On Littlewood–Paley functions

by

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Abstract. We define and study a generalization of the Littlewood–Paley function, $g^*_1$, by using properties of certain approximations of the identity. $L^p$ estimates are obtained and some examples and applications are given.

Introduction. If $f$ is a reasonable function on $\mathbb{R}^n$, the Littlewood–Paley function, $g_1^*(f)$, is defined for $\lambda > 0$ by

$$
g_1^*(f, \omega) = \left( \int_{\mathbb{R}^n} \int_0^\infty \frac{\omega^{\lambda+1}}{(|x|+y)^{n+1}} \left| \text{grad} P_y \ast f(x-s) \right|^2 dy dx \right)^{1/2}
$$

where $P_y(s)$ is the Poisson kernel for the upper half space.

Stein introduced $g_1^*$ in [9], where he showed that the transformation $f \to g_1^*(f)$ is bounded on $L^p(\mathbb{R}^n)$ for $\max \{1, \frac{2n}{\lambda+n} \} < p < \infty$. In [10], he used $g_1^*$ to obtain a characterization of the Lebesgue spaces, $L_p^n$, and in [11], showed how $g_1^*$ and its variants can be used to obtain Hörmander’s version of Mikhlin’s multiplier theorem.

Subsequently, Segovia and Wheeden, [8], introduced a Littlewood–Paley function, which we call $g_1^{**}(f)$, and which is defined for $\lambda > 0$ by

$$
g_1^{**}(f, \omega) = \left( \int_{\mathbb{R}^n} \int_0^\infty \frac{\omega^{\lambda+1}}{(\varphi(x)+y)^{n+1+\lambda}} \left| \frac{\partial}{\partial y} P_y \ast f(x-s) \right|^2 dy dx \right)^{1/2}
$$

where $\varphi(x) = \sum_{i=1}^{n-1} |x_i| + |x_n|^{n/2}$ and $P_y(x)$ is the Poisson type kernel for the upper half space associated with the heat equation $\sum_{\nu=1}^n \nu \sigma_{\nu} y_{\nu} - u_x = 0$. They showed that the transformation $f \to g_1^{**}(f)$ is bounded on $L^p(\mathbb{R}^n)$ for $\max \{1, \frac{2n+2}{\lambda+n+1} \} < p < \infty$.

It is the purpose of this paper to show that Littlewood–Paley functions with similar properties can be defined without recourse to Laplace or heat equations.
Chapter I
PRELIMINARIES

1. Notation and conventions. \( \mathbb{R} \) is the real line whose elements are denoted by \( r, s, t \) and \( \mathbb{R}_+ = \{ t \in \mathbb{R} : t > 0 \} \).

\( \mathbb{R}^n \) is an \( n \)-dimensional real Euclidean space whose elements are denoted by \( x, y, z \) and \( f \).

If \( \Omega \) is a subset of \( \mathbb{R}^n \), \( \overline{\Omega} \) and \( \partial \Omega \) denote the closure and complement of \( \Omega \) in \( \mathbb{R}^n \) respectively. If \( \Omega \) is measurable, \( |\Omega| \) denotes the Lebesgue measure of \( \Omega \).

If \( x \) and \( z \) are elements of \( \mathbb{R}^n \), \( \langle x, z \rangle = \sum_{j=1}^{n} x_j z_j \) and \( |z| = \sqrt{\langle z, z \rangle} \).

The symbols \( a \) and \( b \) will always be used to denote linear transformations on \( \mathbb{R}^n \). \( |a| = \sup \left\{ \frac{|ax_\ell|}{|x_\ell|} : x \in \mathbb{R}^n \right\} \) and \( a^* \) denotes the adjoint of \( a \).

\( I \) denotes the identity matrix.

\( D_j \) denotes the differential operator \( \frac{\partial}{\partial x_j} \), \( j = 1, \ldots, n \).

All functions considered by us are complex valued and all integrals are over \( \mathbb{R}^n \), unless denoted otherwise.

If \( f \) is a measurable function on \( \mathbb{R}^n \), then

\[ \|f\|_p = \left( \int |f(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty, \]

\[ \|f\|_\infty = \text{ess. sup.} \, |f(x)|, \]

and \( L^p(\mathbb{R}^n) = L^p \), is the usual Banach space of functions for which \( \|f\|_p \) is finite.

\( p' \) always denotes the H"older conjugate of \( p \) for \( 1 \leq p \leq \infty \), namely

\[ \frac{1}{p} + \frac{1}{p'} = 1. \]

If \( \Omega \) is an open subset of some Euclidean space, \( C^\infty(\Omega) \) is the set of infinitely differentiable functions on \( \Omega \) and \( C^\infty_0(\Omega) \) is the subset of \( C^\infty(\Omega) \) consisting of functions with compact support. In the case \( \Omega = \mathbb{R}^n \), we simply write \( C^\infty \) and \( C^\infty_0(\mathbb{R}^n) \) instead of \( C^\infty(\mathbb{R}^n) \) and \( C^\infty_0(\mathbb{R}^n) \).

The subspace of \( C^\infty \) consisting of functions which together with all their derivatives tend to zero at infinity faster than any rational function is denoted by \( \mathcal{S} \) and is given the usual topology. Its dual, the space of tempered distributions, is denoted by \( \mathcal{S}' \) and is also given the usual topology. \((T, f)\) denotes the distribution \( T \) acting on \( f \in \mathcal{S} \).

We say that a distribution \( T \) is in \( C^\infty(\mathbb{R}^n \setminus \{0\}) \) (or \( \mathcal{S}(\mathbb{R}^n \setminus \{0\}) \)) if the distribution \((1 - \varphi)T \) is in \( C^\infty \) (or \( \mathcal{S} \)) for every \( \varphi \in C^\infty_0 \) which is identically one in a neighborhood of the origin.

The Fourier transform of \( f \in \mathcal{S} \) is defined by

\[ \mathcal{F}f(\xi) = \hat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x} \, dx, \]

and the inverse Fourier transform of \( f \) is defined by

\[ \mathcal{F}^{-1}f(\xi) = \hat{f}(\xi) = \int f(x) e^{2\pi i \xi \cdot x} \, dx. \]

Plancherel's formula thus reads:

\[ \|f\|_2 = \|\hat{f}\|_2 = \|f\|_2. \]

The Fourier transform is defined on \( \mathcal{S}' \) in the usual manner.

All Fourier transforms and differentiations are to be interpreted in the \( \mathcal{S}' \) sense, unless they make sense otherwise.

The convolution of two functions, \( f \) and \( g \), defined on \( \mathbb{R}^n \) is \( f \ast g \) where

\[ f \ast g (x) = \int f(x - y) g(y) \, dy, \]

whenever this operation makes sense.

The symbol \( C \) will be used generically for constants appearing in certain estimates. Sometimes it will be subscripted to denote the parameters it depends on. At other times the parameters \( C \) depends on will be clear from the proof of the estimate. It need not be the same at different occurrences.

The end of a proof will always be signalled by \( \blacksquare \).

2. Quasi-homogeneous metrics, functions, and distributions. Let \( a \) be a linear transformation on \( \mathbb{R}^n \), and consider the one parameter group \( t^a = e^{-ita}, \quad t > 0 \).

