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An imbedding theorem for $H^\circ(G, \Omega)$ spaces

by

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Abstract. A vector-valued Orlicz space $L(G, \Omega)$ is defined for a convex function, G, of m variables and a domain, Ω, in Euclidean n space. When $m = n$, a norm can be introduced into $C_0^1(\Omega)$ by the taking the $L(G, \Omega)$ norm of the gradient of functions in $C_0^1(\Omega)$. By completion we obtain the space $H^\circ(G, \Omega)$. We prove an imbedding theorem for the space $H^\circ(G, \Omega)$ which includes as a special case an imbedding theorem established by Donaldson and Trudinger for Orlicz — Sobolev spaces.

§ 1. Introduction. In the paper [1], imbedding theorems are established for Orlicz–Sobolev spaces. In the present paper, we consider a more general class of spaces which permit different integral behaviour of derivatives in different directions and we derive the appropriate imbeddings into extended Orlicz spaces. The results extend those in [1] and the techniques we employ are on the whole a refinement and extension of those introduced there. Theorem 1, and its special cases which we treat, can be viewed as interesting extensions of the well-known Sobolev imbedding theorem. The body of the paper is divided into three sections. In Section 2, we discuss the convex functions, called G-functions, and the spaces $L(G, \Omega)$, in terms of which the imbedding theorem is cast. In Section 3, we introduce the $H^\circ(G, \Omega)$ spaces and discuss the imbedding theorem, Theorem 1, together with some applications. The proof of the main theorem is finally supplied in Section 4, along with a brief treatment of possible extensions.

§ 2. G-functions and $L(G, \Omega)$ spaces. Let $\boldsymbol{R}^m$ denote m-dimensional Euclidean space. A function $G\colon \boldsymbol{R}^m \to [0, \infty]$ will be called a *G-function* of m variables if it satisfies the following properties:

(i) $G(0) = 0$;

(ii) $G(\infty) = \infty$, that is, $\lim\limits_{|x|\to\infty} G(x) = \infty$;

(iii) G is *convex*, that is,

$$G\big(\lambda x + (1-\lambda)y\big) \leqslant \lambda G(x) + (1-\lambda) G(y)$$

for all $0 \leqslant \lambda \leqslant 1$, $x, y \in \boldsymbol{R}^m$;

(iv) G is *symmetric*, that is, $G(x) = G(-x)$ for all $x \in \boldsymbol{R}^m$;

(v) $G^{-1}(\infty)$ is bounded away from 0;

(vi) G is lower semicontinuous.

A G-function of one variable will be called a *Young function* (see [9]). Typical examples of Young functions are

(i) $\psi_a(x) = \frac{|x|^a}{a}$, $1 \leqslant a < \infty$,

(ii) $\psi_\infty(x) = \begin{cases} 0 & \text{if } |x| \leqslant 1, \\ \infty & \text{if } |x| > 1, \end{cases}$

(iii) $\psi_e(x) = e^{|x|} - 1$.

G-functions of m variables are readily constructed from Young functions. For example we have the following:

(iv) $G_1(x) = A(|x|)$, $|x| = \left(\sum_{i=1}^m x_i^2\right)^{1/2}$,

(v) $G_2(x) = \sum_{i=1}^m A_i(x_i)$,

(vi) $G_3(x) = \sum_{i=1}^m \psi_{a_i}(x_i)$, $\quad 1 \leqslant a_i \leqslant \infty$

where $A, A_i, i = 1, \ldots, m$ are Young functions.

The class of G-functions essentially includes the class of even generalized N functions treated in [6] (see also [3] and [8]). The latter class consists of G-functions satisfying $G^{-1}(0) = 0$, $G^{-1}(\infty) = \infty$, $\lim_{|x|\to\infty} \frac{G(x)}{|x|} = \infty$ together with a uniformity condition which, for example, excludes the function G_3 unless $1 < a_1 = a_2 = \ldots = a_m < \infty$.

