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Received December 22, 1972 (624)

STUDIA MATHEMATICA, T. L. (1974)

Bases in bornological spaces

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Abstract. This paper presents the fundamentals of the basis theory for bornological spaces. The attention is restricted to complete and regular spaces with a bornology which is either topological or of countable type. Spaces of the latter type are called (LB)-spaces. We begin by introducing the notions of separability and local separability in a bornological space and by showing that they agree for (LB)-spaces, which enables us to give representation theorems for such spaces which are separable. Next, bases and Schauder bases are introduced and a basic lemma which states that a basis of an (LB)-space is also a 'local' basis is proved. Among the many consequences of this fundamental lemma the most important is that every basis of an (LB)-space is a Schauder basis. We then investigate the relationship between bornological and topological Schauder bases and study the properties of a Schauder basis in terms of the dual sequence of bounded linear functionals. Finally, the connection between Schauder bases and reflexivity is given and various types of Schauder bases are analysed.

Introduction. The purpose of this paper is to present the fundamentals of the basis theory for bornological spaces. Attempts have only recently been made to extend to locally convex spaces the classical basis theory for Banach spaces. Here we are concerned with developing a similar theory for regular bornological spaces, the assumption of regularity being imposed by the central role played by duality. We deal essentially with Schauder bases and the fact that bornological spaces with such bases abound in analysis is perhaps motivation enough for a systematic study. However, we make no claim as to the completeness of our discussion.

All notions are used in the bornological sense, unless otherwise specified. For the notions that are not defined here, we refer to [4]. By b.c.s. (l.c.s.) we mean a bornological space (locally convex space) and by (LB)-*space* a complete b.c.s. with a countable base. We are mainly concerned with (LB)-spaces but most of the results obtained can easily be generalized to b.c.s. for which the homomorphism or closed graph theorems hold. If E is a regular b.c.s. with dual $E^{\times}$, the familiar symbols σ and τ are used for the weak and Mackey topologies with respect to the duality $\langle E, E^{\times}\rangle$, unless otherwise stated, and we write then E_{σ} and E_{τ} with obvious meaning. Finally, following Köthe, we denote by ω the product of countably many copies of the scalar field under the product bornology.

The paper is organized as follows. Section 1 deals with separability conditions in bornological spaces, from which representation theorems for (LB)-spaces are deduced in Section 2. In Section 3 the notions of basis and Schauder basis are introduced and in Section 4 the relationship between bornological and topological Schauder bases is investigated. Section 5 studies the properties of a Schauder basis in terms of the dual sequence of bounded linear functionals, and Section 6 the connection between Schauder bases and reflexivity. Various types of Schauder bases are analysed in Section 7.

1. Separability conditions in bornological spaces. A subset of a b.c.s. E is said to be *total* in E if its linear span is dense. E is called *separable* if it has a countable total subset.

Let $\mathfrak{B}$ be the bornology of E and let $\mathfrak{D}$ be a base for $\mathfrak{B}$ consisting of bounded disks. The family $\{E_B\colon B\in\mathfrak{D}\}$ of normed spaces will be called a *representation* of E. Thus we have a biunivocal correspondence between the class of all representations of E and the class of bases for E consisting of bounded disks. We then say that E is *locally separable* if it admits a representation consisting of separable normed spaces. Subspaces and quotient spaces of locally separable spaces are again locally separable, and so for direct sums and inductive limits.

Clearly, every locally separable b.c.s. with a countable base is separable, but the converse is an open question. However we have

PROPOSITION 1. *An* (LB)*-space* $(E,\mathfrak{B})$ *is separable if and only if it is locally separable.*

Proof. In order to prove the necessity, let (x_n) be a total subset of $(E,\mathfrak{B})$ and let $\{F_k\}$ be a representation of $(E,\mathfrak{B})$ consisting of Banach spaces such that $(x_n)\cap F_1\neq\emptyset$. For each k let E_k be the closure in F_k of the subspace spanned by the set $(x_n\colon x_n\in F_k)$. Each E_k is obviously a Banach space and hence $\{E_k\}$ is a representation for a bornology $\mathfrak{B}'\subset\mathfrak{B}$ under which E is an (LB)-space. Thus $\mathfrak{B}'=\mathfrak{B}$ by the isomorphism theorem.

Since we shall deal only with complete b.c.s., a representation will always be assumed to consist of Banach spaces.

2. Representation theorems for (LB)-spaces. If G is a Banach space, we denote by $\oplus_n G$ the direct sum of countably many copies of G and by $C(I)$ the Banach space of continuous functions on $I=[0,1]$.

