > 0$, set $H^\lambda(x) = (1+|x|)^{-n-\lambda}$. The Littlewood–Paley function introduced by Stein is simply $g_\lambda^*(f, x) = \{\sum_{j=0}^n [g_{P_I^j, H_I^\lambda}(f, x)]^2\}^{1/2}$.

Let Γ^0 be the function whose Fourier transform is given by

$$\widehat{\Gamma^0}(\xi) = \Big(\sum_{j=1}^n |\xi_j|^2 + i\xi_n\Big)^{1/2} \exp\Big\{-\Big(\sum_{j=1}^n |\xi_j|^2 + i\xi_n\Big)^{1/2}\Big\}$$

where the principal determination of the square root is indicated. If a is the linear transformation given by $ax = (x_1, \ldots, x_n, 2x_n)$ and $H^\lambda(x) = \big(1+\varrho_a(x)\big)^{-\mathrm{tr}a-\lambda}$, then $g_{\Gamma_a^0}(f, x)$ and $g_{\Gamma_a^0, H_a^\lambda}(f, x)$ are the Littlewood–Paley functions studied by Jones [5] and Segovia and Wheeden [8].

More generally, suppose a is symmetric and let K be a function whose Fourier transform is given by

$$\hat{K}(\xi) = h(\xi) e^{-\varrho_a(\xi)} \tag{32}$$

where h is continuous on $\boldsymbol{R}^n$, in $C^\infty(\boldsymbol{R}^n \setminus \{0\})$, and quasi-homogeneous of degree $\alpha > 0$ with respect to a. The following proposition together with the mean value theorem shows that K satisfies the hypothesis of Theorem 7.

PROPOSITION 5. *If $\hat{K}$ satisfies* (32) *then* $K(x) \leqslant C\big(1+\varrho_a(x)\big)^{-\mathrm{tr}a-\alpha}$.

Proof. Without loss of generality we can assume that a is diagonal (i. e. $ax = (a_1x_1, \ldots, a_nx_n)$). Since K is bounded, it suffices to obtain an estimate for $\varrho_a(x) > 1$. Write

$$\begin{aligned} K(x) &= (2\pi)^{-n/2} \int h(\xi) e^{-\varrho_a(\xi)} e^{i\langle x, \xi\rangle} d\xi \\ &= (2\pi)^{-n/2} \varrho(x)^{-\mathrm{tr}a-\alpha} \int h(\xi) e^{-\varrho_a(\xi)/\varrho_a(x)} e^{i\langle x', \xi\rangle} d\xi, \end{aligned}$$

where $x' = [\varrho_a(x)]^{-a} x$ is on the unit sphere. The last formula implies that it is enough to show that the inverse Fourier transform of $h(\xi)e^{-\varrho_a(\xi)/r}$ is bounded on the unit sphere independent of $r > 1$. To see this, let $\varphi(\xi)$

be a function in $C_0^\infty(\boldsymbol{R}^n)$ such that $\varphi(\xi) = 1$ for $\varrho_a(\xi) \leqslant 1$, $\varphi(\xi) = 0$ for $\varrho_a(\xi) \geqslant 2$ and observe that the inverse Fourier transform of $f(\xi, r) = h(\xi) e^{-\varrho_a(\xi)/r}$ is equal to $\check{f}_1(x, r) + \check{f}_2(x, r)$ where $f_1(\xi, r) = f(\xi, r)\varphi(\xi)$ and $f_2(\xi, r) = f(\xi, r)\big(1-\varphi(\xi)\big)$. Clearly $\check{f}_1(x, r)$ is bounded independent of x and r. To obtain an estimate on $\check{f}_2(x, r)$, observe that

$$D_j^m[f_2(\xi, r)] = \Big\{\sum_{k=0}^{m} h_{a-ma_j+k}(\xi) r^{-k} e^{-\varrho_a(\xi)/r}\Big\}\big(1+\varphi(\xi)\big) + u(\xi, r)$$

where h_{a-ma_j+k} is in $C^\infty(\boldsymbol{R}^n \setminus \{0\})$ and quasi-homogeneous of degree $a - ma_j + k$ with respect to a and $u(\xi, r)$ is supported in $1 \leqslant \varrho_a(\xi) \leqslant 2$ and bounded independent of $r > 1$. From this we see that

$$|D_j^m[f_2(\xi, r)]| \leqslant C[\varrho_a(\xi)]^{a-ma_j}\big(1-\varphi(\xi)\big), \qquad r > 1,$$

and hence, if m is large enough, $\int |D_j^m[f_2(\xi, r)]|\, d\xi$ is bounded independent of $r > 1$. It follows that for sufficiently large m, $|x|^m \check{f}_2(x, r)$ is bounded independent of x and $r > 1$ and we conclude that $\check{f}_2(x, r)$ is bounded independent of $r > 1$ for $|x| = 1$. ∎

Littlewood–Paley functions constructed with kernels of type (32) have applications analogous to that of g_λ^*. Suppose a is diagonal and K^j, $j = 0, 1, \ldots, n$, is defined by $\widehat{K^0}(\xi) = \varrho_a(\xi) e^{-\varrho_a(\xi)}$ and $\widehat{K^j}(\xi) = \xi_j e^{-\varrho_a(\xi)}$, $j = 1, \ldots, n$. For $\lambda > 0$ set $H^\lambda(x) = \big(1+\varrho_a(x)\big)^{-\mathrm{tr}a-\lambda}$, for $\alpha > 0$ define the transformation $f \to f_\alpha$ by $\hat{f}_\alpha(\xi) = \varrho_a(\xi)^{-\alpha}\hat{f}(\xi)$. And for $m = 1, 2, \ldots$ define $\Delta_z^m f(x) = \Delta_z[\Delta_z^{m-1} f(x)]$ where $\Delta_z f(x) = f(x-z) - f(x)$. Finally for $0 < \alpha < \mathrm{tr}\, a$ and $\dfrac{\alpha}{a_0} < m$, a_0 = least eigenvalue of a, define $\mathscr{D}_\alpha^m(f, x)$ for f in $\mathscr{S}$ by

$$\mathscr{D}_\alpha^m(f, x) = \Big\{\int |\Delta_z^m f_\alpha(x)|^2 \varrho_a(z)^{-\mathrm{tr}a-2\alpha}\, dz\Big\}^{1/2}.$$

As in [10], it is not difficult to see that

$$\mathscr{D}_\alpha^m(f, x) \leqslant C_{\lambda,\alpha}\, g_{\alpha,\lambda}^*(f, x) \quad \text{for } 0 < \lambda < 2\alpha \tag{33}$$

where

$$g_{\alpha,\lambda}^*(f, x) = \Big\{\sum_{j=0}^{n} [g_{K_\alpha^j, H_\alpha^\lambda}(f, x)]^2\Big\}^{1/2}.$$

Estimate (33) together with Theorems 5 and 9, imply a characterization of the quasi-homogeneous Lebesgue spaces introduced by Sadosky and Cotlar [7]. The details will appear elsewhere.

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