[26] H. H. Schaefer, *Topological Vector Spaces*, New York–London 1966.
[27] I. Tweddle, *Vector valued measures*, Proc. London. Math. Soc. (3) 20 (1970), pp. 469–489.
[28] D. A. Vladimirov, *Boolean Algebras* (in Russian), Moscow 1969.
[29] B. Walsh, *Mutual absolute continuity of sets of measures*, Proc. Amer. Math. Soc. 29 (1971), pp. 506–510.

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Nuclear spaces on a locally compact group

by

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Abstract. This paper is devoted to a construction of two types of nuclear spaces Φ and Ψ consisting of functions on a locally compact group. These spaces resemble the spaces D and S of Schwartz, respectively, although the construction does not depend on any differential structure on G and no approximation by Lie groups is used. The role of differential operators is played by (unbounded) operators which are the inverse operators to convolution operators by appropriately chosen non-negative L_1-functions. Thus both spaces Φ and Ψ consists of infinitely regularized functions.

1. Introduction. The idea of the construction of a nuclear space of functions on a locally compact group by an infinite process of regularization by "good" functions is due to A. Hulanicki. We would also like to express our gratitude to him for many useful suggestions and the help while this paper was written.

The main idea of the construction of the space Φ was published in [5]. The spaces of the type Φ and Ψ are not unique—they depend on the selection of the sequence of regularizing functions which shall be chosen once for all. Therefore we shall say the space Φ or Ψ rather than a space of the type Φ or Ψ. On the few ocasions will be imposed, this will be clearly stated.

Among the main properties of the spaces Φ and Ψ are the following. Both are non-trivial subspaces of $L_2(G)$ and Ψ is dense in $L_2(G)$. Both are invariant under the left regular representation of G which is jointly continuous on Φ and Ψ. Following [5] Aarnes [1] constructed a space which is invariant under left and right reqular representation of G — a simplification of his construction is given here.

There are many questions which should perhaps be asked about the spaces Φ and Ψ which are not answered in this paper. We would rather postpone considering them to the time when these spaces shall prove (or disprove) to be of any use in harmonic analysis on non-Lie non-commutative locally compact groups.

2. The convolution operator and its inverse. Let G be a locally compact group and let μ be a left invariant Haar measure on it. If $f \in L_1(G)$ and

$g \in L_2(G)$, then the convolution

(1) $$f*g(s) = \int_G f(u)g(u^{-1}s)\,du = \int_G f(su)g(u^{-1})\,du$$

defines a function in $L_2(G)$ for which

(2) $$\|f*g\|_2 \leqslant \|f\|_1 \cdot \|g\|_2.$$

Moreover, if $f \in L_2(G)$ and g satisfies the inequality

(3) $$\int_G \Delta(s)^{-1/2}|g(s)|\,ds < +\infty$$

then $f*g \in L_2(G)$ and

(4) $$\|f*g\|_2 \leqslant \|f\|_2 \cdot \|\Delta^{-1/2} \cdot g\|_1.$$

Convolution $f*g$ is linear with respect to each of its factors and, if $f \in L_2(G)$ and g, h satisfy (3), then $(f*g)*h = f*(g*h)$. If for a fixed element $s \in G$, $f_s(u) = f(us)$, ${}_sf(u) = f(su)$ are the left and right translates of f by s, then

(5) $$\begin{cases} (f*g)_s = f*g_s, \\ {}_s(f*g) = {}_sf*g, \\ f_s*g \;= \Delta(s)(f*{}_sg). \end{cases}$$

If $f, \tilde{g} \in L_2(G)$, where $\tilde{g}(s) = \overline{g(s^{-1})}$ then $f*g$ is continuous. For any function g such that $\Delta^{-1/2} \cdot g \in L_1(G)$ the relation $T_gf = f*g$ defines a bounded operator T_g on the Hilbert space $L_2(G)$. The adjoint T_g^* of T_g is the operator $T_g^*f = T_{\tilde{g}}f = f*\tilde{g}$.

The operator T_g^{-1} inverse to T_g can of course be defined only for some functions g.

A bounded, continuous and integrable function g on G we call an *admissible function* if it satysfies (3) and if $\ker T_g = \ker T_{\tilde{g}} = \{0\}$.

2.1 Lemma. *Let g be an admissible function on G, then $T_g(L_2(G))$ is dense in $L_2(G)$. If G is a non-discrete group then $T_g(L_2(G)) \neq L_2(G)$.*

Proof. The first statement follows from the fact that if an $f \in L_2(G)$ is orthogonal to $T_g(L_2(G))$, then $f \in \ker T_g^*$ and $\ker T_g^* = \ker T_{\tilde{g}} = \{0\}$.

To prove the second part of the lemma we suppose that $T_g(L_2(G)) = L_2(G)$. T_g is then a continuous one-to-one map of $L_2(G)$ onto $L_2(G)$. It follows there is a constant m such that

$$\|f\|_2 \leqslant m\|T_gf\|_2, \quad f \in L_2(G).$$

In particular, for any $f \in L_1(G) \cap L_2(G)$,

(6) $$\|f\|_2 \leqslant m \cdot \|f*g\|_2 \leqslant m\|f\|_1\|g\|_2.$$

Let U_n, $n = 1, 2, \ldots$, be a family of open sets such that $0 \neq \mu(U_n) < 1/n^2$. If $f_n(u) = \{\mu(U_n)\}^{-1}$ for $u \in U_n$ and $f_n(u) = 0$ otherwise, then $f_n \in L_1(G) \cap L_2(G)$, $n = 1, 2, \ldots$, and $\|f_n\|_1 = 1$, $\|f_n\|_2 > n$. But this contradicts (6).

Lemma 2.1 shows that if g admissible then an operator T_g^{-1} inverse to T_g exists, but is unbounded.

2.2 Theorem. *let g be admissible and let $s \in G$, then ${}_sg$, g_s and $\tilde{g}$ are admissible. If g_1, g_2 are admissible then g_1*g_2 is admissible.*

This is an immediate consequence of (5) and the equality $(g_1*g_2)^{\sim} = \tilde{g}_2*\tilde{g}_1$.

The following theorem gives a construction of a large class of admissible functions.

2.3 Theorem ([5], 1). *Let G be a metrizable group and let U be a neighbourhood of the unit of G. There is an admissible function g on G which is real, nonnegative, symmetric (i.e. $\tilde{g} = g$), vanishes outside U, and $\int_G g = 1$.*

The idea of this proof is due to C. Ryll-Nardzewski.

Proof. Let $\{U_n\}_{n=0}^{\infty}$ be a family of symmetric, precompact neighbourhoods of e (i.e. $U_n^{-1} = U_n$, $n = 0, 1, 2, \ldots$) such that $U_0 \subset U$, $U_n^2 \subset U_{n-1}$ for $n = 1, 2, \ldots$ and $\bigcap_{n=0}^{\infty} U_n = \{e\}$.

We consider the function

$$g = \frac{1}{c}\sum_{k=1}^{\infty} a_k(\varphi_k*\varphi_k),$$

where $\varphi_k(u) = \{\mu(U_k)\}^{-1}$ for $u \in U_k$, $\varphi_k(u) = 0$ otherwise and a_k is a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_k \cdot \{\mu(U_k)\}^{-1} < +\infty$$

and $c = \sum_{k=1}^{\infty} a_k$.