If for any two G-functions, G_1, G_2, there exist non-negative numbers c_0 and k such that

$$G_1(x) \leqslant G_2(kx) \quad \text{for all } |x| \geqslant c_0 \tag{1}$$

then we write $G_1 \dashv G_2$. If $G_1 \dashv G_2$ and $G_2 \dashv G_1$, we write $G_1 \sim G_2$ and call G_1 and G_2 equivalent. By virtue of properties (ii) and (iii), there will exist for any G-function, non-negative c_0 and k satisfying

$$|x| \leqslant G(kx) \quad \text{for } |x| \geqslant c_0 \tag{2}$$

so that $\psi_1 \dashv G$ where $\psi_1(x) = |x|$. Any G-function also gives rise to m Young functions, G_i, $i = 1, \ldots, m$, defined by

$$G_i(x_i) = G(0, \ldots, x_i, 0, \ldots, 0), \quad i = 1, \ldots, m, \tag{3}$$

and hence we obtain a further G-function, $\tilde{G}$, given by

$$\tilde{G}(x) = \sum_{i=1}^m G_i(x_i). \tag{4}$$

By the convexity of G, we have

$$G(x) \leqslant \frac{1}{m} \tilde{G}(mx) \tag{5}$$

so that $G \dashv \tilde{G}$. For many G-functions, including the generalized N functions of [6], G and $\tilde{G}$ are in fact equivalent. An example where this is not the case would be

$$G(x_1, x_2) = (x_1 - x_2)^4 + x_2^2.$$

The *conjugate* function G^* of a G-function is defined by

$$G^*(x) = \sup_{y \in \boldsymbol{R}^m} \{x \cdot y - G(y)\}. \tag{6}$$

From properties (i) to (vi) of G, it follows that G^* is also a G-function and $G^{**} = G$ (see [5], Section 12). From (6) we obtain immediately a generalized Young inequality

$$x \cdot y \leqslant G(x) + G^*(y), \quad \text{for all } x, y \in \boldsymbol{R}^m. \tag{7}$$

A further conjugate function, G_+^*, may also be defined by

$$G_+^* = \sup_{y_i \geqslant 0} \{x \cdot y - G(y)\}. \tag{8}$$

Clearly $G_+^* \leqslant G^*$ and in the inequality (7) we may replace G^* by G_+^* provided y satisfies $y_i \geqslant 0$.

Examples of conjugate functions. Referring to the previously given examples of G-functions, we have

$$\begin{cases} \psi_a^* = \psi_\beta \quad \text{where} \quad \frac{1}{a} + \frac{1}{\beta} = 1, \quad 1 \leqslant a \leqslant \infty, \\ \psi_e^*(x) = |x| \log^+ |x|, \\ G_1^*(x) = G_{1+}^*(x) = A^*(|x|), \\ G_2^*(x) = G_{2+}^*(x) = \sum_{i=1}^m A_i^*(x_i), \\ G_3^*(x) = G_{3+}^*(x) = \sum_{i=1}^m \psi_{\beta_i}(x_i), \quad \frac{1}{\beta_i} + \frac{1}{a_i} = 1. \end{cases} \tag{9}$$

The $L(G, \Omega)$ spaces. Let Ω be a domain in $\boldsymbol{R}^n$, $u_1, \ldots, u_m$ be measurable functions on Ω, $u = (u_1, \ldots, u_m)$ and G a G-function. The space $L(G, \Omega)$ is defined by

$$L(G, \Omega) = \left\{u \mid \int_\Omega G(au)\,dx < \infty \text{ for some } a > 0\right\}.$$

More generally, we could replace Ω by any σ finite measure space. A norm, analogous to the Luxemburg norm for Orlicz spaces [3], is introduced into $L(G,\Omega)$ by defining

(10) $$\|u\|_G = \inf\left\{k>0 \mid \int_\Omega G\left(\frac{u}{k}\right)dx \leqslant 1\right\}.$$

By property (vi) of G, we have for $\|u\|>0$

(11) $$\int_\Omega G\left(\frac{u}{\|u\|}\right)dx \leqslant 1.$$

That $\|\ \|_G$ does in fact satisfy the required conditions for a norm follows from properties (i) to (vi) of G. The verification which is similar to the Orlicz space case [3] is left to the reader to supply. Furthermore, from inequality (2) we obtain for $u \in L(G,\Omega)$ and $|\Omega|<\infty$

(12) $$\int_\Omega |u|\,dx \leqslant k(1+c_0|\Omega|)\|u\|_G$$

so that $L(G,\Omega)\to L_1(\Omega)$ if $|\Omega|<\infty$. We use here and in the sequel arrows to indicate continuous imbeddings. One can then conclude that $L(G,\Omega)$ is a Banach space for arbitrary Ω.