THEOREM 1. *Every separable* (LB)*-space is isomorphic to:*

(i) *a quotient space of* $\oplus_n l^1$,

(ii) *a closed subspace of a quotient of* $\oplus_n l^\infty$,

(iii) *a closed subspace of a quotient of* $\oplus_n C(I)$.

Proof. Let E be a separable (LB)-space and let $\{E_n\}$ be a representation of E consisting of separable Banach spaces (Proposition 1). E is a quotient of $\oplus_n E_n$; let $\varphi\colon \oplus_n E_n\to E$.

(i) For each n, E_n is separable and hence there is a homomorphism u_n which takes l^1 onto E_n. The set of mappings (u_n) defines a linear map u of $\oplus_n l^1$ onto $\oplus_n E_n$ which is again a homomorphism. Thus the composition map $\varphi\circ u$ is a homomorphism of $\oplus_n l^1$ onto E, whence E is isomorphic to $(\oplus_n l^1)/H$, where $H=(\varphi\circ u)^{-1}(0)$.

(ii) For each n there is a norm isomorphism u_n of E_n into l^∞; it follows from this the existence of an isomorphism u of $\oplus_n F_n$ onto a closed subspace of $\oplus_n l^\infty$. The subspace $H=u(\varphi^{-1}(0))$ is closed in $\oplus_n l^\infty$. Let ψ be the canonical map $\oplus_n l^\infty\to(\oplus_n l^\infty)/H$. Then the equation

$$v\circ\varphi=\psi\circ u$$

defines a linear map v of E onto a closed subspace of $(\oplus_n l^\infty)/H$. It follows by the isomorphism theorem that v is an isomorphism of b.c.s. structures.

(iii) Embed E_n into $C(I)$ and apply the proof given for (ii).

If E is not separable, Theorem 1 holds with $\oplus_n l^1$, $\oplus_n l^\infty$ and $\oplus_n C(I)$ replaced respectively by $\oplus_n l^1(A_n)$, $\oplus_n l^\infty(A_n)$ and $\oplus_n C(K_n)$, where for each n, A_n is a set with a suitable cardinal and K_n a compact space.

Observe that (ii) and (iii) of Theorem 1 cannot be improved, in the sense that, in general, there is no bounded embedding of a locally separable (LB)-space E into either space $\oplus_n l^\infty$ or $\oplus_n C(I)$, in view of the fact that each of the latter spaces has a separating dual, while the dual of E may reduce to $\{0\}$.

We shall now give a more explicit version of Theorem 1 (ii) by showing that for a regular, separable (LB)-space the quotient of $\oplus_n l^\infty$ can always be chosen to be a suitable co-echelon space. We recall the definition of a (bornological) co-echelon space.

Let (a_k^n) be a double sequence such that $0<a_k^n\leqslant a_k^{n+1}$ for all k, n and let E_n be the space of all sequences (x_k) for which

$$\sup_k\frac{|x_k|}{a_k^n}<\infty.$$

With norm

$$\|(x_k)\|_n=\sup_k\frac{|x_k|}{a_k^n}$$

E_n is a Banach space. Let $a^{(n)}$ denote the sequence (a_k^n) for fixed n; then the sequences $a^{(n)}$ are called *steps*. The *co-echelon space* (of order 1) corresponding to the given system of steps is (bornologically defined to be) the b.c.s. $E=\varinjlim E_n$. It is known that E is a regular (LB)-space. Moreover, since each E_n is norm isomorphic to l^∞, E is isomorphic to a quotient space of $\oplus_n l^\infty$. We have:

THEOREM 2. *Every regular, separable* (LB)-*space is isomorphic to a closed subspace of a co-echelon space.*

Proof. Let E be a regular, separable (LB)-space and let $\{E_n\}$ be a representation of E by separable Banach spaces. For each n, E'_n is a $\sigma(E'_n, E_n)$-separable Banach space. Denote by S'_n the unit sphere of E'_n, by B_n the unit ball of E_n and by p_n the gauge of B_n° in $E^\times$. p_n is a seminorm on $E^\times$ with $p_n(f) = 1$ for $f \in S'_n \cap E^\times$. Since E is regular, $E'_n \cap E^\times$ is $\sigma(E'_n, E_n)$-dense in E'_n for all n by [2], Théorème 1. From this and from the $\sigma(E'_n, E_n)$-separability of E'_n it follows that we can inductively find disjoint sets N_n of integers and elements $(f_k\colon k \in N_n) \subset S'_n \cap E^\times$ satisfying the following conditions

(a) $\{N_n\}$ is a covering for the integers,

(b) the set

$$\bigcup_{i=1}^{n} \bigcup_{k \in N_i} \left\{ \frac{f_k}{p_n(f_k)} \right\}$$

is $\sigma(E'_n, E_n)$-dense in S'_n.

