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On control submeasures and measures

by

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Abstract. Let $\mathscr{R}$ be a σ-ring of sets, X a locally convex space, $\mu\colon \mathscr{R}\to X$ a σ-additive set function. Let $\Gamma(\mu)$ denote the coarsest of the so-called Fréchet–Nikodym topologies on $\mathscr{R}$ making μ continuous. The following question is considered: Under which conditions there exists a control measure for μ, i.e. an additive function $\nu\colon \mathscr{R}\to[0,\infty)$ such that $\Gamma(\mu)=\Gamma(\nu)$? It is proved (Theorem 2.4) that a necessary and sufficient condition is that every family of mutually disjoint non μ-zero sets is at most countable. This is a consequence of Theorem 1.2 concerning semimetrizability of order continuous Fréchet–Nikodym topologies, and evidently constitutes a generalization of a well-known theorem of Bartle, Dunford and Schwartz. In the case X is a normed space, as was first shown by Rybakov, there exists $x_0'\in X'$ such that the variation $v(x_0'\mu)$ of $x_0'\mu$ is a control measure for μ. By a theorem of B. Walsh, the set of all x_0' with that property is norm-dense and G_δ in X'. A part of the present paper is devoted to some generalizations of their results. The above question is discussed also for $\mathscr{R}$ being a ring and μ an exhaustive (= strongly bounded) additive set function ($\mu(E_n)\to 0$ if E_n are disjoint). In particular, a direct proof of a result due to Brooks and Hoffmann–Jørgensen is given. There are also some results on the existence of a control submeasure for group-valued set functions.

Introduction. Suppose that μ is an additive function from a ring of sets $\mathscr{R}$ into a topological abelian group G. We ask if there exists an additive non-negative measure ν on $\mathscr{R}$ which is equivalent in a sense with μ.

The well-known theorem of Bartle, Dunford and Schwartz [2] states that: If $\mathscr{R}$ is a σ-field, X a normed linear space and $\mu\colon \mathscr{R}\to X$ is σ-additive, then a σ-additive measure $\nu\colon \mathscr{R}\to[0,\infty)$ exists such that $\nu(E_n)\to 0$ iff $\|\mu\|(E_n)\to 0$, where $\|\mu\|(:)$ denotes the semivariation of μ. In this statement the function $\|\mu\|(:)$ can be replaced by $\bar{\mu}(:)$, the submeasure majorant for μ with respect to $\|\cdot\|$, defined by $\bar{\mu}(E)=\sup\{\|\mu(F)\|\colon F\subset E,\ F\in\mathscr{R}\}$; in fact, $\bar{\mu}(:)\leqslant\|\mu\|(:)\leqslant 4\bar{\mu}(:)$.

Although the theorem is almost classic, it seems necessary to clarify its 'topological' contents, all the more since we wish to motivate here our approach to the question posed above. In order to do this one should first generalize the concept of a space of measurable sets, introduced over forty years ago by M. Fréchet and O. Nikodym (see e.g. [10; III. 7]). This is realized by considering, instead of the Fréchet–Nikodym semimetric generated by a measure on $\mathscr{R}$, an 'abstract' topology Γ, satisfying certain natural conditions, called an FN-topology on $\mathscr{R}$. Then it is easy to observe that for every additive function $\mu\colon \mathscr{R}\to G$ there exists the weakest FN-topology on $\mathscr{R}$, $\Gamma(\mu)$, under which μ is continuous. If G is semimetrizable then so is $\Gamma(\mu)$. In particular, in the setting of the Bartle–Dunford–Schwartz theorem, it is the subadditive function $\bar{\mu}$ that through the Fréchet–Nikodym type semimetric $\varrho(A,B)=\bar{\mu}(A\Delta B)$ semimetrizes the topology $\Gamma(\mu)$. This explains the role of $\bar{\mu}(:)$ or $\|\mu\|(:)$ in the theorem. It is now clear that the theorem asserts the existence of a finite σ-additive measure ν which also semimetrizes $\Gamma(\mu)$, i.e. $\Gamma(\mu)=\Gamma(\nu)$. Following Brooks [3] we shall call such a measure ν a control measure for μ (and $\bar{\mu}$ a control submeasure for μ). We shall also say that ν controls μ.