The function g is a limit of a uniformly convergent series of continuous functions $\varphi_k*\varphi_k$ and hence it is continuous, is non-negative and vanishes outside $U_0 \subset U$. Symmetry follows from the observation that U_k are symmetric sets.

Now

$$\int_G \varphi_k*\varphi_k(s)\,ds = \int_G\int_G \varphi_k(u)\varphi_k(u^{-1}s)\,du\,ds = \Big(\int_G \varphi_k(u)\,du\Big)^2 = 1,$$

hence $\int_G g(s)\,ds = \frac{1}{c}\sum_{k=1}^{\infty} a_k = 1$.

To prove that g is an admissible function we note that U_n is a basis at e for the topology of G, hence for any $f\in L_2(G)$ we have

$$\lim_{k\to\infty}\|f-f*\varphi_k\|_2 = 0.$$

Let $f\in L_2(G)$ and $f*g = 0$. Since $\tilde{\varphi}_k = \varphi_k$, we have

$$0 = \langle f*g, f\rangle = \frac{1}{c}\sum_{k=1}^{\infty} a_k\langle f*\varphi_k*\varphi_k, f\rangle = \frac{1}{c}\sum_{k=1}^{\infty} a_k\langle f*\varphi_k, f*\varphi_k\rangle,$$

hence $f*\varphi_k = 0$ for $k = 1, 2, \ldots$, and so $f = \lim_{k\to\infty} f*\varphi_k = 0$.

The assumption of metrizability of the group G is clearly essential even for abelian groups. In fact, if g is admissible then $\hat{g}\in L_2(\hat{G})$ and it is different from zero almost everywhere on $\hat{G}$. This means that $\hat{G}$ is a σ-compact group and hence G must be metrizable.

From now on the group G is always assumed to be metrizable. For an admissible g we write τ_g for T_g^{-1}.

It is easy to prove that

$$(7)\qquad \begin{cases} \tau_{g_1*g_2} = \tau_{g_2}\tau_{g_1}, \\ L_s\tau_g = \tau_g L_s, \\ \tau_g(f*h) = f*\tau_g h, \end{cases}$$

where g, g_1 and g_2 are admissible functions on G, $f\in L_1(G)$, $h\in T_g(L_2(G))$. and L_s ($L_s f = {}_{s^{-1}}f$) is the left reqular representation of G on $L_2(G)$.

2.4 Example. Let $G = R$ and let g be the characteristic function of the interval $(0, 1)$. If $f\in C^1(R)$ has compact support and satisfies $\sum_{-\infty}^{+\infty} f'(s+n) = 0$ for $0\leqslant s<1$ (the sum has only finitely many elements different from zero) then f is in the range $D\tau_g = T_g(L_2(G))$ of the operator τ_g and

$$\tau_g f(u) = \sum_{n=0}^{\infty} f'(u-n).$$

Indeed, the function $h(u) = \sum_{n=0}^{\infty} f'(u-n)$ is a bounded continuous function with compact support and

$$T_g h(s) = \int_{-\infty}^{+\infty} h(u)g(s-u)\,du = \int_{s-1}^{s} h(u)\,du = \sum_{n=0}^{\infty}[f(s-n)-f(s-n-1)] = f(s).$$

Hence $h = \tau_g f$.

For an integer k let $g^k = g*g*\ldots*g$. If $f\in D(\tau_g)^k$ then

$$\tau_g^k f(u) = (\tau_g)^k f(u) = \sum_{n=0}^{\infty}\binom{n+k}{n} f^{(k)}(s-n).$$

This example shows that, for some admissible g, τ_g need not be local. Now we give an example of an admissible function g, τ_g being local.

2.5 Example. Let $G = R^n$ and let $g(u) = e^{-|u|}$. If $f\in C^2(R^n)$ and $f, \Delta f\in L_2(R^n)$ then $f\in D\tau_g$ and

$$\tau_g f = \tfrac{1}{2}(f-\Delta f),$$

where Δ is the Laplacian $\left(\Delta = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_n^2}\right)$. In fact, for $y\in R^n$ we have

$$(\tau_g f*g)\hat{}\,(y) = \tfrac{1}{2}[\hat{f}(y)-(\Delta f)\hat{}\,(y)]\hat{g}(y) = \tfrac{1}{2}[\hat{f}(y)+|y|^2\hat{f}(y)]\frac{2}{1+|y|^2} = \hat{f}(y).$$

3. The space Φ. In this section we are going to construct a nuclear space of functions $\Phi = \Phi(G)$ on a first countable locally compact group. This space resembles $D(R^n)$—the space of infinitely differentiable functions with compact supports. The differential operators are replaced by the operators τ_g with suitably chosen admissible functions g. Unfortunately we were unable to select the g's in such a way to make the τ_g local, the possibility of such a choice remaining an open problem. The lack of locality of the τ_g's we use is responsible for several imperfections of our Φ as compaired to the space $D(R^n)$, e.g. we do not know whether Φ is an algebra under multiplication.

Throughout this section, G will always denote a σ-compact and metrizable locally compact group. Let U be a precompact neighbourhood of the unit e in G and $\{U_n\}_{n=0}^{\infty}$ a family of symmetric neighbourhoods of e for which $U_0\subset U$, $U_n^2\subset U_{n-1}$, $n = 1, 2, \ldots$ and $\bigcap_{n=0}^{\infty} U_n = \{e\}$. We fix a sequence $\{g_n\}_{n=1}^{\infty}$ of admissible functions as in Theorem 2.3 with $\operatorname{supp} g_n\subset U_n$. We will write

$$\tau_n = \tau_{g_1}\tau_{g_2}\ldots\tau_{g_n} = \tau_{g_n*g_{n-1}*\ldots*g_1}.$$

3.1 Definition. *Let Φ denote the space of all functions f on G such that $\tau_n f$ is continuous for $n = 0, 1, 2, \ldots$, ($\tau_0 f = f$) and the set*

$$K_f = \bigcup_{n=0}^{\infty}\operatorname{supp}\tau_n f$$

is precompact.

Let $\mathscr{E}$ be the family of all compact subsets of G. For an integer n, $\varepsilon > 0$ and $K \in \mathscr{E}$ we put

$$\mathscr{W}(K, n, \varepsilon) = \{h \in \Phi\colon\ K_h \subset K,\ \sup_{s\in G} |\tau_n h(s)| < \varepsilon\}.$$

We equip the space Φ with the locally convex topology for which the sets

$$\operatorname{conv}\Big[\bigcup_{K\in\mathscr{E}} \mathscr{W}(K, n_k, \varepsilon_k)\Big], \tag{8}$$

where n_k are integers and $\varepsilon_k > 0$, form a base of neighbourhoods of zero.