We now set $a_k^n = p_n(f_k)$ for all k, thus obtaining a double sequence as in the definition of a co-echelon space. Let us form the co-echelon space F corresponding to (a_k^n) and, for each $x \in E$, let the sequence (x_k) be defined by $x_k = f_k(x)$. If $x \in E_n$, condition (b) above yields

$$\|(x_k)\|_n = \sup_k \frac{|x_k|}{a_k^n} = \sup_k \frac{|f_k(x)|}{p_n(f_k)} = \sup \left\{ \frac{|f_k(x)|}{p_n(f_k)} : k \in \bigcup_{i=1}^{n} N_i \right\} = \|x\|_n$$

so that the mapping $x \to (x_k)$ is an isomorphism of E onto a subspace of F. Since this subspace is obviously complete, it is also closed.

3. Bornological spaces with a basis. A *basis* of a b.c.s. E is a sequence (x_n) such that every $x \in E$ has a unique expansion

$$x = \sum_{n=1}^{\infty} a_n x_n, \tag{1}$$

the series being convergent for the bornology of E.

The *dual sequence* associated with a basis (x_n) is the sequence (f_n) in E^* defined by

$$f_n(x_n) = 1 \quad \text{and} \quad f_n(x_k) = 0 \quad \text{for } n \neq k.$$

If $(f_n) \in E^\times$, then (x_n) is called a *Schauder basis*. The normed space in ([11], Example 16.1, p. 160) has a basis which is not a Schauder basis.

A *Schauder basic sequence* is a sequence (x_n) which is a Schauder basis for its closed linear span.

If (x_n) is a Schauder basis, a *Schauder block basic sequence* is a sequence (y_k) of the form

$$y_k = \sum_{i=m_{k-1}+1}^{m_k} a_i x_i,$$

where (m_k) is a strictly increasing sequence of positive integers and $m_0 = 0$. It is easily seen that (y_k) is actually a basic sequence. The following lemma is of fundamental importance for the theory.

LEMMA 1. *Let E be an* (LB)-*space with a basis (x_n). There is a representation $\{E_k\}$ of E such that $(x_n) \cap E_k$ is a basis in E_k for all k.*

Proof. Let $\{F_k\}$ be a representation of E such that $(x_n) \cap F_1 \neq \emptyset$. For each k let $N_k = \{n\colon x_n \in F_k\}$ and

$$G_k = \Big\{ (a_n) \in \omega\colon a_n = 0 \text{ for } n \notin N_k \text{ and } \sum_{n \in N_k} a_n x_n \text{ converges in } F_k \Big\}.$$

By [11], Proposition 3.1, p. 18, G_k is a Banach space under the norm

$$\|(a_n)\|_k = \sup_{n \in N_k} \Big\| \sum_{\substack{i \in N_k \\ i \leq n}} a_i x_i \Big\|_k$$

where the second norm is in F_k. Since (x_n) is a basis for E, the elements $(e^j\colon j \in N_k)$ defined by $e_j^j = 1$ and $e_n^j = 0$ for $n \neq j$ form a basis for G_k. Now $N_k \subset N_{k+1}$ and $\|x\|_{k+1} \leq \|x\|_k$ for all $x \in F_k$ imply

$$\|(a_n)\|_{k+1} \leq \|(a_n)\|_k$$

for all $(a_n) \in G_k$ and hence each embedding $G_k \to G_{k+1}$ is bounded. Let $G = \varinjlim G_k$. Define a linear map $u\colon G \to E$ by $u(a_n) = \sum_n a_n x_n$. Because (x_n) is a basis for E, u is biunivocal and onto. Moreover u is bounded, since

$$\|u(a_n)\|_k = \Big\| \sum_n a_n x_n \Big\|_k \leq \sup_{n \in N_k} \Big\| \sum_{\substack{i \in N_k \\ i \leq n}} a_i x_i \Big\|_k = \|(a_n)\|_k$$

for all $(a_n) \in G_k$ and for all k. Thus u is an isomorphism. But then denoting by E_k the Banach space $u(G_k)$ under the norm of G_k we obtain the desired representation, since $u(e^n) = x_n$ for all n.