We say that \( a \) is good if \( |t^a| \leq 1 \) for \( 0 < t \leq 1 \). If there is an \( \varepsilon > 0 \) such that \( |t^a| \leq 1 \) for \( 0 < t \leq 1 \), we call \( a \) reasonable. Observe that if \( a \) is reasonable, then there is a positive constant, \( b \), such that \( ba \) is good. Also note that if \( a \) is reasonable or good, then \( a^* \) also has that property.

Suppose \( a \) is reasonable and consider the function \( P(x, t) = |t^{-a}x|, \quad t > 0 \).

It is clear that, for fixed \( x \neq 0 \), \( P(x, t) \) is a strictly decreasing continuous function of \( t \), \( \lim_{t \to 0} P(x, t) = \infty \), and \( \lim_{t \to \infty} P(x, t) = 0 \). It follows that \( P(x, t) = 1 \) has a unique solution which we call \( \phi_a(x) \).

Having defined \( \phi_a(x) \) for \( x \neq 0 \), set \( \phi_a(0) = 0 \). The following proposition, whose proof
follows immediately from the definitions, gives us some important properties of the function \( q \).

**Proposition 1.**

(i) \( q(x) \in C^\omega(\mathbb{R}^n \setminus \{0\}) \).
(ii) \( \delta q(t^a x) = t^a \delta q(x) \), \( t > 0 \).
(iii) \( \delta q(x) = 1 \) if and only if \( |x| = 1 \).
(iv) If \( a \) is good, then \( \delta q(x + z) \rightarrow \delta q(x) \) as \( z \rightarrow 0 \).

We call \( \delta q \) the quasi-homogeneous "metric" with respect to \( a \). Note that \( \delta q \) of the above proposition implies that if \( a \) is good then the function \( (x, z) \rightarrow \delta q(x - z) \) is indeed a metric on \( \mathbb{R}^n \).

The above considerations suggest the polar change of variables

\[
  x = r^a z,
\]

where \( r = \delta q(x) > 0 \) and \( z = \delta q(x)^{-a} x \in S^{n-1} \). Writing \( z \) in terms of the usual polar angles of \( \mathbb{R}^n \) and \( \theta_n \), \( \theta_{n-1} \), and computing the Jacobian of (1), we get \( dq = r^{n-1} dr \, d\theta \), where \( r = trc \) is the radial component of \( r \). Therefore, \( \delta q \) is a measure on \( S^{n-1} \). For more details concerning the metric and the polar change of variables see [2] or [6].

Throughout the rest of the section we always assume to be reasonable.

The following formulas can be easily verified using the above change of variables:

\[
  \int_{S^{n-1}} \, d\theta = \frac{n}{\delta q(x)} t^{a-1} \, dt,
\]

where \( \delta q(x) \) is the area of \( S^{n-1} \).

If \( f \) is a measurable function on \( \mathbb{R}^n \) such that for some constant \( C \) and \( m \), \( f(x) \leq C \delta q(x)^m \), then for any \( \varphi \in C^\omega \) such that \( \varphi \equiv 1 \) in a neighborhood of the origin \( \mathbb{R}^n \) is integrable if \( m > -a \) and \( (1 - \varphi(x)) f(x) \) is integrable if \( m < -a \). Also note that if \( a \) is diagonal (i.e. \( a = (a_1, \ldots, a_{n-1}) \) then there is a constant \( C > 0 \) such that

\[
  C^{-1} \delta q(x) \leq \sum_{j=1}^n |a_j|^{1/2} \delta q(x) \leq C \delta q(x)
\]

for all \( x \in \mathbb{R}^n \).

A function \( f \) defined on \( \mathbb{R}^n \) is said to be quasi-homogeneous of degree \( k \), \( k \in \mathbb{R} \), with respect to \( a \) if for every \( t > 0 \) the formula \( f(t^ax) = t^{ka} f(x) \) holds for every \( x \neq 0 \).

The notion of quasi-homogeneity for tempered distributions is analogous to that for functions. For \( t > 0 \), define the operator \( \delta q^a \) acting on \( \varphi \in \mathscr{S} \) by the formula \( \delta q^a \varphi(x) = \varphi(t^{a} x) \). For \( T \in \mathscr{S} \) define the operator \( \delta q^a \)

acting on \( T \) by the formula

\[
  \langle \delta q^a T, \varphi \rangle = \langle T, t^{-a} \delta q^a \varphi \rangle.
\]

It is clear that \( \delta q^a T \) is in \( \mathscr{S} \). Define \( \mathcal{T} \) to be quasi-homogeneous of degree \( k \) with respect to \( a \) if for every \( t > 0 \), \( \delta q^a T = t^{ka} T \).

A simple computation shows that for \( T \in \mathscr{S} \), \( \delta q^a T = t^{-ka} \delta q^a \mathcal{T} \). Now, if \( T \) is quasi-homogeneous of degree \( k \) with respect to \( a \), applying the last formula results in \( \delta q^a \mathcal{T} = t^{-ka} \mathcal{T} \) and hence \( \mathcal{T} \) is quasi-homogeneous of degree \( -ka - k \) with respect to \( a^* \).

We conclude this section with

**Proposition 2.** If the tempered distribution \( T \) is locally integrable, in \( C^\omega(\mathbb{R}^n \setminus \{0\}) \), and quasi-homogeneous of degree \( k \) with respect to \( a \), where \( -ka < k < 0 \), then \( \mathcal{T} \) is locally integrable, in \( C^\omega(\mathbb{R}^n \setminus \{0\}) \), and quasi-homogeneous of degree \( -ka - k \) with respect to \( a^* \).

**Proof.** Let \( \varphi \in C^\omega(\mathbb{R}^n) \) such that \( \int_0^\infty \varphi(t) \frac{dt}{t} = 1 \). Write \( g(\mu) = T(\mu) \varphi(\delta q(a)) \). Clearly \( g \in \mathscr{S} \),

\[
  T(\mu) = \int_0^\infty t^{-k} g(t^a \mu) \frac{dt}{t},
\]

\[
  \mathcal{T}(\mu) = \int_0^\infty t^{-k} g(t^{-a} \mu) \frac{dt}{t},
\]

and the conclusion of the proposition follows.

**3. Vitali families, maximal functions, and "quasi-homogeneous like" kernels.** Let \( (U_n, s > 0) \) be a family of open subsets of \( \mathbb{R}^n \) whose closure is compact.

**Definition.** \( (U_n, s > 0) \) is a Vitali family with constant \( A \) if and only if

(i) for \( s_1 < s_2 \), \( U_{s_1} \subset U_{s_2} \) and \( \bigcup_{s > 0} U_s = \{0\} \),

(ii) \( |U_s - U_{-s}| \leq A |U_s| \) for all \( s \), where \( U_s - U_{-s} \) denotes \( \{x : x = y - z \) where \( y \) and \( z \) are both in \( U_s \} \),

(iii) \( U_s \) is a left continuous function of \( s \).

**Theorem 1.** Suppose \( \Omega \) is a measurable set in \( \mathbb{R}^n \) and let \( x \rightarrow (x) \) be a mapping of \( \Omega \) into \( \mathbb{R}^n \) satisfying:

(i) \( (r(x)) \) is bounded and for every \( r_0 > 0 \) the set \( \{x : x \in \Omega, (x) > r_0 \} \) is a bounded subset of \( \mathbb{R}^n \).