3.2 THEOREM. *A sequence $f_j \in \Phi$ converges to zero in Φ if and only if there is a compact set $K \subset G$ such that $\operatorname{supp} \tau_n f_j \subset K$ for $n, j = 1, 2, \ldots$, and $\tau_n f_j$ tends to zero uniformly for all n.*

To prove that Φ is a nontrivial space we use the following

3.3 LEMMA. *For any $V \in \mathscr{E}$ and $n = 1, 2, \ldots$, the operator $T_n = T_{g_n}$ maps the space $L_2(VU_n)$ into $L_2(VU_{n-1})$ and is of the Hilbert–Schmidt type.*

Proof. Let $T_n'\colon L_2(G) \to L_2(G)$ denote the operator

$$T_n' f(s) = \int_G \mathscr{K}_n(s, u) f(u)\, du,$$

where $\mathscr{K}_n(s, u) = g_n(u^{-1}s)$ for $u \in VU_n$ and $\mathscr{K}_n(s, u) = 0$ otherwise. Since

$$\int_G\int_G |\mathscr{K}_n(s, u)|^2 ds\, du = \int_{VU_n}\Big(\int_G g_n^2(u^{-1}s)\, ds\Big) du = \int_{VU_n}\Big(\int_{U_n} g_n^2(s)\, ds\Big) du$$
$$\leqslant \|g_n\|_\infty^2 \mu(U_n)\mu(VU_n),$$

we see that T_n' is a Hilbert–Schmidt operator. If P_n is the orthogonal projection of $L_2(G)$ onto $L_2(VU_n)$, then $T_n = P_{n-1} T_n' P_n$ and so it is also of Hilbert–Schmidt type.

3.4 THEOREM. *The space Φ is non-zero.*

Proof. First we shall prove (by induction) that for $f \in L_2(U_n)$

$$\int_G T_k T_{k+1} \ldots T_n f(s)\, ds = \int_G f(s)\, ds. \tag{9}$$

Indeed, applying Fubini's theorem to $\int_G T_k h(s)\, ds$, where $h \in L_2(U_k)$, we obtain $\int_G T_k h(s)\, ds = \int_G h(s)\, ds$. Thus

$$\int_G T_k(T_{k+1} \ldots T_n f)(s)\, ds = \int_G T_{k+1} \ldots T_n f(s)\, ds,$$

whence, by induction, (9) follows.

For each $k = 1, 2, \ldots$, the sequence $f_n^k = T_{k+1} \ldots T_n g_{n+1}$, $n = k+1, k+2, \ldots$, is bounded in the space $L_2(U_k)$. Indeed,

$$\|f_n^k\|_2 \leqslant \|f_n^k\|_\infty \{\mu(U_k)\}^{1/2} \leqslant \|g_{k+1}\|_\infty \{\mu(U_k)\}^{1/2}$$

because

$$\|f_n^k\|_\infty = \sup_{s\in G} |(T_{k+1} T_{k+2} \ldots T_n g_{n+1})(s)|$$
$$= \sup_{s\in G}\Big|\int_G g_{k+1}(u^{-1}s) T_{k+2} \ldots T_n g_{n+1}(u)\, du\Big| \leqslant \|g_{k+1}\|_\infty \int_G (T_{k+2} \ldots T_n g_{n+1})(u)\, du$$
$$= \|g_{k+1}\|_\infty \int_G g_{n+1}(u)\, du = \|g_{k+1}\|_\infty.$$

Since T_1 is compact, the sequence $f_n = T_1 f_n^1$ has a convergent subsequence, and for simplicity without any loss of generality we may denote it by f_n. We shall prove that the limit f_∞ of it belongs to Φ.

For a k let $f_{i_n}^k$ denote a convergent subsequence of the sequence $f_n^k = T_{k+1} f_n^{k+1}$ (T_{k+1} is compact) and let $f_\infty^k = \lim_n f_{i_n}^k$. Since $T_1 T_2 \ldots T_k$ is a continuous operator, we have

$$T_1 T_2 \ldots T_k f_\infty^k = \lim T_1 T_2 \ldots T_k f_{i_n}^k = f_\infty.$$

Hence, $f_\infty^k = \tau_k f_\infty$ so $f_\infty \in \Phi$. Finally, $f_\infty \neq 0$ as $\int_G f_\infty(s)\, ds = 1$, the proof is complete.

For a compact set $V \subset G$ let

$$\Phi_{V,n} = T_1 T_2 \ldots T_n\big(L_2(VU_n)\big),$$

$$\Phi_V = \bigcap_{n=1}^{\infty} \Phi_{V,n}.$$

3.5 LEMMA. *If $V_1 \subset V_2$, then $\Phi_{V_1} \subset \Phi_{V_2}$ and for any $s \in G$ the left translation L_s maps Φ_V onto Φ_{sV}.*

Proof. If $V_1 \subset V_2$, then $L_2(V_1 U_n) \subset L_2(V_2 U_n)$ for $n = 1, 2, \ldots$, hence $\Phi_{V_1} \subset \Phi_{V_2}$. The map L_s is an isomorphism of $L_2(VU_n)$ onto $L_2(sVU_n)$. Since

$$L_s T_n = T_n L_s, \quad s \in G,\ n = 1, 2, \ldots, \tag{10}$$

L_s maps $\Phi_{V,n}$ onto $\Phi_{sV,n}$.

3.6 THEOREM. *Φ_V equipped with the topology induced by a system of norms*

$$|f|_{V,n} = \Big\{\int_{VU_n} |\tau_n f(s)|^2 ds\Big\}^{1/2}.$$

is a nuclear space.

Proof. First we note that the space $\Phi_{V,n}$ with the scalar product

$$\langle f, h\rangle_n = \int_{VU_n} \tau_n f(s)\overline{\tau_n h(s)}\,ds$$

is isomorphic to $L_2(VU_n)$, τ_n being an isometry.

To prove the theorem, it suffices to show that for any n there exists an m such that the natural embedding of $\Phi_{V,m}$ into $\Phi_{V,n}$ is nuclear. By Lemma 3.3 the embedding $\iota_n = \tau_n^{-1}T_n\tau_{n+1}$ of $\Phi_{V,n+1}$ into $\Phi_{V,n}$ is of Hilbert-Schmidt type. Thus $\iota_{n+1}\iota_n$ is nuclear, and it suffices to put $m = n+2$.

3.7 Lemma. *Let $\mathscr{E}$ be the family of all compact subsets of G. The space Φ is a strict inductive limit of the spaces Φ_V, $V\in\mathscr{E}$.*

$$\Phi = \liminf_{V\in\mathscr{E}} \Phi_V.$$

Proof. It is clear that $f\in\Phi$ if and only if $f\in\Phi_V$ for some $V\in\mathscr{E}$. We shall prove that the topology of the space Φ coincide with $\liminf_{V\in\mathscr{E}}\Phi_V$.

If $K_f \subset K$ then $f\in\Phi_K$ and

$$|f|_{K,n} = \|\tau_n f\|_2 \leqslant \sup_{s\in K}|\tau_n f(s)|\,\{\mu(K)\}^{1/2}.$$

On the other hand, if $f\in\Phi_V$, then $K_f\subset\overline{VU}$ and

$$\sup_{s\in\overline{VU}}|\tau_n f(s)| = \sup_{s\in\overline{VU}}|(\tau_{n+1}f)*g_{n+1}(s)| \leqslant |f|_{V,n+1}\sup_{s\in G}|g_{n+1}(s)|\,\{\mu(VU_{n+1})\}^{1/2}.$$

Hence $\mathscr{W}$ is a neighbourhood of zero in Φ if and only if $\mathscr{W}\cap\Phi_V$ is a neighbourhood of zero in Φ_V for each $V\in\mathscr{E}$, this means that the topologies coincides.

