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A generalized contraction mapping theorem in E-metric spaces

by

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Abstract. This paper addresses itself to general theorems on the convergence of a sequence generated via $x_{n+1} = Fx_n$ to a fixed point of the operator F; the best known such theorem is the so-called contraction mapping theorem of Banach. Here we prove two main theorems which include as special cases many previous generalizations of Banach's theorem.

1. Introduction. We consider the problem of locating a fixed point of a nonlinear operator F mapping a space X into itself; that is, we seek x^* satisfying $x^* = Fx^*$. Here we assume that X is a complete E-metric space with E-metric $d(\cdot,\cdot)$. That is, E is a partially ordered vector space in which the notion of convergence to zero is defined, with the usual compatibility hypotheses among convergence, vector space operations, and order being assumed, d maps $X \times X$ into $\{e \mid e \in E, e \geqslant 0\} \equiv E^+$ and satisfies (1) $d(x, y) = 0$ if and only if $x = y$ and (2) $d(x, y) \leqslant d(x, z) + d(y, z)$ for all x, y, z in X; in X, a sequence $\{x_n\}$ converges to x if and only if $d(x_n, x)$ converges to zero in E, and it is assumed that every Cauchy sequence in X converges to a point of X. For a derivation of the properties of E-metric spaces, called by some authors (Altman [3], Collatz [5]) "pseudometric" spaces, the reader is referred to [8].

Perhaps the best known fixed point theorem is the contraction mapping theorem which asserts that if $d(Fx, Fy) \leqslant cd(x, y)$ where $c < 1$ and E is the real number system $\boldsymbol{R}$, then there exists a unique fixed point x^* which may be computed as the limit of the sequence defined by $x_{n+1} = Fx_n$, with x_0 arbitrary. We state two main theorems which generalize these results in Section 2, and in Section 3 we show how many previous generalizations are included in ours.

2. The main results. First we state a theorem which gives existence and convergence but not necessarily uniqueness. Let A_k, for each $k \geqslant 0$, be a mapping of E^+ into E^+ satisfying $A_k e_1 \leqslant A_k e_2$ if $e_1 \leqslant e_2$. For nonnegative integers n, define F^n via $F^0 = I$ (the identity mapping) and $F^{n+1}x = F(F^n x)$.

THEOREM 1. *Let F, X, E, d, and $\{A_k\}_0^\infty$ be as above. Suppose that there exist operators $G\colon X\to E^+$ and $P\colon X\to X$, and nonnegative integers r and s such that*

(2.1) $$d(F^{s+1}x, F^s x)\leqslant A_0G(x) \quad \text{for all } x \text{ in } X,$$

(2.2) $$G(F^{rk}x)\leqslant A_kG(x) \quad \text{for all } x \text{ in } X \text{ and all } k\geqslant 1,$$

(2.3) *if x_n converges to x^* in X and $d(F^{s+1}x_n, F^s x_n)$ converges to zero in E, then $F^{s+1}x^* = F^s x^*$,*

(2.4) *the series $\sum_{k=1}^{\infty}A_0A_ke$ converges in E, where $e=\sum_{j=1}^{2r-1}G(x_j)$.*

Then, for any x_0 in X, the sequence generated by $x_{n+1}=Fx_n$ converges to x^* satisfying $F^{s+1}x^*=F^sx^*$, so that $y^*=F^sx^*$ is a fixed point of F. In addition, $G(x_n)$ converges to zero in E. The following error estimate is valid:

(2.5) $$d(x^*, x_i)\leqslant r\sum_{p=k}^{\infty}A_0A_pe,$$

where k is the integer part of i/r and e is as defined in (2.4).

Proof. We shall show first that $\{x_n\}_0^\infty$ is a Cauchy sequence. We write, for $i\geqslant s$, $d(x_{i+1}, x_i)=d(F^{s+1}x_{i-s}, F^sx_{i-s})\leqslant A_0G(x_{i-s})$. Thus, for $i\geqslant s$ and $l\geqslant 0$, we have

$$d(x_{i+l}, x_i)\leqslant\sum_{j=0}^{l-1}d(x_{i+j+1}, x_{i+j})\leqslant\sum_{j=0}^{l-1}A_0G(x_{i+j-s}).$$

