A generalized contraction mapping theorem
in E-metric spaces

by

JAMES W. DANIEL (Austin, Texas)

Abstract. This paper addresses itself to general theorems on the convergence of a sequence generated via $x_{n+1} = Fx_n$ to a fixed point of the operator $F$; the best known such theorem is the well-known contraction mapping theorem of Banach. Here we prove two main theorems which include as special cases many previous generalizations of Banach's theorem.

1. Introduction. We consider the problem of locating a fixed point of a nonlinear operator $F$ mapping a space $X$ into itself; that is, we seek $x^*$ satisfying $x^* = Fx^*$. Here we assume that $X$ is a complete $E$-metric space with $E$-metric $d(\cdot, \cdot)$. That is, $E$ is a partially ordered vector space in which the notion of convergence to zero is defined, with the usual compatibility hypotheses among convergence, vector space operations, and order being assumed, $d$ maps $X \times X$ into $(\varepsilon \in E, \varepsilon \geq 0) = E^*$ and satisfies (1) $d(x, y) = 0$ if and only if $x = y$ and (2) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z$ in $X$; in $X$, a sequence $(x_n)$ converges to $x$ if and only if $d(x_n, x)$ converges to zero in $E$, and it is assumed that every Cauchy sequence in $X$ converges to a point of $X$. For a derivation of the properties of $E$-metric spaces, called by some authors [Altman [3], Collatz [5]] "pseudometric" spaces, the reader is referred to [3].

Perhaps the best known fixed point theorem is the contraction mapping theorem which asserts that if $d(Fx, Fy) \leq c d(x, y)$ where $c < 1$ and $E$ is the real number system $R$, then there exists a unique fixed point $x^*$ which may be computed as the limit of the sequence defined by $x_{n+1} = Fx_n$, with $x_0$ arbitrary. We state two main theorems which generalize these results in Section 2, and in Section 3 we show how many previous generalizations are included in ours.

2. The main results. First we state a theorem which gives existence and convergence but not necessarily uniqueness. Let $A_k$, for each $k \geq 0$, be a mapping of $E^*$ into $E^*$ satisfying $A_k \phi_1 \leq A_k \phi_2$ if $\phi_1 \leq \phi_2$. For non-negative integers $n$, define $F^n$ via $F^n = I$ (the identity mapping) and $F^{n+1} = F(F^n)$.
Theorem 1. Let \( F, X, E, d, \) and \( \{ A_{k} \}^{m}_{k=1} \) be as above. Suppose that there exist operators \( G : X \rightarrow E \) and \( F : X \rightarrow X \), and nonnegative integers \( r \) and \( s \) such that

\[
d(F^{r+1}x,F^{s}x) \leq A_{k}G(x) \quad \text{for all } x \in X.
\]

(2.1)

\[
G(F^{r}x) \leq A_{k}G(x) \quad \text{for all } x \in X \text{ and all } k \geq 1,
\]

(2.2)

if \( x_{n} \) converges to \( x^{*} \) in \( X \) and \( \{ F^{r+1}x_{n} \} \) converges to zero in \( E \), then \( F^{r+1}x^{*} = F^{s}x^{*} \).

(2.3)

(2.4)

Then, for any \( x_{n} \) in \( X \), the sequence generated by \( x_{n+1} = F_{n}x_{n} \) converges to \( x^{*} \) satisfying \( F^{r+1}x^{*} = F^{s}x^{*} \), so that \( y^{*} = F^{r}s^{*} \) is a fixed point of \( F \). In addition, \( G(x_{n}) \) converges to zero in \( E \). The following error estimate is valid:

\[
d(x^{*}, x_{n}) \leq r \sum_{k=1}^{m} A_{k}A_{k}e,
\]

(2.5)

where \( k \) is the integer part of \( \frac{r}{s} \) and \( e \) is as defined in (2.4).

Proof. We shall show first that \( \{ x_{n} \}^{m}_{n=1} \) is a Cauchy sequence. We write, for \( i \geq s, \)

\[
d(x_{i+s}, x_{i}) = d(F^{r+1}x_{i+s-1}, F^{s}x_{i+s-1}) \leq A_{k}G(x_{i+s-1}).
\]

Thus, for \( i \geq s \) and \( l \geq 0 \), we have

\[
d(x_{i+l+s}, x_{i+l}) \leq \sum_{k=1}^{i+l+s-1} d(x_{k+s+i}, x_{k+i+s}) \leq \sum_{k=1}^{i+s} A_{k}G(x_{k+i+s}).
\]