By 3.7 we have at once:

3.8 Theorem. *Φ is a complete locally convex space; it is barreled and bornological.*

This and 3.6 together give

3.9 Theorem. *The space Φ is reflexive.*

Finally we give the principal theorem of this section.

3.10 Theorem. *The space Φ has the following properties:*

(i) *Φ is a nuclear strict $\mathscr{LF}$-space.*

(ii) *Φ is closed under complex conjugation.*

(iii) *The mapping $\mathscr{K}(G)\times\Phi\ni(h,f)\to h*f\in\Phi$ is continuous.*

(iv) *The left regular representation L of G on Φ is jointly continuous.*

Proof. Since G is a σ-compact group, Φ is countable strict inductive limit of Fréchet nuclear spaces Φ_V (i.e. strict $\mathscr{LF}$-space), hence nuclear.

To prove (iii) we note that

$$T_1T_2\ldots T_n(h*\varphi) = h*(T_1T_2\ldots T_n\varphi)$$

hence for $\varphi = \tau_n f$ we get $h*(\tau_n f) = \tau_n(h*f)$ and

$$|h*f|_{KV,n} = \|\tau_n(h*f)\|_2 \leqslant \|h\|_1\cdot\|\tau_n f\|_2 \leqslant \sup_{s\in G}|h(s)|\,\mu(K)\,|f|_{V,n}$$

where $K = \operatorname{supp} h$.

To prove (iv) it suffices to show that for any $V\in\mathscr{E}$ the map $\Phi_V\times G\ni(f,s)\to L_sf\in\Phi$ is continuous. Let $f\in\Phi_V$, $s\in G$ and let $\mathscr{W}$ be a neighbourhood of L_sf. The definition of the topology in Φ shows that for an integer n and $\varepsilon > 0$ we have

$$\mathscr{W} \supset \{g\in\Phi_{s\overline{UV}}\colon\ |L_sf-g|_{s\overline{UV},n} < 2\varepsilon\}$$

(U is a conditionally compact neighbourhood of e).

Since the map $s\to L_sg$ is a uniformly continuous map of G into $L_2(G)$, there is a neighbourhood $W\subset U$ such that $\int_G|(L_sg-L_ug)(t)|^2dt < \varepsilon^2$ for $s^{-1}u\in W$. Replacing g by $\tau_n f$ and applying (10) we obtain $L_uf\in\Phi_{s\overline{UV}}$ and $|L_sf-L_uf|_{s\overline{UV},n} < \varepsilon$. Let $\mathscr{V}$ be a neighbourhood of f in Φ_V of the form

$$\mathscr{V} = \{h\in\Phi_V\colon\ |f-h|_{V,n} < \varepsilon\}.$$

If $h\in\mathscr{V}$ then $|L_uf-L_uh|_{s\overline{UV},n} < \varepsilon$, hence $|L_sf-L_uh|_{s\overline{UV},n} < 2\varepsilon$. This means that $L_uh\in\mathscr{W}$ for all $h\in\mathscr{V}$ and $u\in sW$.

4. Density of Φ in $L_2(G)$. Pursuing the analogy between Φ and D, one would conjecture that $\Phi(G)$ is dense in $L_2(G)$ and in fact it is the case when G is abelian or compact. In general this remains a conjecture of an utmost importance to any further developement of the theory of the spaces Φ. Whatever we can prove towards this aim depends on the following

4.1 Lemma. *Let G be a compact or abelian group and let for a $g\in\mathscr{K}(G)$ $\ker T_g = \{0\}$. Then $\ker T_{\tilde g} = \{0\}$.*

Proof. If G is an abelian group and if $\ker T_g = \{0\}$ then $\hat g \neq 0$ almost everywhere on $\hat G$, thus $\hat{\tilde g} = \overline{\hat g} \neq 0$ almost everywhere and so $\ker T_{\tilde g} = \{0\}$.

If G is a compact group then $L_2(G)$ is the direct sum $\sum_\sigma \oplus H^\sigma$ of finite-dimensional subspaces, each of H^σ being invariant under every T_g, $g\in\mathscr{K}(G)$. Since $\ker T_g = \{0\}$ hence $\ker(T_g|_{H^\sigma}) = \{0\}$, and so the operator $T_g|_{H^\sigma}$ is non-singular and consequently $T_{\tilde g}|_{H^\sigma} = (T_g|_{H^\sigma})^*$ is non-singular, i.e. $\ker(T_{\tilde{g}}|_{H^\sigma}) = \{0\}$ for all σ. But since the projections on H^σ commute with $T_{\tilde{g}}$, $\ker T_{\tilde{g}} = \{0\}$.

4.2 LEMMA. *If Φ contains an admissible function f, then Φ is dense in $L_2(G)$.*

Proof. If $f \in \Phi$ is admissible, then $T_f(\mathscr{K}(G))$ is dense in $L_2(G)$, but by Theorem 3.9 $T_f(\mathscr{K}(G)) \subset \Phi$.

4.3 THEOREM. *If G is a compact or abelian group then Φ is dense in $L_2(G)$.*

Proof. We shall prove that the function f_∞ (cf. the proof of Theorem 3.4) is an admissible function. By 4.1 it is sufficient to show that $\ker T_{f_\infty} = \{0\}$. Let $h \in \ker T_{f_\infty}$. Then

$$0 = T_{f_\infty} h = h * f_\infty = T_1 T_2 \ldots T_n (h * \tau_n f_\infty), \quad n = 1, 2, \ldots$$

Since $\ker(T_1 T_2 \ldots T_n) = \{0\}$ thus $h * \tau_n f_\infty = 0$ for $n = 1, 2, \ldots$ but the sequence $\tau_n f_\infty$ is an approximate identity for $L_2(G)$ $\big($indeed, $\operatorname{supp} \tau_n f_\infty \subset U_{n-1}$, $\tau_n f_\infty = \lim_k f_k^n \geqslant 0$, $\int_G \tau_n f_\infty = \int_G f_\infty = 1\big)$ hence

$$h = \lim_n h * \tau_n f_\infty = 0.$$

4.4 Remark. If the group G has an commutative, symmetric approximate identity $\{u_k\}$ such that $\operatorname{supp} u_k$ tend to $\{e\}$ (for example if G is an [SIN] group), then we can choose g_n in such a way that Φ is dense in $L_2(G)$.

In fact, we may suppose that $\operatorname{supp} u_k \subset U_k$ and we define

$$g_n = \frac{1}{c_n} \sum_{k=n+1}^{\infty} a_k (u_k * u_k),$$

where a_k are as in Theorem 2.3 and $c_n = \sum_{k=n+1}^{\infty} a_k$. The functions g_n commute and consequently the function $f_\infty = \lim_n g_n * g_{n-1} * \ldots * g_1$ is symmetric. Thus f_∞ is admissible and by 4.2 our statement follows.

The sequence $f_n = g_n * \ldots * g_2 * g_1$ has a subsequence which converges in $L_2(G)$ (hence also in $L_1(G)$) to f_∞ (cf. Theorem 3.4). We use this fact to characterize the set of all functions $h \in \Phi$ such that $\sup_n \|\tau_n h\|_1 < \infty$.