Now, we can write l uniquely as $mr+n$ with $0\leqslant n\leqslant r-1$ and i uniquely as $kr+h$ with $s\leqslant h\leqslant s+r-1$. Rearranging the sum in the above inequality yields

$$\begin{aligned} d(x_{i+l}, x_i) &\leqslant \sum_{p=0}^{m}\sum_{q=1}^{r}A_0G(x_{i+pr+q-s}) = \sum_{p=0}^{m}\sum_{q=1}^{r}A_0G(x_{kr+pr+h+q-s}) \\ &= \sum_{p=0}^{m}\sum_{q=1}^{r}A_0G(F^{r(k+p)}x_{h+q-s}) \leqslant \sum_{p=0}^{m}\sum_{q=1}^{r}A_0A_{k+p}G(x_{h+q-s}) \\ &\leqslant \sum_{p=0}^{m}\sum_{q=1}^{r}A_0A_{k+p}e, \end{aligned}$$

where

$$e=\sum_{i=1}^{2r-1}G(x_i).$$

Thus we have

(2.6) $$d(x_{i+l}, x_i)\leqslant r\sum_{p=k}^{k+m}A_0A_pe.$$

Since $\sum_{k=0}^{\infty}A_0A_ke$ converges, we can make $d(x_{i+l}, x_i)$ near zero by making i, and hence k, large; thus $\{x_n\}_0^\infty$ is a Cauchy sequence. Let x^* be its limit. Since $d(x_{i+1}, x_i)$ converges to zero and $d(x_{i+1}, x_i)=d(F^{s+1}x_{i-s}, F^sx_{i-s})$, and since $\{x_{i-s}\}_s^\infty$ converges to x^*, we conclude from (2.3) that $F^{s+1}x^*=F^sx^*$ and hence $y^*=F^sx^*$ is a fixed point of F. The above inequalities show that $G(x_n)$ converges to zero. The error estimate follows by letting l, and hence m, tend to infinity in (2.6).

Remarks. It is easy to modify the above theorem, using (2.6), so that the hypotheses need only hold in the "ball" of "radius" $r\sum_{p=1}^{\infty}A_0A_pe$ about the point x_0. For most applications, the mapping G is given by $G(y)=d(Py, z)$ for some fixed z in X; usually, in fact, one has z equal to zero. Since we want to consider later several instances of this special case and since our uniqueness result is related to this special case, we now state it as a corollary.

COROLLARY 2. *Let F, X, E, d, and $\{A_k\}_0^\infty$ be as above. Suppose that there exists an operator $P\colon X\to X$, an element z in X, and nonnegative integers r and s such that* (2.3) *and*

(2.1′) $$d(F^{s+1}x, F^sx)\leqslant A_0d(Px, z) \quad \text{for all } x \text{ in } X,$$

(2.2′) $$d(PF^{rk}x, z)\leqslant A_kd(Px, z) \quad \text{for all } x \text{ in } X \text{ and } k\geqslant 1,$$

(2.4′) *the series $\sum_{k=1}^{\infty}A_0A_ke$ converges in E, where*

$$e=\sum_{j=1}^{2r-1}d(Px_j, z)$$

are all valid. Then, for any x_0 in X, the sequence generated by $x_{n+1}=Fx_n$ converges to an x^* satisfying $F^{s+1}x^*=F^sx^*$, so that $y^*=F^sx^*$ is a fixed point of F. In addition, Px_n converges to z, and the error estimate of (2.5) *is valid with e defined in* (2.4′).

Under the hypotheses of Theorem 1 or its corollary, the fixed point y^* need not be unique; to give a uniqueness theorem, we thus need stronger hypotheses. The reader should note the similarities of these hypotheses with those of Corollary 2.

THEOREM 3. *In addition to the hypotheses of Corollary 2, assume that for some fixed operator $Q\colon X\to X$ we have*

(2.1″) $$d(F^sy, F^sx)\leqslant A_0d(Qy, Qx) \quad \text{for all } x, y \text{ in } X,$$

(2.2″) $$d(QF^{rk}y, QF^{rk}x)\leqslant A_kd(Qy, Qx) \quad \text{for all } x, y \text{ in } X \text{ and all } k\geqslant 1,$$

(2.4″) $$A_0A_ne \quad \text{converges to zero for every } e \text{ in } E^+.$$

Then the y^* generated in Corollary 2 is the unique fixed point of F.