THEOREM 3. *Let E be an* (LB)-*space. Every basis of E is a Schauder basis.*

Proof. Let (x_n) be a basis of E; by Lemma 1 E has a representation $\{E_k\}$ such that $(x_n) \cap E_k$ is a basis of E_k for all k. Let $\| \ \|_k$ be the norm of E_k. Define a second norm on E_k by

$$|||x|||_k = \sup_n \Big\| \sum_{i=1}^{n} f_i(x) x_i \Big\|_k \qquad (x \in E_k), \tag{2}$$

where $(f_n) \subset E^*$ is the dual sequence of (x_n) and $f_i(x) = 0$ if $x_i \notin E_k$. E_k is complete for the norm (2) and hence, since $\|x\|_k \leqslant |||x|||_k$ for all x in E_k, (2) is an equivalent norm on E_k. It follows that there exists a sequence (c_k) of positive numbers such that

$$|||x|||_k \leqslant c_k \|x\|_k \quad (x \in E_k).$$

Now for every n there exists k_n such that $x_i \in E_{k_n}$ for $1 \leqslant i \leqslant n$. For all $k \geqslant k_n$ we have

$$|f_n(x)|\,\|x_n\|_k = \Big\| \sum_{i=1}^{n} f_i(x) x_i - \sum_{i=1}^{n-1} f_i(x) x_i \Big\|_k \leqslant 2\,|||x|||_k \leqslant 2c_k \|x\|_k$$

for all $x \in E_k$, and hence $f_n \in E^\times$.

Theorem 3 is a generalization of Banach's well-known result ([1], p. 111). The proof given, which generalizes an idea of Banach [1], shows, also through the proof of Lemma 1, the heavy dependence of the conclusion upon the isomorphism theorem (or closed graph theorem), in line with Mityagin's observation ([8], p. 92).

Corollary 1. *Every* (LB)-*space with a basis is regular.*

It follows that every separable (LB)-space with a trivial dual has no basis. (This statement, Theorem 3 and Corollary 1 are the analogues of similar statements valid for complete, metrizable, topological linear spaces.)

Let us now introduce some notations and definitions. If E, F are b.c.s. (l.c.s.), we denote by $L(E,F)$ $(\mathscr{L}(E,F))$ the space of all bounded (continuous) linear maps from E to F. For $L(E,E)$ $(\mathscr{L}(E,E))$ we simply write $L(E)$ $(\mathscr{L}(E))$.

A subset H of $L(E,F)$ is said to be *equibounded* if it is bounded for the natural bornology ([3], Definition 1, p. 239) of $L(E,F)$, i.e. if the set

$$H(B) = \bigcup_{u \in H} u(B)$$

is bounded in F for every bounded subset B of E. H is *simply bounded* if $H(x)$ is bounded in F for every $x \in E$.

If now (x_n) is a Schauder basis of the b.c.s. E with dual sequence (f_n), let

$$s_n(x) = \sum_{k=1}^{n} f_k(x) x_k \quad (x \in E)$$

be the *nth-partial sum operator* from E to E. Then $(s_n) \subset L(E)$ and (x_n) is called an *equi-Schauder basis* if (s_n) is an equibounded sequence in $L(E)$.

It is well known that in a Banach space every Schauder basis is an equi-Schauder basis. This generalizes to complete b.c.s. as follows.

Theorem 4. *If E is an* (LB)-*space or a complete, topological b.c.s., then every Schauder basis of E is an equi-Schauder basis.*

Proof. The sequence of linear operators (s_n) converges pointwise to the identity and hence is simply bounded. If E is an (LB)-space or a complete, topological b.c.s., then E_τ is barreled, whence (s_n) is an equicontinuous subset of $\mathscr{L}(E_\tau)$. But then (s_n) is equibounded in $L(E)$ by [3], Théorème 2, p. 241.

4. Bornological and topological bases. From now we shall simply write "basis" to mean a Schauder basis.

The question arises as to the relationship between bornological and topological bases, that is, under what conditions a bornological (topological) basis for a b.c.s. E (l.c.s. E_τ) is also a topological (bornological) basis for the l.c.s. E_τ (b.s.c. E) associated with $E(E_\tau)$. In one direction there is an easy result:

Proposition 2. *Let E be a regular b.c.s. with a basis (x_n). Then (x_n) is also a basis for E_τ (and E_σ).*

Proof. Clearly, the linear span of (x_n) is dense in E_τ (E_σ) and every $x \in E$ has an expansion of the form (1), the series being convergent in $E_\tau(E_\sigma)$. It remains to show that this expansion is unique. Hence, suppose that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k x_k = 0 \quad \text{in } E_\tau(E_\sigma).$$

Since (x_n) is a Schauder basis for E, the dual sequence (f_n) belongs to $E^\times$ and hence each f_n is a τ-continuous (σ-continuous) linear functional on E. For each n set

$$s_n = \sum_{k=1}^{n} \alpha_k x_k,$$

and let m be fixed. We have, for $n \geqslant m$,

$$\alpha_m = f_m(s_n) = \lim_{n\to\infty} f_m(s_n) = 0,$$

and therefore $\alpha_n = 0$ for all n.

The converse of Proposition 2 is false, since there are non-separable, regular b.c.s. which are τ-separable, e.g. the space E' in [10], Exercise 20, p. 195, under its equicontinuous bornology.