If G is a Young function, we call $L(G,\Omega)$ an *extended Orlicz space*, while if in addition $G^{-1}(0)=0$, $G^{-1}(\infty)=\infty$, $\lim\limits_{|x|\to\infty}\frac{G(x)}{|x|}=\infty$, $\lim\limits_{|x|\to 0}\frac{G(x)}{|x|}=0$, then $L(G,\Omega)$ is an Orlicz space. Referring to the previous examples of G-functions, we obtain

$$L(\psi_a,\Omega) = L_a(\Omega),$$
$$L(G_1,\Omega) = [L(A,\Omega)]^m,$$
$$L(G_2,\Omega) = \prod_{i=1}^m L(A_i,\Omega),$$
$$L(G_3,\Omega) = \prod_{i=1}^m L_{a_i}(\Omega).$$

Equivalent G-functions yield the same $L(G,\Omega)$ space when $|\Omega|$ is finite and for arbitrary Ω if $c_0=0$ in the condition (1). The above examples all reduce to products of extended Orlicz spaces and this will be true for any G-function satisfying $G\sim\tilde G$ and $|\Omega|<\infty$ or $G\sim\tilde G$ with $c_0=0$ in (1). By (5), we always have

$$L(\tilde G,\Omega)\to L(G,\Omega) \quad \text{with } \|u\|_G\leqslant m\|u\|_{\tilde G},\ u\in L(\tilde G,\Omega).$$

From the inequality (7) follows a generalized Hölder inequality

(13) $$\int_\Omega u\cdot v\,dx \leqslant 2\|u\|_G\|v\|_{G^*}$$

for all $u\in L(G,\Omega)$, $v\in L(G^*,\Omega)$. If the components of v are non-negative, we can replace G^* by G^*_+. Although G^*_+ is not necessarily a G-function, formulae (10) and (11) still make sense for $u_i\geqslant 0$.

Now let A be a Young function and suppose that $A(t)<\infty$ for $|t|<N$, $A(t)=\infty$ for $|t|\geqslant N$ where $0<N\leqslant\infty$. By the continuity of A in $(0,N)$, we have for any $u\in L(A,\Omega)$ with $\|u\|_A\neq 0$,

(14) $$\int_\Omega A\left(\frac{u}{\|u\|}\right)dx = 1$$

provided either $N=\infty$ or $|u|=\sup\limits_\Omega|u|$ on a set of positive measure. Furthermore if $N<\infty$, $L(A,\Omega)\subset L_\infty(\Omega)$ and if also $|\Omega|<\infty$, then $L(A,\Omega)=L_\infty(\Omega)$ with

(15) $$A^{-1}(|\Omega|^{-1})\|u\|_A\leqslant \sup_\Omega|u|\leqslant N\|u\|_A.$$

When $A(N)=\infty$ as above, the inverse function A^{-1} is well defined on $(0,\infty)$. On the other hand if $A(t)<\infty$ for $|t|\leqslant N$, $A(t)=\infty$ for $|t|>N$, we take $A^{-1}(t)=N$ for $t\geqslant A(N)$. In general, we will have by virtue of the definition of the conjugate A^*,

(16) $$t\leqslant A^{-1}(t)A^{*-1}(t)\leqslant 2t \qquad \text{for } t>0.$$

Further properties of the $L(G,\Omega)$ spaces such as duality may be developed following the lines of [3] or [6]. The above treatment, however, is sufficient for the purposes of this paper.

§ 3. The imbedding theorem. Let Ω be a domain in $\boldsymbol{R}^n$ and G a G-function of n variables. Then

(17) $$\|u\|_{H^\circ(G,\Omega)} = \|Du\|_G$$

is a norm on $C^1_0(\Omega)$ where we have used Du for the gradient of u. The space $H^\circ(G,\Omega)$ is defined to be the completion of $C^1_0(\Omega)$ under (17). By the estimate (12), the elements of $H^\circ(G,\Omega)$ can be identified as weakly differentiable functions and when $|\Omega|<\infty$ we will have $H^\circ(G,\Omega)\to W^{1,1}_0(\Omega)$ where $W^{1,1}_0(\Omega)$ is the Sobolev space $H^\circ(\psi,\Omega)$, $\psi(x)=|x|$. In fact, for $G(x)=|x|^p$, $1\leqslant p\leqslant\infty$, $H^\circ(G,\Omega)$ coincides with the Sobolev space $W^{1,p}_0(\Omega)$ and more generally when $G(x)=B(|x|)$ where B is an N function, $H^\circ(G,\Omega)$ is the Orlicz–Sobolev space $W^1_0L_B(\Omega)$ (see [1]). The object of our present work is to determine the extended Orlicz spaces into which $H^\circ(G,\Omega)$ is continuously imbedded. Towards this goal we have the following general theorem.