(ii) If \( (x) \) is a sequence which converges to \( x_0 \) and \( (x_0) \rightarrow r_0 \) then \( x_0 \in \Omega \) and \( (x_0) \geq r_0 \).
If \( \{U_t, t > 0\} \) is a Vitali family with constant \( A \), then there exists a sequence \( \{x_0\} \subset \Omega \) such that

1) \( (x_0 + U_{t_0}) \) is disjoint,
2) \( \Omega = \bigcup_{t \in \mathbb{R}} (x_0 + (U_{t_0} - U_{t_0})) \),
3) \( |\Omega| \leq A \sum_{t \in \mathbb{R}} |U_{t_0}| \).

**Theorem 2.** Let \( \{U_t, t > 0\} \) be a Vitali family with constant \( A \). For \( f \in L^p \), define

\[
Mf(x) = \sup_{t > 0} \frac{1}{|U_t|} \int_{U_t} |f(x - z)| \, dz.
\]

Then

(i) \( \|f(z) Mf(z) > 1\| \leq A \|f\|_p \),
(ii) \( \|Mf\|_p \leq C\|f\|_p \) for \( 1 < p \leq \infty \), where \( C \) is a constant depending only on \( A \) and \( p \).

For proofs of the above theorems, see Rivero [8].

In the statements of the next two theorems, we take \( a = 1 \) to be reasonable.

**Theorem 3.** Suppose \( H \in L^1 \) and \( H(tz) = h(ta(t)) \) where \( h(t) \) is a decreasing function on \( R^+ \), and \( |h'(a(t))| \, da \leq C \) constant.

Consider the transformation \( f \rightarrow M_{H^*}f \) where

\[
M_{H^*}f(x) = \sup_{t > 0} \left| \int t^{-n/2} H(t^{-r/2})f(x - z) \, dz \right|.
\]

If \( b \) is reasonable and commutes with \( a \), then

\[
|M_{H^*}f|_p \leq C\|f\|_p \quad \text{for} \quad 1 < p \leq \infty,
\]

where \( C \) is a constant which depends only on \( H, p, \) and \( n \).

**Proof.** Write

\[
\left| \int t^{-n/2} H(t^{-r/2})f(x - z) \, dz \right|
\leq \sum_{b = 0}^{\infty} 2^{b}\|H(2^{a(t)} - 2^{b})\| \left| \int t^{-n/2} f(x - z) \, dz \right|
\leq \sum_{b = 0}^{\infty} 2^{b(2^{a(t)} - 1)} 2^{b t r} \left| \int f(x - z) \, dz \right|
\]

Taking the sup over \( t > 0 \), we get

\[
M_{H^*}f(x) \leq \sum_{b = 0}^{\infty} 2^{b(2^{a(t)} - 1)} 2^{b t r} M_{H}f(x)
\]

where

\[
M_{H}f(x) = \sup_{t > 0} \frac{1}{2^{b t r} 2^{b t r}} \int_{t b^{t r} 2^{b t r}} f(x - z) \, dz.
\]

Let \( \{U_{t}^2, t > 0\} \) be the family of open subsets of \( R^2 \) defined by \( U_{t}^2 \)

\[
= \{z : d(x, y) < 1\}.
\]

Clearly \( \{U_{t}^2, t > 0\} \) is a Vitali family with constant \( 2^2 \). Observing that \( |U_{t}^2| = \frac{2^{2t/2} 2^{t r}}{t r} \) and applying Theorem 2, it follows that \( \|M_{H^*}f\|_p \leq C\|f\|_p \) for \( 1 < p \leq \infty \), where \( C \) is a constant depending only on \( n \) and \( p \). Since this is true for each \( n = 0, 1, 2, \ldots \), we conclude that

\[
\|M_{H^*}f\|_p \leq \sum_{b = 0}^{\infty} 2^{b(2^{a(t)} - 1)} 2^{b t r} \|f\|_p \leq [C \int h(a(t)) \, da] \|f\|_p.
\]

The next theorem is a quasi-homogeneous version of a classical result generally referred to as Sobolev's imbedding theorem. For a proof in the case \( a = I \), see Stein [11].

**Theorem 4.** Suppose \( H \) is locally integrable on \( R^n \) and \( |H(x)| \leq C\|f\|_n \), where \( 0 < a < \text{tr}a \). If \( p \) and \( q \) satisfy \( 1 < p < \frac{\text{tr}a}{a} \) and \( \frac{1}{q} = \frac{1}{p} - \frac{a}{\text{tr}a} \), then for \( f \in L^p \) the transformation \( f \rightarrow H^*f \) is well defined and \( \|H^*f\|_q \leq C\|f\|_p \), where \( C \) depends on \( H, a, p, \) and \( q \).

4. **Distributions whose Fourier transforms are in \( L^p \).**

**Proposition 3.** Let \( \beta_1, \ldots, \beta_n \) be positive integers such that \( \sum_{1}^{n} \frac{1}{\beta_i} < 2 \).

Suppose that \( f \in L^p \) satisfies

(i) \( \|f\|_b \leq B \),
(ii) \( \|H f\|_1 \leq B, \quad j = 1, \ldots, n; \)
then \( f \in L^p \) and \( \|f\|_p \leq CB \), where \( C \) is a constant which depends only on \( \beta_1, \ldots, \beta_n \).

**Proof.** From (i) it is clear that \( f \) is a function in \( L^p \). To compute the \( L^p \) norm of \( f \), let \( b \) be the linear transformation defined by \( b = \left( \frac{1}{\beta_1}, x_1, \ldots, \frac{1}{\beta_n}, x_n \right) \) and write

\[
\left| \int \int f(x) \, dx \right| \leq \left( \int |1 + \varphi_b(x)|^2 \, dx \right)^{1/2} \left( \int |1 + \varphi_b(x)|^{-1} \, dx \right)^{1/2}
\]

where \( \varphi_b(x) \) is the quasi-homogeneous "metric" with respect to \( b \). Recalling
that \( \varphi_k(x) \leq C \sum_{j=1}^{n} |x_j|^{\delta_j} \) and applying Plancherel's formula we have

\[
\left\{ \int |(1 + \varphi_k(x)) f(x)|^p dx \right\}^{1/p} \leq C \left\{ \|f\|_1 + \sum_{j=1}^{n} \|D_j f(x)\|_1 \right\}^{1/p}.
\]

Using the polar change of variables we have

\[
\left\{ \int |(1 + \varphi_k(x)) f(x)|^2 dx \right\}^{1/2} = C \left\{ \sum_{j=1}^{n} (1 + r_j)^{-r_j} \sum_{k=1}^{n} d_k \right\}^{1/2} = C
\]

since \( r_j > 2 \) and \( \beta_j < 2 \). Slipping the last two estimates into (3) gives us the desired result.

**Theorem 5.** Suppose \( a \) is reasonable and let \( \beta_1, \ldots, \beta_n \) be positive integers such that \( \sum_{j=1}^{n} \frac{1}{\beta_j} < 2 \). If \( f \) is in \( L^p \) such that

\[
(1) \quad \int_{\mathbb{R}^n} |D_j f_k(x)|^2 dx \leq B_k^2
\]

for all \( k = 0, \pm 1, \pm 2, \ldots \), and integers \( y_j, 0 \leq y_j \leq \beta_j, j = 1, \ldots, n \), where \( f_k(x) = f(x) \) and the \( B_k \)'s are positive numbers with \( \sum_{k=0}^{n} B_k < \infty \), then \( f \) is in \( L^p \) and the \( L^p \) norm of \( f \) is bounded by a constant which depends only on \( \beta_1, \ldots, \beta_n \) and \( \sum_{k=0}^{n} B_k \).