Theorem 5. *Let E be a complete, topological b.c.s. and let (x_n) be a basis for E_τ. If (x_n) is total in E, then it is also a basis for E.*

Proof. Since $(s_n(x))$ converges to x in E_τ, the sequence (s_n) is simply bounded, hence equicontinuous, for E_τ is barreled. But then (s_n) is equibounded. Now the linear span of (x_n) is dense in E and hence, given

$x \in E$, there is a completant, bounded disk B and a sequence (y_k) of linear combinations of (x_n) which converges to x in the Banach space E_B. Since (s_n) is equibounded, there is a completant, bounded disk C such that (s_n) is a uniformly bounded subset of $L(E_B, E_C)$. It follows, with self-explanatory notation, that

$$\|s_n\|_{BC} = \sup_{x \in B} \frac{\|s_n(x)\|_C}{\|x\|_B} = M < \infty \quad \text{for all } n.$$

Let D be a completant, bounded disk in E containing B and C. Given $\varepsilon > 0$, there exists k such that $\|x - y_k\|_B < \varepsilon(1+M)^{-1}$. Let $y_k = \sum_{i=1}^{m} a_i x_i$; then for every $n \geqslant m$ we have

$$s_n(y_k) = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_i f_j(x_i) = \sum_{j=1}^{m} a_j x_j = y_k.$$

Therefore, for all $n \geqslant m$,

$$\|x - s_n(x)\|_D \leqslant \|x - y_k\|_D + \|s_n(x) - s_n(y_k)\|_D$$
$$\leqslant \|x - y_k\|_B + \|s_n(x) - s_n(y_k)\|_C \leqslant (1+M)\|x - y_k\|_B < \varepsilon,$$

which shows that the sequence $(s_n(x))$ is convergent to x in E.

The proof of the uniqueness of the expansion

$$x = \sum_{n=1}^{\infty} f_n(x) x_n$$

is immediate.

In particular, it follows from the above theorem that every topological basis of a strict inductive limit E of a sequence of Fréchet l.c.s. is also a bornological basis for the topological bornology of E.

Corollary 2. *Let E be a complete, topological* b.c.s. *and let (x_n) be a basis of E_σ. If (x_n) is total in E, then it is also a basis for E.*

Proof. Since E_τ is barreled, the assertion follows from Theorem 5 and the well-known fact that in a barreled space a weak basis is also a basis for the Mackey topology [7].

5. Duality properties. A *biorthogonal system* is a sequence $(x_n, f_n) \in E \times E^\times$ such that $f_n(x_n) = 1$ and $f_k(x_n) = 0$ for $k \neq n$.

Proposition 3. *Let E be a regular b.c.s. and let (x_n, f_n) be a biorthogonal system such that (x_n) is total in E. For (x_n) to be a basis for E it is necessary (and also sufficient when E is complete and topological) that (f_n) be a weak basis for $E^\times$.*

Proof. The necessity is immediate. For the sufficiency, observe that $(s_n(x))$ converges weakly to x for every x in E, and hence (x_n) is a weak basis of E. The hypothesis and Corollary 2 then imply the assertion.

In order to discuss the duality properties when $E^\times$ is given its natural topology, we need the following lemma which the reader can easily establish by a routine argument.

Lemma 2. *Let E, F be regular* b.c.s. *and, for a subset H of $L(E, F)$, denote by H' the set of adjoint maps. Then:*

(i) *$u \in L(E, F)$ implies $u' \in \mathscr{L}(F^\times, E^\times)$,*

(ii) *if H is equibounded in $L(E, F)$, then H' is equicontinuous in $\mathscr{L}(F^\times, E^\times)$.*

Moreover, if F is polar the converses of both these assertions are true.

From now on we assume, unless otherwise stated, that E is either an (LB)-space or a complete, topological b.c.s. $[f_n]$ will denote the closed linear span in $E^\times$ of the dual sequence for (x_n).

For each n, the adjoint s_n' of the operator s_n is given by

$$s_n'(f) = \sum_{k=1}^{n} f(x_k) f_k \qquad (f \in E^\times),$$

and we have:

Proposition 4. *If (x_n) is a basis for E, then (f_n) is an equi-Schauder basic sequence in $E^\times$.*

Proof. By Theorem 4 and Lemma 2, (s_n') is an equicontinuous set in $\mathscr{L}(E^\times)$. Since $s_n'(f)$ converges to f (in the $E^\times$-topology) for every f which is a finite linear combination of the f_n, it converges also to f in the (topological) closure of the set of these linear combinations, hence

$$f = \sum_{n=1}^{\infty} f(x_n) f_n \qquad (f \in [f_n]),$$

the expansion being clearly unique. Thus (f_n) is a basic sequence in $E^\times$, since $(x_n) \subset E^{\times\prime}$.