4.5 THEOREM. *Let $h \in \Phi$. Then the inequality $\sup_n \|\tau_n h\|_1 < \infty$ holds if and only if there exist a measure $\mu \in M(G)$ with compact support such that $h = \mu * f_\infty$. If $h \in \Phi_V$, then $\operatorname{supp} \mu \subset V$.*

Proof. Since $\tau_n h$ can be regarded as functionals on $C_0(G)$, in virtue of the assumption $\sup_n \|\tau_n h\|_1 < \infty$, we may assume that the sequence $\tau_n h$ tends *-weakly to a measure $\mu \in M(G)$. We shall prove that $\operatorname{supp} \mu \subset V$.

If a function $\varphi \in C_0(G)$ vanishes in a neighbourhood of V, then since V is compact, $\operatorname{supp} \varphi \cap V U_{n_0} = \emptyset$ for some n_0 which for $n \geqslant n_0$ gives $\langle \varphi, \tau_n h \rangle = 0$ and so $\langle \varphi, \mu \rangle = 0$.

To prove that $h = \mu * f_\infty$ it is sufficient to show that for any $\varphi \in C_0(G)$ the equality

$$\langle \varphi, \mu * f_\infty \rangle = \langle \varphi, h \rangle$$

holds.

For any integer n we have

$$\langle \varphi, h \rangle = \langle \varphi, \tau_n h * f_n \rangle = \langle \varphi * \tilde{f}_n, \tau_n h \rangle.$$

Since $\varphi * \tilde{f}_n \to \varphi * \tilde{f}_\infty$ uniformly on G, we have

$$\langle \varphi, h \rangle = \lim_n \langle \varphi * \tilde{f}_n, \tau_n h \rangle = \langle \varphi * \tilde{f}_\infty, \mu \rangle = \langle \varphi, \mu * f_\infty \rangle.$$

The converse implication is obvious.

4.6 COROLLARY. *The set $\Phi_0 = \{h \in \Phi : \sup_n \|\tau_n h\|_1 < \infty\}$ is dense in $L_2(G)$ if and only if f_∞ is an admissible function.*

Proof. If f_∞ is admissible then $\Phi_0 \supset T_{f_\infty}(\mathscr{K}(G))$, which is dense in $L_2(G)$. On the other hand, if Φ_0 is dense in $L_2(G)$ then, since $\ker T_{f_\infty} = \{0\}$, it suffices to show that $\ker T^*_{f_\infty} = \{0\}$. Let $h \in \ker T^*_{f_\infty} = \ker T_{\tilde{f}_\infty}$ then

$$\langle h, \mu * f_\infty \rangle = \langle h * \tilde{f}_\infty, \mu \rangle = 0$$

for any measure $\mu \in M(G)$ with compact support. Hence h is orthogonal to Φ_0 and so $h = 0$.

4.7 EXAMPLE. For a non-discrete group G there is a function $h \in \Phi$ such that $\sup_n \|\tau_n h\|_1 = \infty$.

In G we choose a convergent sequence s_n of distinct elements such that

$$\|L_{s_n} \tau_n f_\infty - L_{s_{n+1}} \tau_n f_\infty\|_2 < \frac{1}{2^n}.$$

Then for $k < n$

$$\|L_{s_n} \tau_k f_\infty - L_{s_{n+1}} \tau_k f_\infty\|_2 = \|T_{k+1} \ldots T_n (\tau_n L_{s_n} f_\infty - \tau_n L_{s_{n+1}} f_\infty)\|_2$$
$$\leqslant \|\tau_n L_{s_n} f_\infty - \tau_n L_{s_{n+1}} f_\infty\|_2 < \frac{1}{2^n}.$$

Put

$$h_N = \sum_{n=1}^{2N} (-1)^n L_{s_n} f_\infty.$$

Obviously, $h_N \in \Phi$ and $\tau_k h_N = \sum_{n=1}^{2N} (-1)^n L_{s_n} \tau_k f_\infty$ $(k, N = 1, 2, \ldots)$. Since for $k \leqslant 2N$

$$|h_{N+1} - h_N|_{V,k} = \|\tau_k h_{N+1} - \tau_k h_N\|_2 = \|L_{s_{2N+2}} \tau_k f_\infty - L_{s_{2N+1}} \tau_k f_\infty\|_2 < \frac{1}{2^{2N+1}},$$

where $V = \{s_1, s_2, \ldots\} \cup \{\lim s_n\}$, is a compact set, h_N converges in Φ and $\lim\limits_N h_N = h \in \Phi$.

We are going to prove that $\|\tau_k h\|_1$ is an unbounded sequence. By the Schwartz inequality we get

$$\|\tau_k h_N - \tau_k h\|_1 \leqslant [\mu(VU_k)]^{1/2} \cdot |h_N - h|_{V,k}, \qquad k, N = 1, 2, \ldots,$$

hence

$$\|\tau_k h\|_1 = \lim_N \|\tau_k h_N\|_1, \qquad k = 1, 2, \ldots$$

For an integer M we can find a k such that the sets $s_1 U_k, s_2 U_k, \ldots$ $\ldots, s_M U_k\, (V - \{s_1, s_2, \ldots, s_M\}) U_k$ are pairwise disjoint, hence for $N \geqslant M$

$$\|\tau_k h_N\|_1 \geqslant \sum_{n=1}^{M} \|L_{s_n} f_\infty\|_1 = M$$

and so $\|\tau_k h\|_1 \geqslant M$.

5. The construction of a bi-invariant space. Let $\Phi^\vee = \{f^\vee : f \in \Phi\}$, where $f^\vee(s) = f(s^{-1})$. In $\Phi^\vee$ we transfere the topology from Φ by the map $f \to f^\vee$.

The bilinear map $Y: \Phi \times \Phi^\vee \to \mathscr{K}(G)$ defined by $Y(f, g^\vee) = f * g^\vee$ is obviously separately continuous, hence by Proposition 13 ([2], § 3 n°1) it determines the unique continuous linear map $Y: \Phi \overline{\otimes} \Phi^\vee \to \mathscr{K}(G)$ (if E and F are locally convex linear spaces, then $E \hat{\otimes} F$ and $E \overline{\otimes} F$ denote the completion of $E \otimes F$ in the projective or the inductive topology, respectively, cf. [2]). We define $\mathscr{Y}$ to be the space $Y(\Phi \overline{\otimes} \Phi^\vee)$ equipped with the quotient topology.

5.1 Lemma ([1], 3.6). *Each function $f \in \mathscr{Y}$ is a sum of an absolutely convergent series*

(11) $$f = \sum_{i=1}^{\infty} \lambda_i f_i * h_i^\vee,$$

where $\{\lambda_i\}$ is a sequence of scalars such that $\sum\limits_{i=1}^{\infty} |\lambda_i| \leqslant 1$ and the sequences $\{f_i\}$ and $\{h_i\}$ converge to zero in Φ. Conversely, every function of the form (11) belongs to $\mathscr{Y}$.