**Proof.** Let \( \psi \) be a positive function in \( G^\infty(\mathcal{R}) \) with support in \( \{ \mathbb{R} \} \) and such that \( \psi(t) > 0 \) for \( \frac{1}{2} < t < V \). Set \( \varphi(t) = \psi(t) / \sum_{m} \psi(2^{-m} t) \) and \( \varphi(x) = \Phi(x) \). Observe that \( \psi \in G^\infty, \varphi(2^{-m} x) \) has support in \( \{ x : 2^{-m} \leq \varphi(x) \leq 2^{m+1} \} \) and \( \sum_{m} \varphi(2^{-m} x) = 1 \) for \( x \neq 0 \). Write

\[
(2) \quad f(x) = \sum_{m} \varphi(2^{-m} x) f(x) - \sum_{m} \varphi(2^{-m} x) \delta(x).
\]

It is clear that the \( \varphi_k \)'s are in \( L^p \). To obtain an estimate on their \( L^p \) norms, using the same method as in proof of Proposition 3, write

\[
(3) \quad \int |\varphi(x)|^p dx \leq \int \left\{ (1 + \varphi(x)) |\varphi(x)|^p dy \right\}^{1/p} \left\{ (1 + \varphi(x))^{-1} dx \right\}^{1/p} \leq C 2^{-1/p}.
\]

Now, using Plancherel's formula and (1), we have

\[
\left\| \varphi \right\|_p = \left\{ \int |f(\xi)|^p d\xi \right\}^{1/p} \leq 2^{1/p} B_k
\]

and

\[
\left\| \varphi(x) \right\|_p = \left\{ \int |g(\xi)|^p d\xi \right\}^{1/p} \leq C \left\{ \int \left| \varphi(x) \right| f(x) \right\}^{1/p} \leq C 2^{-1/p} \sum_{j=1}^{n} \left\| D_j f_j(x) \right\|_1 \leq C 2^{-1/p} \sum_{j=1}^{n} \left\| D_j f_k(x) \right\|_1 \leq C 2^{-1/p} B_k.
\]

Slipping the last two estimates into (4), we conclude that

\[
(4) \quad \int |\varphi(x)|^p dx \leq \int \left\{ (1 + \varphi(x)) |\varphi(x)|^p dy \right\}^{1/p} \left\{ (1 + \varphi(x))^{-1} dx \right\}^{1/p} \leq C 2^{-1/p}.
\]

Since \( \sum_{k=0}^{n} B_k < \infty \), it follows from (5) that there is an \( F \) in \( L^p \) such that \( \lim_{k} \left\| f_k \right\|_p = 0 \) and whose \( L^p \) norm is bounded by \( C \sum_{k=0}^{n} B_k \). Observe that (3) implies that \( \lim \sum_{k=0}^{n} \left\| f_k \right\|_p = \left\| f \right\|_p \) in \( L^p \), and using the fact that the Fourier transform is continuous on \( L^p \), we conclude that \( \hat{f} = F \).

The following corollary of Theorem 5 can be used for most applications.

**Corollary.** Suppose \( a \) is diagonal (i.e. \( a = (a_1, \ldots, a_n) \)) and \( a_j > 0, j = 1, \ldots, n \), and let \( \beta_1, \ldots, \beta_n \) be as in Theorem 5.

If \( f \) is in \( L^p \) and sufficiently smooth with

\[
\sup_{\mathbb{R}^n} \left| \frac{d^n f(x)}{dx^n} \right| \leq B_k
\]

for \( k = 0, \pm 1, \pm 2, \ldots \), and integers \( y_j, 0 \leq y_j \leq \beta_j, j = 1, \ldots, n \), where the \( B_k \)'s are positive numbers with \( \sum_{k=0}^{n} B_k < \infty \), then \( f \) is in \( L^p \) and \( L^q \) norm of \( f \) depends only on \( \beta_1, \ldots, \beta_n \) and \( \sum_{k=0}^{n} B_k \).

**5. Vector valued singular integrals.** In this section we assume that \( a \) is a good linear transformation on \( \mathbb{R}^n \). In this case, the function \( (x, a) \rightarrow \varphi_k(x, a) = x \) is a metric on \( \mathbb{R}^n \).

If \( \mathcal{H} \) is a Hilbert space, then \( |\cdot|_\mathcal{H} \) denotes the norm of the element \( u \in \mathcal{H} \) and \( L^p(\mathcal{H}) \), \( 1 \leq p < \infty \), denotes the space of strongly measurable
$\mathcal{X}$ valued functions defined on $\mathbb{R}^n$ such that $f(x)|_{\mathcal{X}}$ belongs to $L^p$ with norm $\|f\|_{L^p(\mathcal{X})} = L^p$ norm of $f(x)|_{\mathcal{X}}$. If $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert spaces, then $L^p(\mathcal{X}, \mathcal{Y})$ denotes the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ and if $L$ is in $L^p(\mathcal{X}, \mathcal{Y})$ then $\|L\|_{L^p(\mathcal{X}, \mathcal{Y})} = \sup\{\|Lx\|_{\mathcal{Y}} : x \in \mathcal{X}, \|x\|_{\mathcal{X}} \leq 1\}$.

The following theorem is a generalization of the Calderón– Zygmund inequality and its proof can be found in Rivière [6].

Theorem 6. Let $\mathcal{X}(\delta)$ be a function on $\mathbb{R}^n$ with values in $L^p(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X}$ is measurable and integrable on compact subsets of $\mathbb{R}^n \setminus \{0\}$. Suppose $\mathcal{X}$ has the following properties:

(i) $\lim_{\epsilon \to 0} \int_{\epsilon < \|x\| < \delta} \mathcal{X}(x)dx \|\mathcal{X}(x)\|_{L^p(\mathcal{X})} \leq C_1$, where $C_1$ is independent of $\delta$ and $\epsilon$, $0 < \epsilon < \delta < \infty$, and for each $x \in \mathcal{X}$, $C_1 \mathcal{X}(x)dx$ exists.

(ii) For $x \in \mathcal{X}$, $\int_{\epsilon < \|x\| < \delta} \mathcal{X}(x)dx \|\mathcal{X}(x)\|_{L^p(\mathcal{X})} \leq C_2$, where $C_2$ is independent of $\delta$ and $\epsilon$, $0 < \epsilon < \delta < \infty$, and for each $x \in \mathcal{X}$, $C_2 \mathcal{X}(x)dx$ exists.

(iii) For $x \in \mathcal{X}$, $\int_{\epsilon < \|x\| < \delta} \mathcal{X}(x)dx \|\mathcal{X}(x)\|_{L^p(\mathcal{X})} \leq C_3$, where $C_3$ is independent of $\delta$ and $\epsilon$, $0 < \epsilon < \delta < \infty$, and for each $x \in \mathcal{X}$, $C_3 \mathcal{X}(x)dx$ exists.

Also assume that $\mathcal{X}^*(x)$ enjoys the same properties as $\mathcal{X}(x)$.