Corollary 3. *If (x_n) is a basis for E and E is reflexive, then (f_n) is a basis for $E^\times$.*

Proposition 5. *Let E be a polar b.c.s. whose dual $E^\times$ is barreled, and let (x_n, f_n) be a biorthogonal system with (x_n) total in E. If (f_n) is a basis for $E^\times$, then (x_n) is a basis for E.*

Proof. Since (s_n') is equicontinuous, (s_n) is equibounded by Lemma 2. The rest of the proof proceeds as in Theorem 5.

We shall not here pursue further the investigation of duality properties.

6. Bases and reflexivity. A basis (x_n) for E is called *boundedly complete* if $\left(\sum_{k=1}^{n} a_k x_k\right)$ converges in E whenever it is a bounded set, and *shrinking* if (f_n) is a basis for $E^\times$.

The following theorem generalizes a well-known result of James ([5], Theorem 1, p. 519) for Banach spaces with a basis, and it is the bornological analogue of the corresponding topological generalization of Retherford ([9], Theorem 2.3, p. 281) to a barreled l.c.s. with a basis.

THEOREM 6. *If (x_n) is a basis for E, then E is reflexive if and only if (x_n) is shrinking and boundedly complete.*

Proof. If E is reflexive, then (x_n) is shrinking by Corollary 3. Now suppose that $\left(\sum_{k=1}^{n} a_k x_k\right)$ is a bounded sequence in E; the reflexivity of E implies the existence of a weakly adherent point $x \in E$. Clearly, $x = \sum_{k=1}^{\infty} f_k(x) x_k$ weakly, with $f_k(x) = a_k$ for all k. But this expansion holds bornologically, and so (x_n) is boundedly complete.

Conversely, let (x_n) be a boundedly complete and shrinking basis for E. Let E be an (LB)-space and let $\{E_k\}$ be a representation of E such that $(x_n) \cap E_k$ is, for all k, a basis for E_k (see Lemma 1). For each k let p_k be the projection of $E^\times$ into E'_k and let F_k be the closed linear span of $(p_k(f_n))$ in E'_k. Since (x_n) is shrinking, we have

$$E^\times = \varprojlim E'_k = \varprojlim F_k \text{ (topologically)}. \tag{3}$$

Let F'_k be the dual of the Banach space F_k under the Mackey topology $\tau(F'_k, F_k)$. Since the second projective limit in (3) is reduced, its dual $E^{\times\prime}$, under its Mackey topology $\tau(E^{\times\prime}, E^\times)$, is the topological inductive limit of the sequence $\{F'_k\}$ ([10], (4.4), p. 139). Algebraically we have

$$E^{\times\prime} = \bigcup_{k=1}^{\infty} F'_k, \quad \text{with } F'_k \subset F'_{k+1}.$$

Let $z \in E^{\times\prime}$ and let k be such that $z \in F'_k$. By [11], Theorem 12.5 c, p. 129, F'_k is isomorphic (as a Banach space) under the mapping $z \to (z(f_n))$, to the Banach space of sequences of scalars

$$\Big\{(a_n)\colon \sup_{n \in N_k} \Big\| \sum_{\substack{i \in N_k \\ i \leq n}} a_i x_i \Big\|_k < \infty\Big\},$$

where $N_k = \{n\colon p_k(f_n) \in E'_k\}$, and the norm is in E_k. With this convention in mind, we see that the sequence $\Big(\sum_{\substack{i, n \in N_k \\ i \leq n}} z(f_i) x_i\Big)$ is bounded in E_k and hence in E. It therefore converges to an $x \in E$, since (x_n) is boundedly complete, and clearly $f_i(x) = z(f_i)$ for all $i \in N_k$. Since $F'_k = E''_k / F_k^{\circ}$, we have $z = x$, and so E is semi-reflexive, whence reflexive (isomorphism theorem).

Suppose now that E is a complete, topological b.c.s. If $z \in E^{\times\prime}$ then

$$f\Big(\sum_{i=1}^{n} z(f_i) x_i\Big) = z\Big(\sum_{i=1}^{n} f(x_i) f_i\Big) \to z(f) \quad (f \in E),$$

since (x_n) is shrinking; the sequence $\left(\sum_{i=1}^{n} z(f_i) x_i\right)$ is therefore weakly bounded in E. But then it is also bounded, and since (x_n) is boundedly complete there exists an x in E such that

$$x = \sum_{i=1}^{\infty} z(f_i) x_i.$$

It follows that $x = z$ and, consequently, E is reflexive.