Proof. Let $G = \bigcup\limits_{i=1}^{\infty} V_i$ where V_i, $i = 1, 2, \ldots$, is an increasing sequence of compact subsets of G. If $\Phi_{V_i}^\vee$ is the image of Φ_{V_i} by the map $f \to f^\vee$, then, by proposition 14, [2], § 3 n°1, $\Phi \overline{\otimes} \Phi^\vee = \operatorname{limind}\limits_i \Phi_{V_i} \overline{\otimes} \Phi_{V_i}^\vee$, but the spaces Φ_{V_i}, $i = 1, 2, \ldots$, and consequently $\Phi_{V_i}^\vee$ are nuclear Fréchet spaces so $\Phi_{V_i} \overline{\otimes} \Phi_{V_i}^\vee = \Phi_{V_i} \hat{\otimes} \Phi_{V_i}^\vee$ and hence

(12) $$\Phi \overline{\otimes} \Phi^\vee = \operatorname{limind}_i \Phi_{V_i} \hat{\otimes} \Phi_{V_i}^\vee.$$

Now let $f \in \mathscr{Y}$, then $f = Y(\xi)$ for some $\xi \in \Phi \overline{\otimes} \Phi^\vee$. But $\xi \in \Phi \overline{\otimes} \Phi^\vee$ if and only if for some k, $\xi \in \Phi_{V_k} \hat{\otimes} \Phi_{V_k}^\vee$ and hence by Theorem 1 ([2], § 2 n°1) it is of the form $\xi = \sum\limits_{i=1}^{\infty} \lambda_i f_i \otimes h_i^\vee$ where $\sum\limits_{i=1}^{\infty} |\lambda_i| \leqslant 1$ and $\{f_i\}$, $\{h_i\}$ are convergent to zero in Φ_{V_k}.

On the other hand, if f_i and h_i converge to zero in Φ, then for a k they converge to zero in Φ_{V_k}. Hence, passing to the quotient space, the lemma follows.

5.2 Theorem ([1], 3.9). *The space $\mathscr{Y} \subset \mathscr{K}(G)$ is a complete nuclear $\mathscr{LF}$-space. $\mathscr{Y}$ is a two-sided ideal (with respect to the convolution) in $\mathscr{K}(G)$ and is closed with respect to taking of complex conjugates and the operation $^\vee$. The operation $^\vee$ and the mapping*

$$\mathscr{K}(G) \times \mathscr{Y} \ni (f, \varphi) \to f * \varphi \in \mathscr{Y}$$

are continuous. The left and right regular representation of G on $\mathscr{Y}$ are jointly continuous.

Proof. In virtue of (12) $\Phi \overline{\otimes} \Phi^\vee$ is a nuclear $\mathscr{LF}$-space, consequently $\mathscr{Y}$, as a quotient space is a complete nuclear $\mathscr{LF}$-space.

If $\varphi \in \mathscr{K}(G)$ and $f \in \mathscr{Y}$, is of the form (11), then, since $(f_i * h_i^\vee) * \varphi = f_i * (h_i^\vee * \varphi) = f_i * (\varphi^\vee * h_i)^\vee$ and $(f_i * h_i^\vee)^\vee = h_i * f_i^\vee$ we see that

$$\varphi * f = \sum_{i=1}^{\infty} \lambda_i (\varphi * f_i) * h_i^\vee,$$

$$f * \varphi = \sum_{i=1}^{\infty} \lambda_i f_i * (\varphi^\vee * h_i)^\vee$$

and

$$f^\vee = \sum_{i=1}^{\infty} \lambda_i h_i * f_i^\vee$$

are in $\mathscr{Y}$. Moreover, if $s \in G$, then $L_s(f_i * h_i^\vee) = L_s f_i * h_i^\vee$ and $R_s(f_i * h_i^\vee) = f_i * R_s h_i^\vee = f_i * (L_s h_i)^\vee$, thus

$$L_s f = \sum_{i=1}^{\infty} \lambda_i L_s f_i * h_i^\vee,$$

$$R_s f = \sum_{i=1}^{\infty} \lambda_i f_i * (L_s h_i)^\vee$$

are in $\mathscr{Y}$.

The map $P: \Phi \times \Phi^\vee \to \Phi \overline{\otimes} \Phi^\vee$ defined by

$$P(f, g^\vee) = g \otimes f^\vee$$

is a separately continuous bilinear map, so we may extend it to a continuous linear operator P in $\Phi\hat{\otimes}\Phi^{\check{}}$. Since P^2 is the identity operator, we see that P is an isomorphism "onto". Moreover,

$$(Y\xi)^{\check{}} = YP\xi$$

hence $\check{}$ is continuous.

To prove that the left regular representation L of the group G on $\mathscr{Y}$ is jointly continuous, it is sufficient to show that

(a) $L_s\colon \mathscr{Y}\to\mathscr{Y}$ is continuous,

(b) $G\ni s\to L_s f$ is continuous.

(a) follows from the observation that

$$L_s Y = Y(L_s\otimes I) \tag{13}$$

(I is the identity operator on $\Phi^{\check{}}$) is continuous and that Y is an open map.

To prove (b) we observe first that, by (13), it suffices to show that $s\to(L_s\otimes I)\xi$, $s\in G$, is continuous for all $\xi\in\Phi\hat{\otimes}\Phi^{\check{}}$.

Let $\xi_0 = \sum_{i=1}^{\infty}\lambda_i f_i\otimes h_i^{\check{}}$ (note that for a k we have $\xi_0\in\Phi_{V_k}\hat{\otimes}\Phi_{V_k}^{\check{}}$). Let $s\in G$ and let $\mathscr{W}$ be a neighbourhood of $(L_s\otimes I)\xi_0$. Let finally U be a conditionally compact neighbourhood of e and l such that $sUV_k\subset V_l$ (then $(L_u\otimes I)\xi_0\in\Phi_{V_l}\hat{\otimes}\Phi_{V_l}^{\check{}}$ for all $u\in sU$). The definition of the topology in $\Phi\hat{\otimes}\Phi^{\check{}}$ shows that for a pair of integers m, n and $\varepsilon>0$

$$\mathscr{W}\supset\{\xi\in\Phi_{V_l}\hat{\otimes}\Phi_{V_l}^{\check{}}\colon p\otimes q[\xi-(L_s\otimes I)\xi_0]<\varepsilon\},$$

where $p(f) = |f|_{V_l,n}$ and $q(f^{\check{}}) = |f|_{V_l,m}$, $f\in\Phi_{V_l}$. Since $\lim f_i = \lim h_i = 0$, there is an M such that for $i>M$ $p(f_i)\cdot q(h_i^{\check{}})<\varepsilon/2$; then for $u\in sU$

$$p(L_u f_i - L_s f_i)\cdot q(h_i^{\check{}})<\varepsilon.$$

Next, by Theorem 3.9, there exists a neighbourhood $W\subset U$ such that

$$p(L_u f_i - L_s f_i)<\varepsilon[q(h_i^{\check{}})]^{-1}$$

for $i=1,2,\ldots,M$ and $s^{-1}\in W$. Hence

$$p\otimes q[(L_u\otimes I)\xi_0-(L_s\otimes I)\xi_0] = p\otimes q[(L_u-L_s)\otimes I\xi_0]$$

$$\leqslant\sum_{i=1}^{\infty}|\lambda_i| p(L_u f_i - L_s f_i)\cdot q(h_i^{\check{}})<\varepsilon\sum_{i=1}^{\infty}|\lambda_i|\leqslant\varepsilon.$$

This means that $(L_u\otimes I)\xi_0\in\mathscr{W}$ for all $u\in sW$ and thus (b) is proved.