THEOREM 1. *Let $f = (f_1, \dots, f_n)$ where f_i are continuous, non-negative, non-decreasing functions on $[0, \infty)$ satisfying*

$$G_+^*(f(s)) \leqslant s, \tag{18}$$

$$\int_0^1 \frac{ds}{m(s)} < \infty \tag{19}$$

where

$$m(s) = \Big(s \prod_{i=1}^n f_i(s)\Big)^{1/n}. \tag{20}$$

Then $H^\circ(G, \Omega) \to L(A, \Omega)$ for any Young function A satisfying

$$\int_0^{|t|} \frac{ds}{m(s)} \leqslant kA^{-1}(|t|) \tag{21}$$

for some constant k. Furthermore there exists a constant c depending only on n such that

$$\|u\|_A \leqslant ck\|Du\|_G \tag{22}$$

for all $u \in H^\circ(G, \Omega)$.

An immediate consequence of Theorem 1 is the following

COROLLARY 1. *If in addition to the hypotheses of Theorem 1 we also have*

$$\int_1^\infty \frac{ds}{m(s)} < \infty \tag{23}$$

then $H^\circ(G, \Omega) \to C^\circ(\bar{\Omega})$. Furthermore, for any $u \in H^\circ(G, \Omega)$,

$$\sup_\Omega |u| \leqslant c\|Du\|_G \int_0^\infty \frac{ds}{m(s)}. \tag{24}$$

Corollary 1 follows by virtue of the estimate (15) and the completeness of $L_\infty(\Omega)$.

There is an optimal way of choosing the function f in Theorem 1. Namely we require that equality hold in (18) and that

$$f_i \frac{\partial G_+^*}{\partial x_i}(f) = \text{constant}. \tag{25}$$

Frequently, as will be evidenced in the examples below, a Young function equivalent to any obtained through this procedure, may be found by simpler means.

For $|\Omega| < \infty$, the condition (19) is not required to establish the imbedding $H^\circ(G, \Omega) \to L(A, \Omega)$ as this condition can be effected by the replacement of G by an equivalent G function. Of course then the form of inequality (22) would change. For $|\Omega| = \infty$, the restriction (19) will be necessary.

Applications

(i) Let $G(x) = B(|x|)$ where B is a Young function. We have then $G_+^*(x) = B_+^*(|x|)$ so that we may choose

$$f_i(s) = \frac{1}{\sqrt{n}} B^{*-1}(s).$$

Consequently

$$m(s) = \frac{1}{\sqrt{n}} s^{\frac{1}{n}} B^{*-1}(s).$$

Defining

$$\overline{m}(s) = \frac{s^{1+\frac{1}{n}}}{B^{-1}(s)} \tag{26}$$

we have, by inequality (16),

$$\frac{1}{\sqrt{n}}\, \overline{m}(s) \leqslant m(s) \leqslant \frac{2}{\sqrt{n}}\, \overline{m}(s), \qquad s > 0,$$

so that provided

$$\int_0^1 \frac{ds}{\overline{m}(s)} < \infty,$$

$H^\circ(G, \Omega) \to L(A, \Omega)$ for

$$A^{-1}(|t|) = \int_0^{|t|} \frac{ds}{\overline{m}(s)}.$$

Also by the estimate (22), we will have

$$\|u\|_A \leqslant C\sqrt{n}\, \|Du\|_G$$

for any $u \in H^\circ(G, \Omega)$. This result agrees with Theorem 2.4 in [1].

(ii) Let $G(x) = \sum_{i=1}^n B_i(x_i)$ where B_i, $i = 1, \dots, n$ are Young functions. Then

$$G_+^*(x) = \sum_{i=1}^n B_i^*(x_i)$$

so that we may choose

$$f_i(s) = B_i^{*-1}\Big(\frac{s}{n}\Big).$$

Consequently

$$m(s) = \left[s \prod_{i=1}^{n} B_i^{*-1}\left(\frac{s}{n}\right)\right]^{\frac{1}{n}}$$

and hence defining this time

$$\overline{m}(s) = s^{1+\frac{1}{n}}\left[\prod_{i=1}^{n} B_i^{-1}\left(\frac{s}{n}\right)\right]^{-\frac{1}{n}} \tag{27}$$

we have, by inequality (16),

$$\frac{1}{n}\,\overline{m}(s) \leqslant m(s) \leqslant \frac{2}{n}\,\overline{m}(s), \qquad s > 0.$$