7. Regular, bounded and normal bases. The definitions and results in this and the next section should be compared with [6].

Denote by μ the Mackey-closure topology of a b.c.s. E. A sequence (x_n) in E is said to be *regular* if it is disjoint from some μ-neighbourhood of 0, so that no subsequence of a regular sequence converges to 0. We say that (x_n) can be *regularized* if there is a sequence (a_n) of scalars such that $(a_n x_n)$ is regular. The natural basis (x_n) of ω cannot be regularized (for every sequence (a_n) of scalars, the sequence $(a_n x_n)$ converges to zero in ω_τ, and so bornologically since ω_τ is metrizable).

PROPOSITION 6. *Let (x_n) be a basis of E with dual sequence (f_n). Then (x_n) is regular if and only if (f_n) is equibounded.*

Proof. If (f_n) is equibounded, then it is τ-equicontinuous (E_τ is barreled) and hence $V = \frac{1}{2}(f_n)^\circ$ is a τ-neighbourhood of 0. Since τ is coarser than μ and $x_n \notin V$ for all n, (x_n) is regular. Conversely, let (x_n) be regular. Since $x = \sum_{n=1}^{\infty} f_n(x) x_n$ for all $x \in E$, we have $\lim_{n\to\infty} f_n(x) x_n = 0$ in E and so $\lim_{n\to\infty} f_n(x) = 0$. Thus (f_n) is weakly bounded, hence τ-equicontinuous and so equibounded.

We recall that a sequence (x_n) is *regular* in a l.c.s. if it is disjoint from some neighbourhood V of 0 ([6], Definition 1.1). A priori, a basis in E could be regular without being τ-regular; that this is not the case is shown by the following:

PROPOSITION 7. *A basis (x_n) of E is regular if and only if it is τ-regular.*

Proof. Sufficiency is obvious. For the necessity observe that (f_n) is equibounded by Proposition 6, hence τ-equicontinuous, which implies the τ-regularity of (x_n) as in the proof of Proposition 6.

Next we show how regularizability of a basis gives information about the space.

THEOREM 7. *If E has a basis (x_n) then the following assertions are equivalent.*

(a) *$(a_n x_n)$ is regular for some sequence (a_n),*

(b) *$(\beta_n x_n)$ is equibounded for some sequence (β_n) of non-zero scalars,*

(c) *there exists on E a bounded norm,*

(d) *no closed subspace of E is isomorphic to ω,*

(e) *there is no subsequence (x_{n_k}) of (x_n) such that $\sum_{k=1}^{\infty} a_k x_{n_k}$ exists for all sequences (a_k).*

Proof. (a) implies (b) by Proposition 6, since $(\beta_n f_n)$ is the dual sequence of $(a_n x_n)$ when $\beta_n = a_n^{-1}$.

If (b) holds, then it is immediately seen that $\|x\| = \sup_n |\beta_n f_n(x)|$ is a bounded norm on E.

Obviously, (c) implies (d), since there is no bounded norm on ω.

Now suppose that (d) holds and that there is a subsequence (x_{n_k}) of (x_n) such that $\sum_{k=1}^{\infty} a_k x_{n_k}$ exists for all (a_k). Let

$$F = \Big\{\sum_{k=1}^{\infty} a_k x_{n_k}:\ (a_k) \in \omega\Big\};$$

it is easily seen that F is closed in E and hence complete. Define a linear mapping u from ω to F by

$$u(a_k) = \sum_{k=1}^{\infty} a_k x_{n_k}.$$

Then u is one-to-one, onto and bounded (direct verification) and hence an isomorphism by ([4], Corollaire 2, p. 43). Thus (d) implies (e).

Finally, to show that (e) implies (a), let us assume, first of all, that E is complete and topological. Let (a_n) be an arbitrary sequence and suppose that for every subsequence $(a_{n_k} x_{n_k})$ and μ-neighbourhood V of 0 there exists k_V such that $a_{n_k} x_{n_k} \in V$ for all $k \geqslant k_V$. If follows that $(a_{n_k} x_{n_k})$ converges to 0 for μ, and so it contains a subsequence (bornologically) convergent to 0 ([4], Théorème 1, p. 15). But then $(a_n x_n)$ converges to 0 for μ ([4], Théorème 1, p. 15) and so is a bounded sequence, which implies that $\sum_{n=1}^{\infty} a_n x_n$ exists, since (a_n) was arbitrary. Thus (e) implies the existence of a sequence (a_n) with a subsequence (a_{n_k}) such that $(a_{n_k} x_{n_k})$ is regular and an induction process finishes the proof for the case of E complete and topological. If now E is an (LB)-space, then it is clear that the sequence $(\|x_n\|_n^{-1} x_n)$ is regular (here we take a representation $\{E_n\}$ of E such that $x_n \in E_n$ for all n) by $\|x_n\|_n |f_n(x)| \leqslant 2|||x|||_n$ and Proposition 6.