Under these conditions, the transformation $f \mapsto \mathcal{X}$ given by $\mathcal{X}(f) = \lim_{\epsilon \to 0} \int \mathcal{X}(x)f(x)dx$ is well defined on $L^p(\mathcal{X})$ to $L^p(\mathcal{X})$ for $1 < p < \infty$ and $\|\mathcal{X}(f)\|_{L^p(\mathcal{X})} \leq C\|f\|_{L^p(\mathcal{X})}$, where $C$ depends only on $C_1$, $C_2$, $C_3$ and $p$.

**CHAPTER II**

**LITTLEWOOD–PALEY FUNCTIONS**

1. $g_{\mathcal{K}}$. Suppose that $K$ is in $L^1$ and $a$ is good; then for $f \in L^p$, $1 \leq p < \infty$, the Littlewood–Paley function $g_{\mathcal{K}}(f)$ is defined by the formula

$$g_{\mathcal{K}}(f, x) = \int_0^\infty |K_{\mathcal{K}}(x)|^2dt \frac{dt}{t^{1/2}}$$

where $K_{\mathcal{K}}(x) = t^{-a}K(tx/a)$.

The following theorem gives some conditions on $K$ which imply that the semi-linear transformation, $f \mapsto g_{\mathcal{K}}(f)$, maps $L^p$ boundedly into $L^p$.

**Theorem 7.** Suppose $K$ has the following properties:

(i) $|K(x)| \leq h(|x|)$ where $h(t)$ is a decreasing function on $R_+$ and $h(t) \leq C_1t^{-\alpha} + |t|^\alpha$ for some $\delta > 0$.

(ii) $K(x)dx = 0$.

(iii) $|K(x) - K(x)|dx \leq C_2|x|^\alpha$ for some $\alpha > 0$.

**Under these conditions it follows that**

$$\|g_{\mathcal{K}}(f)\|_p \leq C\|f\|_p, \quad 1 < p < \infty,$$

where $C$ depends only on $C_1$, $C_2$, $\delta$, $\alpha$, and $p$.

**Proof.** This result is an easy consequence of Theorem 5. To see this, let $\mathcal{X}$ be the complex numbers and let

$$\mathcal{X} = \{x \in \mathbb{C} : \psi \text{ measurable on } R_+ \text{ with } \int \frac{|\psi(t)|^2}{t} \frac{dt}{t} < \infty\}.$$ 

Let $\mathcal{X}(x)$ be the linear operator transforming the complex number $\xi$ into the element $K_{\mathcal{K}}(x)\xi$ of $\mathcal{X}$. Now, if $\mathcal{X}$ satisfies the hypothesis of Theorem 5, observe that $\mathcal{K}_{\mathcal{K}}(f) = \mathcal{X}(f)|_{\mathcal{X}}$ and hence

$$\|g_{\mathcal{K}}(f)\|_p = \|\mathcal{X}(f)\|_{L^p(\mathcal{X})} \leq C\|f\|_{L^p(\mathcal{X})} = C\|f\|_p.$$ 

The calculations showing that $\mathcal{X}(x)$ and $\mathcal{X}^*(x)$ satisfy (i), (ii), and (iii) are analogous to those in Benedek, Calderón, and Panzone [1], so we omit them. 

Observe that if $K$ satisfies the hypothesis of Theorem 6, then

$$\int \frac{|K_{\mathcal{K}}(x)|^2}{t} \frac{dt}{t} < \infty$$

and recall that $K_{\mathcal{K}}(x) = K(tx/a)$. Now, if $K$ is a function which depends only on $a_\theta(x)$ (i.e., $K_{\mathcal{K}} = a_\theta(x)$ where $a_\theta$ is a function on $R_+$), then

$$\int \frac{|K_{\mathcal{K}}(x)|^2}{t} \frac{dt}{t} = \int |\mathcal{K}_{\mathcal{K}}(t)|^2 \frac{dt}{t} = C = \text{constant independent of } \delta.$$ 

In this case, for $f_1$ and $f_2$ in $\mathcal{X}$, we have

$$\int \frac{|K_{\mathcal{K}}(x)f_1(x)|^2}{t} \frac{dt}{t} = \int \frac{|K_{\mathcal{K}}(x)f_1(x)f_2(x)|^2}{t} \frac{dt}{t} = C\int \frac{|f_1(x)f_2(x)|^2}{t} \frac{dt}{t}.$$ 

Applying Hölder’s inequality to the last formula results in

$$\|f_1\|_p \leq C\|g_{\mathcal{K}}(f_1)\|_p.$$ 

The last inequality together with Riesz representation and a simple limiting argument give us the following:

**Corollary.** Suppose $K$ satisfies the hypothesis of Theorem 6 and $K_{\mathcal{K}}$ depends only on $a_\theta(x)$, then if $K$ is not identically zero

$$\|f\|_p \leq C\|g_{\mathcal{K}}(f)\|_p, \quad 1 < p < \infty,$$

where $C$ is a constant depending only on $K$ and $p$. 


2. Suppose $H$ is a non-negative function in $L^1$ and $b$ is reasonable, if $K$ and $a$ satisfy the hypothesis of Theorem 7 and $f$ is in $L^p$, $1 \leq p < \infty$, the Littlewood-Paley function $g_{K,a}(f)$ is defined by the formula

$$
(7) \quad g_{K,a}(f, x) = \left\{ \int \left( \int_{\mathbb{R}} H_{b}(x - z) K_a f(z) \right)^2 dz \right\}^{1/2} d\mu(z)
$$

where $H_b(x) = \int_{\mathbb{R}} H(t - b) dt$. Observe that if $f$ is in $L^2$ then

$$
(8) \quad \|g_{K,a}(f)\|_2 \leq C \|f\|_2
$$

where $C = \left\{ \|H\|_1 \right\} \left\{ \|K_a f\|_2 \right\}^{1/2}$. The following theorems show that with somewhat better $H$, the semi-linear operator $f \mapsto g_{K,a}(f)$ maps $L^p$ boundedly into $L^p$ for other values of $p$ also.

Consider the transformation $f \mapsto M_{K,a} f$ defined by the formula $M_{K,a} f(x) = \sup \{H_{b}(x - z) \mid f(z)\}^{1/2}$, clearly this operation maps $L^p$ boundedly into $L^p$ and recall that Theorem 3 gives conditions on $H$ which insure that it maps $L^p$ boundedly into $L^p$ for $p > 0$, less than 1.

**Theorem 8.** Suppose $f \mapsto M_{K,a} f$ maps $L^p$ boundedly into $L^p$ for $1 < p < \infty$, then there is a constant, $C_p$, independent of $f$, such that

$$
\|g_{K,a}(f)\|_p \leq C_p \|f\|_p
$$

for $2 \leq p < \infty$.