Now, we can write \( l \) uniquely as \( nr + m \) with \( 0 \leq m \leq r - 1 \) and \( i \) uniquely as \( k = h + l \) with \( s \leq h \leq s + r - 1 \). Rearranging the sum in the above inequality yields

\[
\sum_{k=1}^{i} A_{k}G(x_{k+i+s}) = \sum_{k=1}^{i} A_{k}G(x_{k+i+s}) \leq \sum_{k=1}^{i} A_{k}G(x_{k+i+s}).
\]

where

\[
e = \sum_{k=1}^{i-1} G(x_{k}).
\]

Thus we have

\[
d(x_{i+l+s}, x_{i+l}) \leq e \sum_{k=1}^{i+s} A_{k}A_{k}e,
\]

(2.6)

Since \( \sum_{k=1}^{m} A_{k}A_{k}e \) converges, we can make \( d(x_{i+l+s}, x_{i+l}) \) near zero by making \( e \) and hence \( x \), large; thus \( \{ x_{n} \}^{m}_{n=1} \) is a Cauchy sequence. Let \( x^{*} \) be its limit. Since \( d(x_{n+1}, x_{n}) \) converges to zero and \( d(x_{n+1}, x_{n}) = d(F^{r+1}x_{n+1}, F^{s}x_{n+1}) \), and since \( \{ x_{n+1} \} \) converges to \( x^{*} \), we conclude from (2.3) that \( F^{r+1}x^{*} = F^{s}x^{*} \) and hence \( y^{*} = F^{r}s^{*} \) is a fixed point of \( F \). The above inequalities show that \( G(x_{n}) \) converges to zero. The error estimate follows by letting \( i \), and hence \( m \), tend to infinity in (2.6).

Remarks. It is easy to modify the above theorem, using (2.6), so that the hypotheses need only hold in the “ball” of “radius” \( r \sum_{k=1}^{m} A_{k}A_{k}e \) about the point \( x_{0} \). For most applications, the mapping \( G \) is given by \( G(y) = d(Fy, z) \) for some fixed \( z \) in \( X \); usually, in fact, one has \( z \) equal to zero. Since we want to consider later several instances of this special case and since our uniqueness result is related to this special case, we now state it as a corollary.

Corollary 2. Let \( F, X, E, d, \) and \( \{ A_{k} \}^{m}_{k=1} \) be as above. Suppose that there is an operator \( P : X \rightarrow X \), an element \( z \) in \( X \), and nonnegative integers \( r \) and \( s \) such that

\[
d(F^{r+1}x,F^{s}x) \leq A_{k}d(Pz, x) \quad \text{for all } x \in X,
\]

(2.1')

\[
d(P^{r+1}x,F^{s}x) \leq A_{k}d(Pz, x) \quad \text{for all } x \in X \text{ and } k \geq 1.
\]

(2.2')

(2.4')

Then, for any \( x_{n} \) in \( X \), the sequence generated by \( x_{n+1} = P_{n}x_{n} \) converges to \( x^{*} \) satisfying \( F^{r+1}x^{*} = F^{s}x^{*} \), so that \( y^{*} = F^{r}s^{*} \) is a fixed point of \( F \). In addition, \( P_{n}x_{n} \) converges to \( z \), and the error estimate of (2.5) is valid with \( e \) defined in (2.4').

Under the hypotheses of Theorem 1 or its corollary, the fixed point \( y^{*} \) need not be unique; to give a uniqueness theorem, we thus need stronger hypotheses. The reader should note the similarities of these hypotheses with those of Corollary 2.

Theorem 3. In addition to the hypotheses of Corollary 2, assume that for some fixed operator \( Q : X \rightarrow X \) we have

\[
d(Q^{r+1}x,Q^{s}x) \leq A_{k}d(Qy, Qz) \quad \text{for all } x, y \text{ in } X,
\]

(2.1'')

\[
d(Q^{r+1}x,Q^{s}x) \leq A_{k}d(Qy, Qz) \quad \text{for all } x, y \text{ in } X \text{ and all } k \geq 1.
\]

(2.2'')

(2.4'')

Then the \( y^{*} \) generated in Corollary 2 is the unique fixed point of \( F \).
Proof. Suppose \( y \) is also a fixed point of \( F \); then \( y \) and \( y^* \) are both fixed points of \( F^n \) for all \( n \geq 1 \). Thus \( d(y, y^*) = d(F^n y, F^n y^*) \leq A_0 d(Q F^n y, Q F^n y^*) = A_0 A_d Q(y, y^*) \) which converges to zero.