Proof. Suppose y is also a fixed point of F; then y and y^* are both fixed points of F^n for all $n \geqslant 1$. Thus $d(y, y^*) = d(F^{s+rn}y, F^{s+rn}y^*) \leqslant A_0 d(QF^{rn}y, QF^{rn}y^*) \leqslant A_0 A_n d(Qy, Qy^*)$ which converges to zero.

Remarks. As before, this result can be stated locally rather than globally. Essentially the idea here is that we think of P in Corollary 2 as having the form $P = QF - Q$; we see that (2.1″) and (2.2″) essentially give (2.1′) and (2.2′) then when we take $y = Fx$.

3. Some special cases. We now wish to show that many previous fixed point theorems can be deduced from our general result in Theorem 1 and from its corollary. Since for this purpose Corollary 2 is sufficient, in the interest of brevity we present only the special cases in that setting; the more general results analogous to Theorem 1 are obvious extensions of those we present.

Throughout this section, let F, X, E, d, and $\{A_k\}_0^\infty$ be as described at the start of Section 2, and let X be a vector space. For our first special case, we merely take $P = F^{s+1} - F^s$.

COROLLARY 4. *Suppose that there are nonnegative integers r and s and z in X such that* (2.3′) *and* (2.4) *holds with $P = F^{s+1} - F^s$, $A_0 = I$, and suppose that*

$$d(F^{rk+s+1}x - F^{rk+s}x, z) \leqslant A_k d(F^{s+1}x - F^s x, z) \quad \text{for all } x \text{ in } X. \tag{3.1}$$

Then, the conclusion of Corollary 2 holds.

The corresponding uniqueness result is as follows.

COROLLARY 5. *Suppose that there is z in X and nonnegative integers r and s such that* (2.3), (2.4′), *and* (2.4″) *hold with $P = F^{s+1} - F^s$. Suppose also that*

$$d(F^{rk+s}y, F^{rk+s}x) \leqslant A_k d(F^s y, F^s x) \quad \text{for all } x, y \text{ in } X. \tag{3.2}$$

Then the conclusions of Theorem 3 follow.

Proof. In Theorem 3, merely set $Q = F^s$.

A further special case of the above situation is the following; for brevity we hereafter refrain from stating the uniqueness results.

COROLLARY 6. *Suppose that there is an operator $B\colon E^+ \to E^+$ with $Be_1 \leqslant Be_2$ whenever $e_1 \leqslant e_2$, that there is a z in X and nonnegative integers r and s such that* (2.3) *holds and*

$$d(F^{r+s+1}x - F^{r+s}x, z) \leqslant Bd(F^{s+1}x - F^s x, z) \text{ for all } x \text{ in } X, \tag{3.3}$$

$$\sum_{k=1}^{\infty} B^k e \text{ converges in } E, \text{ where } e = \sum_{j=1}^{2r-1} d(F^{s+1}x_j - F^s x_j, z). \tag{3.4}$$

Then the conclusions of Corollary 2 follow.

Proof. We merely use Corollary 4. We have $d(F^{rk+s+1}x, F^{rk+s}x) = d(F^{r+s+1}F^{(k-1)r}x, F^{r+s}F^{(k-1)r}x) \leqslant \mathrm{Bd}(F^{r(k-1)+s+1}x, F^{r(k-1)+s}x)$. By simple induction, we deduce that (3.2) and (2.4′) hold with $A_0 = I$, $A_k = B^k$.

The corresponding uniqueness result simply requires $d(F^{r+s}y - F^{r+s}x, z) \leqslant \mathrm{Bd}(F^s y - F^s x, z)$.

In the case of $E = \boldsymbol{R}$, Corollary 6 with $s = 0$, $z = 0$, and $r = 1$ gives the iterated contraction theorem of Ortega–Rheinboldt (1970), [7] p. 401; the version of the theorem giving uniqueness yields the original contraction theorem of Banach (1922), [4]. Corollary 6 with $s = 0$, $z = 0$, and $r = 1$ for arbitrary E-metric spaces is mentioned in [7], p. 440; these results are important for deriving global convergence theorems for nonlinear Jacobi and SOR methods ([7], Ch. 13). Corollary 5 with $s = 0$, $z = 0$, and $r = 1$ first appeared in [9]. The uniqueness theorem corresponding to Corollary 6 with $E = \boldsymbol{R}$, $z = 0$, and $s = 0$ first appeared in [6]. Corollary 5 with $E = \boldsymbol{R}$ and $s = 0$ first appeared in [6], p. 383.