**Proof.** Since we already have the result for $p = 2$, assume that $2 < p < \infty$. Let $g$ be any function in $L^q$, where $q = \left( \frac{p}{2} \right)$, and write

$$
\left\{ \|\varphi(\cdot) g_{K,a}(f, \cdot)\|_2 \right\}^{1/2} \leq \left\{ \int \left( \int_{\mathbb{R}} H_{b}(x - z) \varphi(z) \right)^2 dz \right\}^{1/2} \leq \int M_{K,a} \varphi(x) \langle g_{K,a}(f, \cdot) \rangle^2 dz
$$

where $\varphi(x) = |\varphi(-x)|$. Applying the hypothesis and Hölder’s inequality to the right-hand side of the last estimate we get

$$
\left\{ \|\varphi(\cdot) g_{K,a}(f, \cdot)\|_2 \right\}^{1/2} \leq C \|g\|_p \|g_{K,a}(f)\|_p.
$$

The desired result now follows from Riesz representation and Theorem 7. ■

**Theorem 9.** Suppose $H$ is a non-negative function on the open interval, $\varphi_0(x)$ is in $L^1$, and $\varphi(t) \leq C\langle t \rangle^{\gamma - 1}$ for $\gamma > \frac{1}{2}$, then there is a constant, $C_p$, independent of $f$ such that $\|g_{K,a}(f)\|_p \leq C_p \|f\|_p$ for $1 \leq p < \infty$.

This theorem is a simple corollary of the following lemma and the Marcinkiewicz interpolation theorem. The proof presented here is a modification of the calculation used by Fefferman, [3], in obtaining the corresponding estimate for $\mathcal{G}_1$.

**Lemma 1.** Suppose $H$ and $K$ satisfy the hypothesis of Theorem 9 and $\gamma$ satisfies $0 < \gamma < \alpha$. Then, for $p = \frac{2 \alpha x}{\gamma + x - a}$ and any $f$ in $L^p$,

$$
\left\{ \{x : g_{K,a}(f, x) > \delta\} \right\} \leq C \alpha^{-\gamma} \|f\|_p^p
$$

for each $\delta > 0$, where $C$ is a constant dependent of $f$ and $a$.

**Proof.** Let $f$ be a function in $L^p$, $p = \frac{2 \alpha x}{\gamma + x - a}$. To prove the lemma it suffices to show that

$$
\left\{ \{x : g_{K,a}(f, x) > 1\} \right\} \leq C \|f\|_p^p
$$

where $C$ is a constant independent of $f$.

Let $B(x, s) = \{z : \varphi_0(x - z) < s\}$ and observe that $\{B(0, s), s > 0\}$ is a Vitali family with constant $2^\gamma$. Now set

$$
F(x) = \frac{1}{\mu(B(0, s))} \int_B \left| f(z) \right|^p dz
$$

and define a mapping of $B$ into $B_0$ defined by $r(x) = \frac{1}{\delta^\gamma} \mu(B(x, \delta))$. Since $\delta$ is open and of finite measure, $r(x)$ satisfies the hypothesis of Theorem 1 and hence there exists a sequence in $\Omega$ such that $B(x, r(x))$ is disjoint and $\bigcup B(x, 2r(x)) \supseteq \Omega$. Let $\{V_j\}$ be a disjoint family of measurable sets such that $B(x, r(x)) \subseteq V_j \subseteq B(x, 2r(x))$, $j = 1, 2, \ldots$, $\infty \in V_j = \Omega$. Define the functions $f'$ and $f''$ by

$$
\begin{cases}
  f'(x) = \frac{1}{|V_j|} \int_{V_j} f(z) dx, & x \in V_j;
  \quad \text{and} \quad f''(x) = f(x) - f'(x), \quad x \in \Omega
\end{cases}
$$

and observe that the estimates

$$
\left\{ \{x : g_{K,a}(f', x) > \frac{1}{2}\} \right\} \leq C \|f\|_p^p
$$

and

$$
\left\{ \{x : g_{K,a}(f'', x) > \frac{1}{2}\} \right\} \leq C \|f\|_p^p
$$

imply (9).
To see (10), note that, by definition, \(|f'(x)| \leq C'\) for almost all \(x\), where \(C'\) is independent of \(f\). This together with the fact that \(|f'|_p \leq ||f||_p\) imply that \(f'\) is in \(L^2\) and \(||f'||_p \leq C||f||_p\). Hence, applying (8), we have

\[ |\{x : g_{\nu_{\nu_{\nu_{\nu}}}}(f', x) > \frac{1}{2}\}| \leq 4||g_{\nu_{\nu_{\nu_{\nu}}}}(f', x)||_p \leq C||f||_p,\]

which proves (10).

Let \(B_j = B_{\nu_{\nu_{\nu_{\nu}}}}(\nu_{\nu_{\nu_{\nu}}})\), \(f_j(x) = f'(x)\chi_{\nu_{\nu_{\nu_{\nu}}}}(x)\), and write \(K_{\nu_{\nu_{\nu_{\nu}}}f_j}(x) = \sum_{\nu_{\nu_{\nu_{\nu}}}} K_{\nu_{\nu_{\nu_{\nu}}}f_j}(x) = A_1(x) + A_2(x)\), where

\[ A_1(x, t) = \sum_{\nu_{\nu_{\nu_{\nu}}}} [K_{\nu_{\nu_{\nu_{\nu}}}f_j}(x)] \chi_{\nu_{\nu_{\nu_{\nu}}}}(x), \quad A_2(x, t) = \sum_{\nu_{\nu_{\nu_{\nu}}}} [K_{\nu_{\nu_{\nu_{\nu}}}f_j}(x)] \chi_{\nu_{\nu_{\nu_{\nu}}}}(x),\]

and \(\chi_{\nu_{\nu_{\nu_{\nu}}}}\) always denotes the characteristic function of the set \(E\). Now, if

\[ g_1(x) = \left\{ \int_0^x H_{\nu_{\nu_{\nu_{\nu}}}}(x-s)|A_1(x, t)|^2 ds \frac{dt}{t} \right\}^{1/2},\]

and

\[ g_2(x) = \left\{ \int_0^x H_{\nu_{\nu_{\nu_{\nu}}}}(x-s)|A_2(x, t)|^2 ds \frac{dt}{t} \right\}^{1/2},\]

then

\[ \|g_{\nu_{\nu_{\nu_{\nu}}}}(f', x) \| = g_1(x) + g_2(x) \]

and it is clear that (11) will follow if we can show

\[ |\{x : g_1(x) > \frac{1}{2}\}| \leq C||f||_p^2\]

and

\[ |\{x : g_2(x) > \frac{1}{2}\}| \leq C||f||_p.\]

To see (12), write

\[ |\{x : g_1(x) > \frac{1}{2}\}| \leq 16||g_{\nu_{\nu_{\nu_{\nu}}}}||_p = 16 \int_0^\infty \int_0^x |A_1(x, t)|^2 ds \frac{dt}{t}.\]

To obtain an estimate on \(|A_1(x, t)|\), recall that \(|K(x)| \leq h_{\nu_{\nu_{\nu}}}(x)|\), where \(h(x)\) is a decreasing function on \(R_+\) and \(\int h_{\nu_{\nu_{\nu}}}(x)dx < \infty\), and observe that

\[ \chi_{\nu_{\nu_{\nu}}}(x) \sup_{\nu_{\nu_{\nu_{\nu}}}} |K_{\nu_{\nu_{\nu}}}(-y)| \leq \frac{1}{|\nu_{\nu_{\nu_{\nu}}}|} \int_{\nu_{\nu_{\nu_{\nu}}}} \nu_{\nu_{\nu_{\nu}}} e^{-\nu_{\nu_{\nu}}(x-s-y)} ds.\]