Now we are in a position to make precise the question stated at the beginning. Actually, we should ask two questions:

1° Under which conditions the topology $\Gamma(\mu)$ is semimetrizable? And, if we know that for some μ the answer is affirmative,

2° does there exist an additive non-negative measure ν on $\mathscr{R}$ which controls μ?

But while the first question seems to be sensible for μ taking values in an arbitrary group G, there is no reason—in such a general setting—to expect any valuable answer for the second question. In fact, it is rather hardly to imagine another way of obtaining an additive measure ν required in 2°, than by constructing ν from scalar valued additive functions which in some natural manner may be associated with μ. (See however [22].) Therefore, as concerns 2°, we restrict ourselves to the case where G is a Hausdorff locally convex space X, since then we have a bijective correspondence between additive functions $\mu\colon \mathscr{R}\to X$ and families $\{x'\mu\colon x'\in X'\}$ of scalar additive functions. Nevertheless, it turns out that a condition both necessary and sufficient for semimetrizability of $\Gamma(\mu)$ and for the existence of a control measure for μ is the same:

(ccc) *every family of pairwise disjoint non-μ-zero sets from $\mathscr{R}$ should be at most countable,*

provided $\mathscr{R}$ is a σ-ring and μ is σ-additive. This condition is due to Dubrovskiĭ [9].

The contents of the present paper can be roughly described as follows. Section 0 is preliminary. In Section 1 we consider the question of semimetrizability of order continuous FN-topologies on σ-rings. Then in Section 2 we give complete answers for the questions 1° and 2° in the case $\mathscr{R}$ is a σ-ring and μ is σ-additive. Here it is also observed that a measure or submeasure which controls μ does not depend on the particular choice of a topology on X or G under which μ is σ-additive. Section 3 is devoted to some generalizations or improvements of recent results of Rybakov [25] and Walsh [29]. Finally, in Section 4 we deal with the problem of the existence of a control measure for exhaustive additive set functions.

A number of results analogous to those in Sections 2 and 3 is presented, but neither 1° nor 2° is solved in a satisfactory manner. Section 4 contains also a direct proof of a theorem obtained via the Stone representation theorem by Hoffman–Jørgensen [15] and Brooks [3]. (¹)

The author is indebted to the referee for a number of very useful remarks on this paper.

0. Terminology and notation used in this paper is in principle that of [4], [7]. A brief description of such notions as Fréchet–Nikodym (FN-) topologies, exhaustive (exh.) and order continuous (o.c.) FN-topologies or set functions etc. is given in [6].

Everywhere in the sequel $\mathscr{R}$ denotes a ring of sets (σ-ring if explicitely stated), G a Hausdorff topological abelian group, X a Hausdorff locally convex vector space.

A set function μ on $\mathscr{R}$ is called in this paper:

1) a *submeasure* if $\mu\colon \mathscr{R}\to\bar{R}_+$ and $\mu(\emptyset)=0$, $E\subset F$ implies $\mu(E)\leqslant\mu(F)$, $\mu(E\cup F)\leqslant\mu(E)+\mu(F)$ for any $E,F\in\mathscr{R}$;

2) a *measure* if $\mu\colon \mathscr{R}\to\bar{R}_+$ or $\mu\colon \mathscr{R}\to G$ and μ is finitely additive. A σ-measure is a σ-additive measure.

We say that μ is *exhaustive* (exh.) (resp., *order continuous*, o.c.) if $\mu(E_n)\to 0$ whenever E_n are disjoint (resp., $E_n\searrow\emptyset$). Evidently, a G-valued measure is o.c. iff it is σ-additive. A scalar valued measure is exh. iff it is bounded.