The joint continuity of the right regular representation follows immediately from the continuity of operation $\check{}$.

6. The space Ψ. Now we are going to construct another nuclear space consisting of rapidly decreasing functions on a locally compact group G. As the space Φ is similar to Schwartz's space D, the new space Ψ resembles the space S, and, accordingly, it is a nuclear Fréchet space. The most important feature of Ψ is that it is dense in $L_2(G)$ (the fact which we were unable to prove for Φ). The construction of Ψ is analogous to that of Φ—this also consists of infinitely regularized functions, but as the functions in Φ have compact support, the behavior of a function in Ψ at infinity is measured by submultiplicative functions (in the case of S it is done by the functions $\varrho_n(x) = (1+|x|^2)^{-n}$).

A non-negative, continuous function φ on G is called *submultiplicative* if

$$\varphi(su)\leqslant\varphi(s)\varphi(u)\quad\text{for all } s, u\in G.$$

A continuous function is called *rapidly decreasing* if the product of it and any submultiplicative function is a bounded function. Let $\mathscr{E}(G)$ denote the space of all rapidly decreasing functions. The set of pseudo-norms

$$\|f\|_{\varphi} = \sup_{s\in G}|f(s)\varphi(s)|$$

define a locally convex topology in $\mathscr{E}(G)$.

6.1 Lemma ([4], 1.1). *For a compactly generated group G there exists a submultiplicative function φ_0 on G such that $\varphi_0^{-1}\in L_1(G)$ and for any submultiplicative function φ there exist integers M and k such that $\varphi\leqslant M\varphi_0^k$.*

For the proof see [4].

We now suppose that G is a compactly generated metrizable group and we fix a sequence $\{g_n\}$ of admissible functions which are symmetric and rapidly decreasing. We write T_n for T_{g_n} and also τ_n for $\tau_{g_n}\tau_{g_{n-1}}\cdots\tau_{g_1}$.

6.2 Definition. *Let Ψ denote the space of all functions f on G such that $\tau_n f$ is rapidly decreasing for all n. The topology in Ψ is defined by the set of norms*

$$p_{\varphi,n}(f) = \sup_{s\in G}|\varphi(s)\tau_n f(s)|.$$

The space Ψ is metrizable and a sequence $f_j\in\Psi$ converges to zero in Ψ if and only if for any fixed n and submultiplicative function φ the sequence $\varphi\tau_n f_j$ converges uniformly to zero.

If g_n are the same as in Section 3, then $\Psi\supset\Phi$.

6.3 Lemma. *Let g be a rapidly decreasing admissible function and let $a\geqslant 0$. The mapping $L_2(G)\ni h\to\varphi_0^a T_g(h\varphi_0^{-a-1/2})\in L_2(G)$ is a Hilbert–Schmidt operator.*

Proof. We have $\varphi_0^a T_g(h\varphi_0^{-a-1/2})(s) = \int_G\mathscr{K}_a(u,s)h(u)\,du$ where $\mathscr{K}_a(u,s) = \varphi_0^a(s)\varphi_0^{-a-1/2}(u)g(u^{-1}s)$. It suffices to show that $\mathscr{K}_a(u,s)$ is square summable on $G\times G$. We have

$$\int_G\int_G |\mathscr{K}_a(u,s)|^2\,du\,ds = \int_G\int_G \varphi_0^{2a}(s)\varphi_0^{-2a-1}(u)|g(u^{-1}s)|^2\,ds\,du$$

$$= \int_G\int_G \varphi_0^2(us)\varphi_0^{-2a-1}(u)|g(s)|^2\,ds\,du \leqslant \int_G\int_G \varphi_0^{-1}(u)\varphi_0^{2a}(s)|g(s)|^2\,ds\,du$$

$$= \|\varphi_0^{-1}\|_1 \int_G \varphi_0^{-1}(s)\varphi_0^{2a+1}(s)|g(s)|^2\,du \leqslant \|\varphi_0^{-1}\|_1^2 \sup_{s\in G}|\varphi_0^{a+1/2}(s)g(s)|^2.$$

6.4 Corollary. *The operator* $S_n\colon L_2(G)\to L_2(G)$, $n=1,2,\ldots$, *defined by*

$$S_n f = \varphi_0^n T_n(f\varphi_0^{-n-1})$$

is of Hilbert–Schmidt type.

Now we give a simple characterization of the space Ψ.

6.5 Proposition. *A function f belongs to Ψ if and only if $\varphi_0^n\tau_n f\in L_2(G)$ for $n=1,2,\ldots$ The topology of Ψ coincides with the topology given by the set of norms*

$$|f|_n = \|\varphi_0^n\tau_n f\|_2.$$

Proof. The first part of the proposition follows from the inequality

$$\|\varphi_0^n\tau_n f\|_2 = \|\varphi_0^{-1}\varphi_0^{n+1}\tau_n f\|_2 \leqslant \|\varphi_0^{-1}\|_2 \sup_{s\in G}|\varphi_0^{n+1}(s)\tau_n f(s)|.$$

To prove the second part, we first show that for any two integers n and m there is a constant $c_{n,m}$ such that

$$\|\varphi_0^{n+m}\tau_n f\|_2 \leqslant c_{n,m}\|\varphi_0^{n+2m}\tau_{n+2m}f\|_2. \tag{14}$$

By Lemma 6.2 the mapping

$$h\to\varphi_0^{n+m}[T_{n+1}T_{n+2}\ldots T_{n+2m}(h\varphi_0^{-(n+2m)})],\qquad h\in L_2(G)$$

is continuous. Letting h to be $\varphi_0^{n+2m}\tau_{n+2m}f$ and $c_{n,m}$ the norm of this mapping we get (14).

Now, if φ is a submultiplicative function on G, then

$$p_{\varphi,n}(f) = \sup_{s\in G}|\varphi(s)\tau_n f(s)| = \sup_{s\in G}\int_G \varphi(s)|\tau_{n+1}f(u)g_{n+1}(u^{-1}s)|\,du$$

$$\leqslant \sup_{s\in G}\int_G \varphi(u)\varphi(u^{-1}s)|\tau_{n+1}f(u)g_{n+1}(u^{-1}s)|\,du \leqslant \|g_{n+1}\|_\varphi\|\varphi\cdot\tau_{n+1}f\|_1$$

but, by Lemma 6.1 and by the Schwartz inequality, we have

$$\|\varphi\tau_{n+1}f\|_1 \leqslant M\|\varphi_0^k\tau_{n+1}f\|_1 \leqslant M\|\varphi_0^{-1}\|_2\|\varphi_0^{k+1}\tau_{n+1}f\|_2.$$

This together with (14) shows that there is an integer m and a constant N such that

$$p_{\varphi,n}(f) \leqslant N|f|_m$$

6.6 Lemma. *Let φ be a submultiplicative function. If $\varphi\cdot f\in L_1(G)$ and $\varphi\cdot h\in L_2(G)$, then $\varphi\cdot(f*h)\in L_2(G)$ and the inequality*

$$\|\varphi(f*h)\|_2 \leqslant \|\varphi\cdot f\|_1\|\varphi\cdot h\|_2 \tag{15}$$

holds.