Also note that

\[ \int_{\nu_{\nu_{\nu}}}|f'(y)|dy \leq C|\nu_{\nu_{\nu_{\nu}}}|,\]

where \(C\) is a constant depending only on \(a\). Using (15) and (16), we have

\[ A_1(x, t) \leq \sum_{\nu_{\nu_{\nu_{\nu}}}} [K_{\nu_{\nu_{\nu}}}f_j(x)] |\chi_{\nu_{\nu_{\nu}}}(x)| \]

\[ \leq \sum_{\nu_{\nu_{\nu_{\nu}}}} |\chi_{\nu_{\nu_{\nu}}}(x)| \sup_{\nu_{\nu_{\nu_{\nu}}}} |K_{\nu_{\nu_{\nu}}}(-y)| \int_{\nu_{\nu_{\nu}}}|f'(y)|dy \]

\[ \leq C \int_{\nu_{\nu_{\nu}}} h_{\nu_{\nu_{\nu}}}(x)dx.\]

To obtain an estimate on \(|A_2(x, t)|\), use (17) and write

\[ \int_0^\infty \int_{\nu_{\nu_{\nu_{\nu}}}} |A_2(x, t)|^2 ds \frac{dt}{t} \leq C \int_{\nu_{\nu_{\nu}}} \int_{\nu_{\nu_{\nu}}} |K_{\nu_{\nu_{\nu}}}f_j(x)|^2 dx \frac{dt}{t}.\]

Since \(K\) satisfies the hypothesis of Theorem 7, it is easy to see that

\[ \int_{\nu_{\nu_{\nu}}} \int_{\nu_{\nu_{\nu}}}|K_{\nu_{\nu_{\nu}}}(-y)| \chi_{\nu_{\nu_{\nu}}}(x)dx \frac{dt}{t} \leq C\]

for any \(y \in V_j\),

where \(x_j\) is the center of \(B_j\). Hence, using the fact that \(\int f'(-y)dy = 0\) and (16) we have

\[ \int_{\nu_{\nu_{\nu}}} \int_{\nu_{\nu_{\nu}}}|K_{\nu_{\nu_{\nu}}}f_j(x)|^2 dx \frac{dt}{t} \leq C.\]

Slipping (19) into (18) results in

\[ \int_{\nu_{\nu_{\nu}}} \int_{\nu_{\nu_{\nu}}}|A_2(x, t)|^2 ds \frac{dt}{t} \leq C \sum_{\nu_{\nu_{\nu_{\nu}}}} |V_j| = C|\nu_{\nu_{\nu_{\nu}}}|.\]

Estimate (12) now follows from (14) and (20).

To complete the proof of the lemma, it remains to show (13). Since \(|\nu_{\nu_{\nu}}| \leq C||f||_p\), (13) will follow from

\[ |\{x : g_2(x) > \frac{1}{2}\}| \leq C||f||_p.\]
Recall that 
\[ g_t(x) = \left\{ \begin{array}{ll} \int_0^t H_\mu(x-s) |A_\lambda(x,t)|^2 \frac{dt}{t} & \text{for} \ x \neq 0 \\ \frac{t^\nu}{\nu} & \text{for} \ x = 0 \end{array} \right. \]
and observe that 
\[ A_\lambda(t) = \sum_{j=1}^\infty (K_\mu \ast f_j)(x) I_B_j(x) \]
is identically zero for \( x \in \Omega' \), since \( B_j \subset \Omega \) for all \( j \). It follows that 
\[ [g_t(x)]^2 = \sum_{n=1}^\infty \int_\Omega \int_\Omega \chi_{n}(x \cdot x_n) H_\mu(x-x_n) |A_\lambda(x,t)|^2 \frac{dt}{t} \frac{dx}{x} \]
Now, if \( x_n \) is the center of \( B_n \) and \( x \in \Omega' \), then for any \( x \in V_n \), \( \phi_n(x-x_n) \leq \frac{1}{2} \tilde{\phi}(x-x_n) \) and hence, since \( H_\mu(x) \leq C \phi_n(x-x_n)^{-\frac{1}{2} - \frac{1}{2}} \), \( H_\mu(x-x_n) \leq C \phi_n(x-x_n)^{-\frac{1}{2} - \frac{1}{2}} \). Using this fact together with the last formula for \( g_t(x) \) gives us 
\[ [g_t(x)]^2 \leq \sum_{n=1}^\infty \phi_n(x-x_n)^{-\frac{1}{2} - \frac{1}{2}} \int_\Omega \int_\Omega \chi_{n}(x \cdot x_n) |A_\lambda(x,t)|^2 \frac{dt}{t} \frac{dx}{x} \]
for all \( x \in \Omega' \).

To obtain an estimate on \( \int_\Omega \int_\Omega \chi_{n}(x \cdot x_n) |A_\lambda(x,t)|^2 \frac{dt}{t} \frac{dx}{x} \), first observe that the integrand can be non-zero only if \( x \) is contained in \( V_n \cap B_j \) for some \( j \). Now recall that \( B_j(x) = V_j \cap B_j(x) \cdot 2\mu(x) \) and \( B_j = B_j(x) \cdot 2\mu(x) \cdot r_j \) where \( r_j = \frac{1}{2} \inf_{x \in V_n \cap B_j} \phi_n(x-x_n) \). It is easy to see that if \( B_j \cap V_n \neq \emptyset \) then \( B_j \subset B_{n_j} \), where \( B_{n_j} = B_j(x) \cdot 2\mu(x) \cdot 3 \) and if \( N \) denotes the number of \( B_j \)'s whose intersection with \( V_n \) is non-empty then \( N \leq 44 \). Hence if \( \sum_{j < n} \) denotes the sum of those \( j \)'s where \( B_j \cap V_n \neq \emptyset \), then 
\[ \chi_{n}(x \cdot x_n) |A_\lambda(x,t)|^2 \leq N \sum_{j < n} |K_\mu \ast f_j(x)|^2 \]
Using the last inequality, we have 
\[ \int_\Omega \int_\Omega \chi_{n}(x \cdot x_n) |A_\lambda(x,t)|^2 \frac{dt}{t} \frac{dx}{x} \leq N \sum_{j < n} \int \int |K_\mu \ast f_j(x)|^2 \frac{dt}{t} \frac{dx}{x} \]
Now, using Plancherel, Fubini, and the fact that \( \hat{K}(\xi) \leq C (1 + \xi^\gamma) \) for \( \gamma > \frac{1}{2} \), write 
\[ \int_\Omega \int_\Omega \int_{\Omega} |K_\mu \ast f_j(x)|^2 \frac{dt}{t} \frac{dx}{x} \]
\[ \leq C \int \int |(\hat{f}_j(x))/|(|\xi|)|^2 \frac{dt}{t} \frac{dx}{x} \]
\[ \leq C \int \int |(\hat{f}_j(x))/|(|\xi|)|^2 \frac{dt}{t} \frac{dx}{x} \]
\[ \leq C \int \int |(\hat{f}_j(x))/|(|\xi|)|^2 \frac{dt}{t} \frac{dx}{x} \]
where \( R \) is the inverse Fourier transform of \( \phi_\mu(\cdot)^{-1} \). Applying Prop..
Slipping the last two estimates into (31) gives us
\[ \int_0^\infty \frac{\|\mathcal{F}_p \Phi R_* T_0 f(x)\|_t}{t} dt \leq C \int_0^\infty \int_0^\infty \left( \frac{1}{|x-y|^{m-1}} + 1 \right) \Phi_{p} \Phi R_* f(x-y)^2 |x-y|^2 d\beta d\lambda, \]
which is the desired result. \( \square \)

As an immediate corollary, we see that if \( h \) satisfies the hypothesis of Proposition 2 then the transformation \( f \rightarrow T_0 f \) maps \( L^p \) boundedly into \( L^q \) for \( 1 < p < \infty \).