Remarks. As before, this result can be stated locally rather than globally. Essentially the idea here is that we think of \( P \) in Corollary 2 as having the form \( P = Q - F \); we see that (3.1') and (3.2') essentially give (2.1') and (2.2') then when we take \( y = F x \).

3. Some special cases. We now wish to show that many previous fixed point theorems can be deduced from our general result in Theorem 1 and from its corollary. Since for this purpose Corollary 3 is sufficient, in the interest of brevity we present only the special cases in that setting; the more general results analogous to Theorem 1 are obvious extensions of those we present.

Throughout this section, let \( P, X, E, d, \) and \( (A_d) \) be as described at the start of Section 2, and let \( X \) be a vector space. For our first special case, we merely take \( P = F^{n+1} - F^n \).

Corollary 4. Suppose that there are nonnegative integers \( r \) and \( s \) and \( x \in X \) such that (3.1') and (3.2') hold with \( P = F^{n+1} - F^n, A_d = I \), and suppose that

\[
\tag{3.1} d(F^{n+1} x, F^n x, s) \leq A_d d(F^{n+1} x, F^n x, s) \quad \text{for all } x \in X.
\]

Then, the conclusion of Corollary 2 holds.

The corresponding uniqueness result is as follows.

Corollary 5. Suppose that there is \( x \in X \) and nonnegative integers \( r \) and \( s \) such that (2.3), (2.4), and (2.4') hold with \( P = F^{n+1} - F^n \). Suppose also that

\[
\tag{3.2} d(F^{n+1} x, F^n x, s) \leq A_d d(F^n y, F^s x) \quad \text{for all } x, y \in X.
\]

Then the conclusions of Theorem 3 follow.

Proof. In Theorem 3, merely set \( Q = F^n \).

A further special case of the above situation is the following; for brevity we hereafter refrain from stating the uniqueness results.

Corollary 6. Suppose that there is an operator \( B: E^+ \rightarrow E^+ \) with \( B \leq B \) whenever \( e \leq e \), that there is a \( r \in X \) and nonnegative integers \( r \) and \( s \) such that (2.3) holds and

\[
\tag{3.3} d(F^{n+1} x, F^n x, s) \leq B d(F^{n+1} x, F^n x, s) \quad \text{for all } x \in X,
\]

\[
\tag{3.4} \sum_{i=1}^{\infty} B^i e \text{ converges in } E, \quad e = \sum_{i=1}^{\infty} d(F^{n+1} x, F^n x, s).
\]

Then the conclusions of Corollary 2 follow.
Bases in bornological spaces

by

V. B. MOSCATELLI (Brighton, England)

Abstract. This paper presents the fundamentals of the basis theory for bornological spaces. The attention is restricted to complete and regular spaces with a bornology which is either topological or of countable type. Spaces of the latter type are called (LB)-spaces. We begin by introducing the notions of separability and local separability in a bornological space and by showing that they agree for (LB)-spaces, which enables us to give representation theorems for such spaces which are separable. Next, bases and Schauder bases are introduced and a basis lemma which states that a basis of an (LB)-space is also a 'local' basis is proved. Among the many consequences of this fundamental lemma the most important is that every basis of an (LB)-space is a Schauder basis. We then investigate the relationship between bornological and topological Schauder bases and study the properties of a Schauder basis in terms of the dual sequence of bounded linear functionals. Finally, the connection between Schauder bases and reflectivity is given and various types of Schauder bases are analyzed.

Introduction. The purpose of this paper is to present the fundamentals of the basis theory for bornological spaces. Attempts have only recently been made to extend to locally convex spaces the classical basis theory for Banach spaces. Here we are concerned with developing a similar theory for regular bornological spaces, the assumption of regularity being imposed by the central role played by duality. We deal essentially with Schauder bases and the fact that bornological spaces with such bases abound in analysis is perhaps motivation enough for a systematic study. However, we make no claim as to the completeness of our discussion. All notions are used in the bornological sense, unless otherwise specified. For the notions that are not defined here, we refer to [4]. By b.c.s. (i.e.c.s.) we mean a bornological space (locally convex space) and by (LB)-space a complete b.c.s. with a countable base. We are mainly concerned with (LB)-spaces but most of the results obtained can easily be generalized to b.c.s. for which the homomorphism or closed graph theorems hold. If $E$ is a regular b.c.s. with dual $E'$, the familiar symbols $\sigma$ and $\tau$ are used for the weak and Mackey topologies with respect to the duality $(E, E')$, unless otherwise stated, and we write then $E_1$ and $E_2$ with obvious meaning. Finally, following Köthe, we denote by $\omega$ the