Proof. We may suppose that f and h are non-negative. Since for $s,u\in G$, we have $\varphi(s)\leqslant\varphi(u)\cdot\varphi(u^{-1}s)$, therefore

$$\varphi(s)(f*h)(s) = \int_G \varphi(s)f(u)h(u^{-1}s)\,du$$

$$\leqslant \int_G \varphi(u)f(u)\varphi(u^{-1}s)h(u^{-1}s)\,du = (\varphi f)*(\varphi h)(s)$$

and by (2), inequality (15) follows.

6.7 Corollary. *If $h\in\Psi$ and f is a rapidly decreasing function, then $f*h\in\Psi$.*

6.8 Theorem. *Let G be a compactly generated, metrizable group. The space Ψ is a nuclear B_0-space; it is invariant under left regular representation L, and closed under convolution from the left by functions of $\mathscr{E}(G)$. Both operations*

$$G\times\Psi\ni(s,f)\to L_s f\in\Psi \tag{16}$$

and

$$\mathscr{E}(G)\times\Psi\ni(h,f)\to h*f\in\Psi \tag{17}$$

are continuous.

Proof. Let Ψ_n be the completion of Ψ in the norm $|\ |_n$; then $\Psi_1\supset\Psi_2\supset\ldots$ and $\Psi=\bigcap_{n=1}^\infty\Psi_n$, thus Ψ is a complete space.

If $\sigma_n f = \varphi_0^n\tau_n f$ denotes the isometric map of Ψ_n into $L_2(G)$ then by 6.4 the natural embedding $\iota_n = \sigma_n^{-1}S_n\sigma_{n+1}$ of Ψ_{n+1} into Ψ_n is of Hilbert–Schmidt type, which proves that Ψ is a nuclear space.

Continuity of the map (17) follows by the inequality

$$|f*h|_n \leqslant \|\varphi_0^n f\|_1\cdot|h|_n,\qquad n=1,2,\ldots$$

which is an immediate consequence of (15).

To prove (16) we apply the inequality

$$|L_s h|_n \leqslant \varphi_0^n(s)\cdot|h|_n,\qquad n=1,2,\ldots,\qquad s\in G,\ h\in\Psi$$

which is obtained by replacing the function f in Lemma 6.6 by a measure concentrated at a point s.

7. Density of Ψ in $L_2(G)$. We shall prove that it is possible to obtain a space Ψ which is dense in $L_2(G)$. The proof is based on the following

7.1 THEOREM ([4], 2.1). *Let G be compactly generated metric group. There exists a sequence of functions p_j, $j=1,2,\ldots$ in $\mathscr{E}(G)$ which has the following properties:*

(i) $p_j(s)\geqslant 0$,

(ii) $p_j^*=p_j$ (*i.e.* $\Delta(s)\overline{p_j(s^{-1})}=p_j(s)$),

(iii) $p_i*p_j=p_j*p_i$ *for all* $i,j=1,2,\ldots$,

(iv) *for every submultiplicative function* φ *on* G *there is a constant* C_φ *such that* $\|p_j*f\|_\varphi\leqslant C_\varphi\|f\|_\varphi$ *for all* f *in* $\mathscr{E}(G)$ *and* $j=1,2,\ldots$,

(v) *for every* f *in* $\mathscr{E}(G)$ *the sequence* p_j*f *is convergent to* f *in the topology of* $\mathscr{E}(G)$.

For the proof see [4].

7.2 COROLLARY. *Let G be compactly generated metric group. There exists a sequence g_j, $j=1,2,\ldots$ of admissible rapidly decreasing functions such that*

(i) $\tilde{g}_j=g_j$ (*i.e.* $\overline{g_j(s^{-1})}=g_j(s)$),

(ii) $g_i*g_j=g_j*g_i$ *for all* $i,j=1,2,\ldots$,

(iii) *for every submultiplicative function* φ *on* G *there is a constant* C_φ *such that* $\|g_j*f\|_\varphi\leqslant C_\varphi\|f\|_\varphi$ *for all* f *in* $\mathscr{E}(G)$ *and* $j=1,2,\ldots$,

(iv) $\|g_n*g_{n+1}-g_n\|_{\varphi_0^n}<\varepsilon_n C_{\varphi_0^n}^n$ *with* $\sum_{n=1}^{\infty}\varepsilon_n<\|g_1\|_{\varphi_0}$.

Proof. First we observe that, by using the trick from remark 4.4 (the sequence a_j being such that $\sum_{j=1}^{\infty}a_j\|p_j*p_j\|_{\varphi_0^j}<\infty$), we may suppose that the sequence p_j, $j=1,2,\ldots$, consists of admissible functions.

It is easy to prove that the functions

$$g_j=\Delta^{-1/2}p_j$$

are admissible and satisfy (i)–(iii). The property (iv) is obtained by applying 7.1 (v) to select a suitable subsequence of $\{g_j\}$.

7.3 THEOREM. *Let G be a compactly generated metric group. There is a space Ψ which is dense in $L_2(G)$.*

Proof. First we prove that for any integer k the sequence $h_n^k=g_{k+1}*g_{k+2}*\ldots*g_n$, $n=k+1,k+2,\ldots$, is convergent in $\mathscr{E}(G)$.

Let φ be any submultiplicative function on G. In virtue of 6.1 for sufficiently large n we have $\varphi\leqslant M\varphi_0^n$. Hence

$$\|h_{n+1}^k-h_n^k\|_\varphi\leqslant M\|g_{k+1}*\ldots*g_{n-1}*(g_n*g_{n+1}-g_n)\|_{\varphi_0^n}$$
$$\leqslant MC_{\varphi_0^n}^n\|g_n*g_{n+1}-g_n\|_{\varphi_0^n}\leqslant M\varepsilon_n.$$

Since $\sum_{n=1}^{\infty}\varepsilon_n<+\infty$, the sequence h_n^k is convergent. Let $h_\infty=\lim h_n^0$; then $\tau_k h_\infty=\lim_n h_n^k$ is a rapidly decreasing function for $k=0,1,2,\ldots$, thus $h_\infty\in\Psi$. The function h_∞ is non-zero because

$$\|h_\infty\|_{\varphi_0}\geqslant\|g_1\|_{\varphi_0}-\lim\|g_1-h_n^0\|_{\varphi_0}\geqslant\|g_1\|_{\varphi_0}-\sum_{n=1}^{\infty}\varepsilon_n>0.$$

Since g_n commute and are admissible, therefore h_∞ is admissible too, and consequently $T_{h_\infty}(\mathscr{K}(G))$ is dense in $L_2(G)$, but by 6.7 $T_{h_\infty}(\mathscr{K}(G))\subset\Psi$.

References

[1] J. F. Aarnes, *A large bi-invariant nuclear function-space on a locally compact group* (to appear).

[2] A. Grothendieck, *Produits tensoriels topologique et espaces nucléaires*, Memoirs of Amer. Math. Soc. 16 (1955).

[3] E. Hewitt and K. Ross, *Abstract harmonic analysis*, Berlin 1963.

[4] A. Hulanicki and T. Pytlik, *On commutative approximate identities and cyclic vectors of induced representations*, Studia Math. 48 (1973), pp. 189–199.

[5] T. Pytlik, *A nuclear space of functions on a locally compact group*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astr. et Phys. 3 (1969), pp. 161–166.

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