Hence, provided

$$\int_0^1 \frac{ds}{\overline{m}(s)} < \infty,$$

we have $H^\circ(G, \Omega) \to L(A, \Omega)$ for

$$A^{-1}(|t|) = \int_0^{|t|} \frac{ds}{\overline{m}(s)}.$$

Furthermore, for any $u \in H^\circ(G, \Omega)$,

$$\|u\|_A \leqslant cn\|Du\|_G. \tag{28}$$

Utilizing Corollary 1, we see that if $|\Omega| < \infty$ and

$$\int_1^\infty \frac{ds}{\overline{m}(s)} < \infty,$$

then $H^\circ(G, \Omega) \to C^\circ(\overline{\Omega})$.

(iii) In the previous example, let us take $B_i = P_{a_i}$, $1 \leqslant a_i \leqslant \infty$, where

$$\begin{aligned} P_{a_i}(x) &= |x|^{a_i}, \quad 1 \leqslant a_i < \infty, \\ P_\infty(x) &= \psi_\infty(x), \end{aligned} \tag{29}$$

that is, the ith partial derivative of u, $D_i u \in L_a(\Omega)$.

In fact, we clearly have

$$\|Du\|_G \leqslant \sum_{i=1}^{n} \|D_i u\|_{a_i} \leqslant n\|Du\|_G. \tag{30}$$

Let us define a, satisfying $1 \leqslant a \leqslant \infty$, by

$$\frac{n}{a} = \sum_{i=1}^{n} \frac{1}{a_i}. \tag{31}$$

Then

$$\overline{m}(s) = n^{\frac{1}{a}} s^{1+\frac{1}{n}-\frac{1}{a}}.$$

Consequently if $a < n$, $H^\circ(G, \Omega) \to L_q(\Omega)$ where

$$\frac{1}{q} = \frac{1}{a} - \frac{1}{n}$$

and the estimate

$$\|u\|_q \leqslant cqn^{1-\frac{1}{a}}\|Du\|_G \leqslant cqn^{1-\frac{1}{a}} \sum \|D_i u\|_{a_i} \tag{32}$$

holds by (22) and (30) for any $u \in H^\circ(G, \Omega)$.

If $a = n$, we have

$$\int_0^1 \frac{ds}{\overline{m}(s)} = \infty, \qquad \int_1^s \frac{ds}{\overline{m}(s)} = n^{-\frac{1}{a}} \log s$$

so that for $|\Omega| < \infty$, $H^\circ(G, \Omega) \to L(\psi_e, \Omega)$ where $\psi_e(x) = e^{|x|} - 1$.

Finally, if $a > n$, we have

$$\int_0^1 \frac{ds}{\overline{m}(s)} = \infty, \qquad \int_1^\infty \frac{ds}{\overline{m}(s)} < \infty$$

so that for $|\Omega| < \infty$, $H^\circ(G, \Omega) \to C^\circ(\overline{\Omega})$. In the last two cases the bounds on the imbedding operators depend on the choice of an equivalent G-function agreeing with G for values of $|x_i|$ bounded away from zero. Consequently these bounds will depend on $|\Omega|$ as well as n and a. To determine the explicit dependence on $|\Omega|$, we consider a transformation of coordinates T: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ given by

$$x_i = |\Omega|^{\frac{1}{n}+\frac{1}{a_i}-\frac{1}{a}} y_i. \tag{33}$$

Then if $T\tilde{\Omega} = \Omega$ we have $|\tilde{\Omega}| = 1$. Furthermore

$$\|Dy_i u\|_{L_{a_i}(\tilde{\Omega})} = |\Omega|^{\frac{1}{n}-\frac{1}{a}} \|Dy_i u\|_{L_{a_i}(\Omega)}. \tag{34}$$

Hence we obtain for $a > n$, $u \in H^\circ(G, \Omega)$

$$\sup_\Omega |u| \leqslant c_1(a, n)\,|\Omega|^{\frac{1}{n}-\frac{1}{a}} \sum_{i=1}^{n} \|D_i u\|_{a_i} \tag{35}$$

while for $a = n$ we have

$$\int_\Omega \exp\left(\frac{u}{c_2(n)\sum \|D_i u\|_{a_i}}\right) dx \leqslant c_3(n)\,|\Omega| \tag{36}$$

where $c_1(a, n)$, $c_2(n)$, $c_3(n)$ are positive constants depending on their indicated arguments. The imbeddings for the cases $a < n$, $a > n$ were derived by Nikolskii [4] by completely different means.