Since (a) does not depend on (e) when E is an (LB)-space, we have also proved:

COROLLARY 4. (a)–(e) *of Theorem* 7 *hold in every* (LB)-*space with a basis.*

COROLLARY 5. *Every basis of an* (LB)-*space can be regularized.*

This reflects a well-known property of bases in Banach spaces.

COROLLARY 6. *If E is a separable* (LB)-*space on which there is no bounded norm, then E has no basis.*

The following dual form of Proposition 6 holds.

PROPOSITION 8. *A basis (x_n) of E is bounded if and only if the dual sequence (f_n) is a topologically regular basic sequence in $E^\times$.*

Proof. If (x_n) is bounded, the set $V = \frac{1}{2}(x_n)^\circ$ is a neighbourhood of 0 in $E^\times$ and $f_n \notin V$ for all n. Thus (f_n) is regular.

Conversely, suppose (f_n) regular; then there is a bounded set B in E with

$$\sup\{|f_n(y)|:\ y \in B\} > 1 \quad \text{for all } n,$$

and hence B contains a sequence (y_n) such that $|f_n(y_n)| > 1$. By Theorem 4 there is a bounded subset C of E with

$$\bigcup_{n=1}^{\infty} s_n(B) - \bigcup_{n=1}^{\infty} s_n(B) \subset C.$$

Since

$$f_n(y_n) x_n = s_n(y_n) - s_{n-1}(y_n) \in C$$

for all n, the sequence $(f_n(y_n) x_n)$ is bounded and hence so must be (x_n), for $|f_n(y_n)| > 1$.

A sequence which is both bounded and regular will be called *normal*. Then we have:

PROPOSITION 9. *A basis (x_n) of E is normal if and only if the dual sequence (f_n) is a topologically normal basic sequence in $E^\times$.*

Proof follows from Propositions 6 and 8 and the fact that E_τ is barreled.

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Received April 4, 1973 (669)

STUDIA MATHEMATICA, T. L. (1974)

Decompositions of operator-valued functions in Hilbert spaces

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Abstract. In the present paper we will prove some theorems concerning the canonical decompositions of operator-valued functions in Hilbert spaces. We consider positive definite, completely positive, completely contractive functions and representations of subalgebras of C^*-algebras. Moreover we give some corollaries about dilatable functions.

To begin with we introduce some notation and definitions. We denote by H the Hilbert space with the inner product $(\ ,\ )$. $L(H)$ stands for the algebra of all linear, bounded operators in H. For $A \in L(H)$ we write $R(A) = \{Ax, x \in H\}$. I_H stands for the identity operator in H. If M is a closed subspace of H then $M^\perp$ denotes the orthogonal complement of M. An operator $P \in L(H)$ such that $P = P^2 = P^*$ is called a *projection*. If P is a projection onto the subspace M then P denotes the projection onto $M^\perp$. $A \in L(H)$ is a contraction (or contractive operator) if $\|A\| \leqslant 1$. Every involution preserving homomorphism of involutive Banach algebra B into $L(H)$ is called **-representation* of B. Every homomorphism of B into $L(H)$ is called a *representation* of B.

It is well known (see [11], ch. I.3.2) that for a contraction $T \in L(H)$ there are subspaces $H_0, H_1 = H_0^\perp$ reducing T such that the operator $T_0 = T|_{H_0}$ is the unitary operator in H_0 and $T_1 = T|_{H_1}$ is completely non-unitary. The decomposition $T = T_0 \oplus T_1$ is uniquely determined. It is called the *canonical decomposition of* T.

Every contraction $T \in L(H)$ induces a representation Π (by the J. von Neumann inequality) of the disc algebra $A(\Gamma)$ into $L(H)$ such that $\Pi(1) = I_H$, $\Pi(z) = T$ and $\|\Pi\| \leqslant 1$. $A(\Gamma)$ consists of all holomorphic functions in the open unit disc $\{|z| < 1\}$ continuous in its closure $\{|z| \leqslant 1\}$; $\Gamma = \{|z| = 1\}$. The representation Π has the following property: T is unitary if and only if there is a *-representation $\hat{\Pi}\colon C(\Gamma) \to L(H)$ which is an extension to $C(\Gamma)$ of the representation Π. If such $\hat{\Pi}$ exists then it is unique.

The reinterpretation of the canonical decomposition reads as follows: There exist subspaces $H_0, H_1 = H_0^\perp \subset H$ reducing $\Pi(u)$ for all $u \in A(\Gamma)$