Let \( P_j, j = 0, 1, \ldots, n \), be the \( L^2 \) functions whose Fourier transform is given by \( \hat{P}_j(\xi) = [\xi_j e^{-|\xi_j|}, \hat{P}_j(\xi) = e^{i\xi_j}, j = 0, 1, \ldots, n \) and \( f \) for \( \lambda > 0 \).

The Littlewood–Paley function introduced by Stein is simply \( g^j(f, x) = \left( \sum_{|j| < \lambda} g_{p_j}^j(f, x) \right)^{1/2} \).

Let \( \rho^u \) be the function whose Fourier transform is given by
\[ \hat{\rho}^u(\xi) = \left( \sum_{|j| < \lambda} |\xi_j|^2 + i \xi_j \right)^{1/2} \exp \left\{ -\left( \sum_{|j| < \lambda} |\xi_j|^2 + i \xi_j \right)^{1/2} \right\}, \]
where the principal determinant of the square root is indicated. If \( a \) is the linear transformation given by \( x = (x_1, \ldots, x_n, 2\pi) \) and \( H^u(x) = (1 + \omega_{\infty})^{-u-1} \), then \( g_{p_j}^j(f, x) \) and \( g_{p_j}^j(f, x) \) are the Littlewood–Paley functions studied by Jones [5] and Segovia and Wheeden [8].

More generally, suppose \( a \) is symmetric and let \( K \) be a function whose Fourier transform is given by
\[ \hat{K}(\xi) = h(|\xi|) e^{-\omega_{\infty}|\xi|} \]
where \( h \) is continuous on \( \mathbb{R}^n \), in \( C^m(\mathbb{R}^n \setminus \{0\}) \), and quasi-homogeneous of degree \( \alpha > 0 \) with respect to \( a \). The following proposition together with the mean value theorem shows that \( K \) satisfies the hypothesis of Theorem 7.

**Proposition 5.** If \( \hat{K} \) satisfies (32) then \( K(x) \leq (1 + \omega_{\infty})^{-u-1} \).

**Proof.** Without loss of generality we can assume that \( a \) is diagonal (i.e. \( a = (a_1, 0, \ldots, 0, a_n) \)). Since \( K \) is bounded, it suffices to obtain an estimate for \( \omega_{\infty}(x) > 1 \). Write
\[ K(x) = (2\pi)^{-n/2} \int h(\xi) e^{i\xi x} d\xi, \]
where \( \omega_{\infty}(x) = \left( \sum_{|j| < \lambda} |\xi_j|^2 + i \xi_j \right)^{1/2} \) is on the unit sphere. The last formula implies that it is enough to show that the inverse Fourier transform of \( h(\xi) e^{i\xi x} \) is bounded on the unit sphere independent of \( r > 1 \). To see this, let \( \varphi(\xi) \)
be a function in $C^n_0(\mathbb{R}^n)$ such that $\varphi(\xi) = 1$ for $\varphi_a(\xi) \leq 1$, $\varphi(\xi) = 0$ for $\varphi_a(\xi) \geq 2$ and observe that the inverse Fourier transform of $f(\xi, r) = h(\xi)e^{-i\varphi(\xi) + \varphi_a(\xi)}$ is equal to $f_1(\xi, r) + f_2(\xi, r)$ where $f_1(\xi, r) = f(\xi, r)\varphi(\xi)$ and $f_2(\xi, r) = f(\xi, r)(1 - \varphi(\xi))$. Clearly $f_1(\xi, r)$ is bounded independent of $x$ and $r$. To obtain an estimate on $f_2(\xi, r)$, observe that

$$DF_0^{\alpha}[f_2(\xi, r)] = \sum_{k=0}^m \frac{\partial^n}{\partial \xi_1^{m-k}}[\varphi(\xi)\xi^n - \varphi_a(\xi)](1 + \varphi(\xi))] + u(\xi, r),$$

where $h_{m-n+\alpha}$ is in $C^m(\mathbb{R}^n \setminus \{\theta\})$ and quasi-homogeneous of degree $\alpha - m\varphi_a + \varepsilon$ with respect to $\alpha$ and $u(\xi, r)$ is supported in $1 \leq \varphi_a(\xi) \leq 2$ and bounded independent of $r > 1$. From this we see that

$$|DF_0^{\alpha}[f_2(\xi, r)]| \leq C[(\varphi_a(\xi))^{-m\varphi_a 1 - \varphi(\xi))],$$

and hence, if $m$ is large enough, $|DF_0^{\alpha}[f_2(\xi, r)]| |dr|$ is bounded independent of $r > 1$. It follows that for sufficiently large $m$, $|f_2(\xi, r)|$ is bounded independent of $x$ and $r > 1$ and we conclude that $f_2(\xi, r)$ is bounded independent of $r > 1$ for $|x| = 1$.

Littlewood–Paley functions constructed with kernels of type (32) have applications analogous to that of $g^i_\lambda$. Suppose $n$ is even and let $\mathcal{K}^j(\xi) = \alpha_0(\xi)\varphi(\xi)$ and $\mathcal{K}^j(\xi) = \beta_0(\xi)\varphi(\xi)$, $j = 1, \ldots, m$. For $\lambda > 0$ set $H(\lambda) = (1 + \alpha_0(\xi))^{-m\varphi_a - \varepsilon}$, for $\alpha > 0$ define the transformation $f \rightarrow f_a$ by $f_a(x) = \varphi(x)^{-m\varphi_a}(\xi^j)$. And for $m = 1, 2, \ldots$ define $DF_0^{\alpha}[f_a(\xi)] = \varphi_a(\xi)^{-m\varphi_a}(\xi^j)$ where $A_{\alpha}(f) = f(x - x) - f(x)$. Finally for $0 < \alpha < \varepsilon$ and $a < \xi$, $a_\xi$ is least eigenvalue of $a$, define $\mathcal{A}_{\alpha}(f, x)$ for $f$ in $\mathcal{A}$ by

$$\mathcal{A}_{\alpha}(f, x) = \left\{ \int |A_{\alpha}(f, x)|^2 \varphi_a(\xi)^{m\varphi_a - \varepsilon} d\xi \right\}^{1/2}.$$

As in [10], it is not difficult to see that

$$\mathcal{A}_{\alpha}(f, x) \leq C_{\alpha, \varphi_a}(f, x)$$

for $0 < \lambda < 2a$

where

$$g_{\alpha, \lambda}(f, x) = \left\{ \frac{\int |A_{\alpha}(f, x)|^2 \varphi_a(\xi)^{m\varphi_a - \varepsilon} d\xi}{} \right\}^{1/2}.$$

Estimate (33) together with Theorems 5 and 9, imply a characterization of the quasi-homogeneous Lebesgue spaces introduced by Sadosky and Cotlar [7]. The details will appear elsewhere.