An obvious question to ask in relation to Theorem 1, is for what G-functions is the result optimal, that is, $L_A(\Omega)$ is the smallest extended Orlicz space into which $H^\circ(G, \Omega)$ is imbedded. We know from the Sobolev space case that the result is optimal for $G(x) = |x|^a$, $a \neq n$ but not for $G(x) = |x|^n$. In the latter case, $H^\circ(G, \Omega) \to L(A, \Omega)$ where

$$A(t) = e^{|t|^{\frac{n}{n-1}}} - 1 \tag{37}$$

and this imbedding is sharp [2], [7].

Spaces of higher order derivatives may be defined analogously to the $H^\circ(G, \Omega)$ spaces. Let k be a non-negative integer and G a G-function of n^k variables. Writing p_a, where a is a multi-index of length k, for a generic point in $\boldsymbol{R}^{n^k}$, we can define a norm on $C_0^k(\Omega)$ by

$$\|u\|_{H_k^\circ(G,\Omega)} = \|D^a u\|_G \tag{38}$$

and complete $C_0^k(\Omega)$ under (38) to get a Banach space $H_k^\circ(G, \Omega)$. An imbedding theorem for $H_k^\circ(G, \Omega)$ would then follow by iteration of the case $k = 1$. A general formula would be exceedingly complicated to write down but special cases can be readily derived. For example, if

$$G(D^a u) = \sum_{|a|=k} P_{\beta_a}(|D^a u|), \qquad 1 \leqslant \beta_a \leqslant \infty, \tag{39}$$

where P is defined by (29), then setting

$$\frac{1}{p} = \frac{1}{n^k} \sum_{|a|=k} \frac{1}{\beta_a} \tag{40}$$

we have $H_k^\circ(G, \Omega) \to L_q(\Omega)$ if $kp < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$, $H_k^\circ(G, \Omega) \to L(\psi_e, \Omega)$ if $kp = n$ and $|\Omega| < \infty$, and $H_k^\circ(G, \Omega) \to C^\circ(\bar{\Omega})$ if $kp > n$ and $|\Omega| < \infty$.

§ 4. Proof of Theorem 1. The key calculus lemma in the proof is the following extension of the Sobolev imbedding theorem for the case $G(x) = |x|$.

LEMMA 1. *For* $0 < N \leqslant \infty$, *let* $B = (B_1, \ldots, B_n)$ *where* $B_i \in C^1[0, N)$, $B_i(0) = 0$, $\lim_{t \to N} B_i(t) = \infty$, $B_i' \geqslant 0$, $1 \leqslant i \leqslant n$. *Writing*

$$\bar{B} = \Big(\prod_{i=1}^{n} B_i\Big)^{\frac{1}{n}}$$

we have, for any $u \in C_0^1(\boldsymbol{R}^n)$ *with* $\sup |u| < N$,

$$\|\bar{B}(|u|)\|_{\frac{n}{n-1}} \leqslant \frac{1}{2n} \int \sum_{i=1}^{n} B_i'(|u|)\,|D_i u|\,dx. \tag{41}$$

Proof. Since

$$B_i(|u|) \leqslant \int_{-\infty}^{x_i} B_i'(|u|)\,|D_i u|\,dx_i$$

and

$$B_i(|u|) \leqslant \int_{x_i}^{\infty} B_i'(|u|)\,|D_i u|\,dx_i,$$

we have

$$B_i(|u|) \leqslant \tfrac{1}{2} \int_{-\infty}^{\infty} B_i'(|u|)\,|D_i u|\,dx_i$$

for every i. Hence

$$\big(2\bar{B}(|u|)\big)^{\frac{n}{n-1}} \leqslant \Big(\prod_{i=1}^{n} \int_{-\infty}^{\infty} B_i'(|u|)\,|D_i u|\,dx_i\Big)^{\frac{n}{n-1}}. \tag{42}$$

We now integrate the estimate (42) successively over each variable x_i, applying at each stage the following extension of the Schwarz inequality

$$\int \prod_{i=1}^{n-1} \chi_i \leqslant \Big(\prod_{i=1}^{n-1} \int |\chi_i|^{n-1}\Big)^{\frac{1}{n-1}}. \tag{43}$$