These theorems can be used to apply to the problem of solving general nonlinear equations; we give one simple example of a very special case. Suppose that X_1 and X_2 are linear E_1-metric and linear E_2-metric spaces with metrics d_1 and d_2 respectively; suppose that, for $i = 1, 2$, $d_i(x, y) = d_i(x - y, 0)$. We seek to solve $Px = 0$, where $P\colon X_1 \to X_2$. Suppose that for each x in X_1 we have a mapping $H_x\colon X_2 \to X_1$. We shall try to solve our problem via the iteration $x_{n+1} = x_n - H_{x_n}Px_n$; clearly this is related to seeking fixed points of $Fx \equiv x - H_x Px$.

COROLLARY 7. *Suppose that there are mappings $B\colon E_2^+ \to E_2^+$ and $C\colon E_2^+ \to E_1^+$ such that $e \leqslant e'$ implies $Be \leqslant Be'$ and $Ce \leqslant Ce'$, such that $d_1(H_x y, 0) \leqslant Cd_2(y, 0)$ for all x in X_1 and y in X_2, and such that $d_2(P(x - H_x Px), 0) \leqslant Bd_2(Px, 0)$ for all x in X_1. Suppose that the series $\sum_{k=1}^{\infty} CB^k d_2(Px_1, 0)$ converges in E_1. Then the sequence generated by $x_{n+1} = x_n - H_{x_n}Px_n$ converges to an x^* and Px_n converges to zero. If P is closed or continuous at x^*, then $Px^* = 0$.*

Proof. Defining $Fx = x - H_x Px$, we see that $d_1(Fx, x) \leqslant Cd_2(Px, 0)$ and $d_2(PFx, 0) \leqslant Bd_2(Px, 0)$. It follows that $d_2(PF^k x, 0) \leqslant B^k d_2(Px, 0)$. The result then follows from Theorem 1, with $G(y) = d_2(Py, 0)$, $A_0 = C$, and $A_k = B^k$.

This theorem extends results in [1], [2], [3]; further extensions can easily be made via Theorem 1.

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Bases in bornological spaces

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Abstract. This paper presents the fundamentals of the basis theory for bornological spaces. The attention is restricted to complete and regular spaces with a bornology which is either topological or of countable type. Spaces of the latter type are called (LB)-spaces. We begin by introducing the notions of separability and local separability in a bornological space and by showing that they agree for (LB)-spaces, which enables us to give representation theorems for such spaces which are separable. Next, bases and Schauder bases are introduced and a basic lemma which states that a basis of an (LB)-space is also a 'local' basis is proved. Among the many consequences of this fundamental lemma the most important is that every basis of an (LB)-space is a Schauder basis. We then investigate the relationship between bornological and topological Schauder bases and study the properties of a Schauder basis in terms of the dual sequence of bounded linear functionals. Finally, the connection between Schauder bases and reflexivity is given and various types of Schauder bases are analysed.

Introduction. The purpose of this paper is to present the fundamentals of the basis theory for bornological spaces. Attempts have only recently been made to extend to locally convex spaces the classical basis theory for Banach spaces. Here we are concerned with developing a similar theory for regular bornological spaces, the assumption of regularity being imposed by the central role played by duality. We deal essentially with Schauder bases and the fact that bornological spaces with such bases abound in analysis is perhaps motivation enough for a systematic study. However, we make no claim as to the completeness of our discussion.

All notions are used in the bornological sense, unless otherwise specified. For the notions that are not defined here, we refer to [4]. By b.c.s. (l.c.s.) we mean a bornological space (locally convex space) and by (LB)-*space* a complete b.c.s. with a countable base. We are mainly concerned with (LB)-spaces but most of the results obtained can easily be generalized to b.c.s. for which the homomorphism or closed graph theorems hold. If E is a regular b.c.s. with dual $E^{\times}$, the familiar symbols σ and τ are used for the weak and Mackey topologies with respect to the duality $\langle E, E^{\times}\rangle$, unless otherwise stated, and we write then E_σ and E_τ with obvious meaning. Finally, following Köthe, we denote by ω the