Consequently, we obtain

$$\int \big(2\bar{B}(|u|)\big)^{\frac{n}{n-1}} dx \leqslant \Big(\prod_{i=1}^{n} \int B_i'(|u|)\,|D_i u|\,dx\Big)^{\frac{1}{n-1}}$$

so that

$$\|\bar{B}(|u|)\|_{\frac{n}{n-1}} \leqslant \frac{1}{2} \Big(\prod_{i=1}^{n} \int B_i'(|u|)\,|D_i u|\,dx\Big)^{\frac{1}{n}}$$

$$\leqslant \frac{1}{2n} \int \sum_{i=1}^{n} B_i(|u|)\,|D_i u|\,dx. \ \blacksquare$$

The proof of Theorem 1 will now be accomplished by making a specific choice of the functions B_i in Lemma 1. It suffices to choose A to satisfy

$$kA^{-1}(t) = \int_0^t \frac{ds}{m(s)}, \qquad t > 0. \tag{44}$$

Then differentiating (44), we obtain

$$A' = km(A) = kA^{\frac{1}{n}}\Big(\prod_{i=1}^{n} f_i(A)\Big)^{\frac{1}{n}}$$

so that

$$(A^{1-\frac{1}{n}})' = k\Big(1-\frac{1}{n}\Big)\Big(\prod_{i=1}^{n} f_i(A)\Big)^{\frac{1}{n}}.$$

Hence

$$A^{1-\frac{1}{n}}(t) = k\Big(1-\frac{1}{n}\Big)\int_0^t \Big(\prod_{i=1}^{n} f_i(A)\Big)^{\frac{1}{n}} ds \tag{45}$$

$$\leqslant k\Big(1-\frac{1}{n}\Big)\Big(\prod_{i=1}^{n}\int_0^t f_i\big(A(s)\big)\,ds\Big)^{\frac{1}{n}}.$$

Now let us define

$$B_i(t) = \int_0^t f_i\big(A(s)\big)\,ds. \tag{46}$$

Then by inequality (45)

$$A^{1-\frac{1}{n}} \leqslant k\Big(1-\frac{1}{n}\Big)\bar{B} \tag{47}$$

and by inequality (18)

$$G_+^*(B') \leqslant A. \tag{48}$$

The functions B_i will clearly satisfy the hypotheses of Lemma 1 for

$$N = \frac{1}{k}\int_0^\infty \frac{ds}{m(s)}. \tag{49}$$

Hence if $u \in C_0^1(\Omega)$ and $\sup|u| < N$, we have

$$\|\bar{B}(|u|)\|_{\frac{n}{n-1}} \leqslant \frac{1}{2n}\int_\Omega \sum_{i=1}^{n} B_i'(|u|)\,|D_i u|\,dx \tag{50}$$

$$\leqslant \frac{1}{n}\|B'(|u|)\|_{G_+^*}\|Du\|_G \quad \text{by Hölder's inequality (13).}$$

Let us now replace u by $\frac{u}{\lambda}$ where $\lambda = \|u\|_A$ and suppose that

$$\sup_\Omega |u| < N\|u\|_A. \tag{51}$$

Using inequalities (48) and (11), we obtain

$$\int_\Omega G_+^*\Big(B'\Big(\frac{|u|}{\lambda}\Big)\Big)\,dx \leqslant \int_\Omega A\Big(\frac{u}{\lambda}\Big)\,dx \leqslant 1$$

so that

$$\Big\|B'\Big(\frac{|u|}{\lambda}\Big)\Big\|_{G_+^*} \leqslant 1.$$

Also by (47)

$$\Big\|\bar{B}\Big(\frac{|u|}{\lambda}\Big)\Big\|_{\frac{n}{n-1}} \geqslant \frac{n}{(n-1)k}\Big\|A^{1-\frac{1}{n}}\Big(\frac{u}{\lambda}\Big)\Big\|_{\frac{n}{n-1}}$$

$$= \frac{n}{(n-1)k}\Big(\int_\Omega A\Big(\frac{u}{\lambda}\Big)\,dx\Big)^{1-\frac{1}{n}} = \frac{n}{(n-1)k}$$

by (14) if either $N = \infty$ or $|u| = \sup_\Omega |u|$ on a set of positive measure. We then have by (50)

$$\frac{n}{(n-1)k} \leqslant \frac{\|Du\|_G}{n\lambda}$$

so that

$$\|u\|_A = \lambda \leqslant \frac{(n-1)k}{n^2}\|Du\|_G. \tag{52}$$

The condition on u when $N < \infty$ is removed by replacing u by the function

$$u_l = \begin{cases} l & \text{if } u \geqslant l, \\ u & \text{if } |u| \leqslant l, \\ -l & \text{if } u \leqslant -l \end{cases} \tag{53}$$

for $0 < l < \sup|u|$. Inequality (51) will then hold for u_l and clearly $|u_l| = \sup|u_l|$ on a set of positive measure. Thus the estimate (52) holds for u_l and by letting l tend to $\sup|u|$, we obtain (52) for arbitrary $u \in C_0^1(\Omega)$. Theorem 1 now follows from the density of $C_0^1(\Omega)$ in $H^\circ(G, \Omega)$ and the completeness of $L(A, \Omega)$. The constant C in estimate (22) clearly satisfies $C \leqslant \frac{n-1}{n^2}$.

The above proof is extendable to other spaces of weakly differentiable functions. Let G now be a G-function of $n+1$ variables $u, p_1, \ldots, p_n$ and denote by $W^1(G, \Omega)$ the set of weakly differentiable functions u on Ω for which the vector function $u, D_1 u, \ldots, D_n u$ belongs to $L(G, \Omega)$. For u belonging to $W^1(G, \Omega)$ we define

$$\|u\|_{W^1(G,\Omega)} = \|(u, Du)\|_G \tag{54}$$

and note that $W^1(G, \Omega)$ will be a Banach space under the norm (54). We also define $W_0^1(G, \Omega)$ to be the closure of $C_0^1(\Omega)$ in $W^1(G, \Omega)$. Then Theorem 1 and Corollary 1 will hold for $W_0^1(G, \Omega)$, instead of $\overset{\circ}{H}(G, \Omega)$, provided in inequality (18) we replace the function $G_+^*(f)$ by $G_+^*(0, f)$. The imbedding theorem also extends to the spaces $W^1(G, \Omega)$ for certain domains Ω but this situation will be the subject of a further investigation. The imbedding of Theorem 1 will also be compact if the Young function A increases strictly less rapidly than a function which satisfies the hypotheses of the Theorem (see [1]).

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Recognition and limit theorems for L_p-multipliers

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Abstract. The main theorem gives a necessary and sufficient condition for a function in $L^\infty(\Gamma)$ to be the Fourier transform of an $L_p(G)$-multiplier. Three applications of this theorem are given: to an extension of Hahn's theorem, to a limit theorem of Lévy type for L_p-multipliers, and to the study of the maximal ideal space of an algebra of L_p-multipliers.

1. Introduction. Let G denote a locally compact abelian (LCA) group with dual group Γ. Let $M_p(G)$ denote the algebra of bounded, translation invariant linear operators on $L_p(G)$, $1 \leqslant p < \infty$. It is well known that $M_p(G) = M_{p'}(G)$ when p' is the conjugate index to p and that the inclusion $M_p \subset M_2$ is continuous if $p \leqslant 2$. $M_2(G)$ is isometrically isomorphic with $L^\infty(\Gamma)$ via the Fourier transform and $M_1(G)$ is isometrically isomorphic with $M(G)$, the bounded Borel measures on G by $T(f) = \mu * f$; [10]. An element T of $M_p(G)$ has a Fourier transform $\hat{T}(\xi)$ which is assigned by letting $\hat{T}(\xi)$ be the Fourier transform of T regarded as an operator on $L_2(G)$.

In this paper we shall consider the following pair of questions:

(1) *When is $\varphi \in L^\infty(\Gamma)$ the Fourier transform of an operator T in $M_p(G)$?*

(2) *If $\{T_a\}$ is a net of operators in $M_p(G)$, which converges in the weak operator topology over $L_2(G)$, when does $\{T_a\}$ converge in the weak operator topology over $L_p(G)$?*

To answer the first question we shall give a criterion on φ in $L^\infty(\Gamma)$ which is similar to the criterion given in Schoenberg's theorem [3], [11] which characterizes the Fourier transforms of measures. The theorem of Schoenberg says that φ in $L^\infty(\Gamma)$ is the Fourier transform of a bounded Borel measure μ on G if and only if there is a real number $M \geqslant 0$ such that for every H in $L_1(\Gamma)$,

$$\Big| \int_\Gamma \varphi(\gamma) H(\gamma)\, d\gamma \Big| \leqslant M \|\hat{H}\|_\infty;$$

$\hat{H}$ denotes the Fourier transform of H and $\| \ \|_\infty$ is the sup-norm. If $\varphi = \hat{\mu}$,

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