Let M be a family of set functions on $\mathscr{R}$, each $\mu\in M$ being a submeasure or a topological group valued measure. Then $\Gamma(M)$ denotes the weakest FN-topology on $\mathscr{R}$ under which every $\mu\in M$ is continuous; if $M=\{\mu\}$, we write simply $\Gamma(\mu)$. If Γ is an FN-topology then $\nu\ll\Gamma$ means that ν is Γ-continuous, i.e. $\Gamma(\nu)\subset\Gamma$; we write $\nu\ll\mu$ if $\Gamma=\Gamma(\mu)$. We say that submeasures or measures μ,ν are equivalent, in symbols $\mu\sim\nu$, iff $\mu\ll\nu$ and $\nu\ll\mu$, i.e. $\Gamma(\mu)=\Gamma(\nu)$. $\mathscr{N}(\Gamma)$ denotes the ideal of all Γ-zero sets $E\in\mathscr{R}$, that is $\mathscr{N}(\Gamma)=$ the closure of $\{\emptyset\}$ in $(\mathscr{R},\Gamma)$. If M is as above, we write $\mathscr{N}(M)$ instead of $\mathscr{N}(\Gamma(M))$; in particular $\mathscr{N}(\mu)=\mathscr{N}(\Gamma(\mu))$. Sets $E\in\mathscr{N}(\mu)$ are called *μ-zero*. Clearly, E is μ-zero iff $\mu(F)=0$ for every $F\subset E$, $F\in\mathscr{R}$ (G is supposed Hausdorff). $\mathscr{N}(M)=\bigcap_{\mu\in M}\mathscr{N}(\mu)$.

Remark. Let us note that every FN-topology Γ on $\mathscr{R}$ can be represented as $\Gamma(\mu)$, where μ is a measure. Indeed, the canonical mapping of $\mathscr{R}$ onto the quotient topological group $G=(\mathscr{R},\Delta)/\mathscr{N}(\Gamma)$ can serve as such μ.

Everywhere below countable means the same as at most countable.

(¹) The main results of Sections 1, 2 and 4 were presented at the meeting of the Polish Mathematical Society, Poznań Branch, held at 1 February 1972, and then at Professor Orlicz's seminar.

1. Semimetrizability of FN-topologies. Let Γ be an FN-topology on a ring $\mathscr{R}$. It is known that the properties (i) Γ has a countable base at $\emptyset$, (ii) Γ is semimetrizable and (iii) there exists a submeasure η such that $\Gamma = \Gamma(\eta)$, are equivalent [4]. (Recall that if Γ is exh. or o.c., then so is η in (iii), and conversely.)

In this section we give another condition assuring semimetrizability of Γ.

Definition. We say that Γ satisfies the countable chain condition, (ccc), if every family of pairwise disjoint sets from $\mathscr{R} \setminus \mathscr{N}(\Gamma)$ is (at most) countable.

The name for this property is motivated by the fact that Γ satisfies (ccc) iff the quotient Boolean ring $\mathscr{R}/\mathscr{N}(\Gamma)$ does so. A family M of submeasures or measures is said to satisfy (ccc) iff the corresponding FN-topology $\Gamma(M)$ satisfies (ccc).

In the remainder of this section we shall assume that $\mathscr{R}$ is a σ-ring and $\Gamma, \Gamma_1, \Gamma_2$ are FN-topologies on $\mathscr{R}$.

1.1. Theorem. *If Γ_1 is o.c., Γ_2 is semimetrizable and $\mathscr{N}(\Gamma_2) \subset \mathscr{N}(\Gamma_1)$, then $\Gamma_1 \subset \Gamma_2$.*

In view of Remark in Section 0, this is simply a reformulation of theorem 2.9(c) in [6].

The next is the main theorem of this section; it is closely related to some results of Vladimirov ([28]; III. 4).

1.2. Theorem. *Suppose that Γ is o.c., and let M be any family of (necessarily o.c.) submeasures on $\mathscr{R}$ such that $\Gamma = \Gamma(M)$. Then the following are equivalent:*

(a) *Γ satisfies* (ccc).

(b) *There exists a countable subfamily H of M with $\Gamma(H) = \Gamma(M)$.*

(c) *Γ is semimetrizable, i.e. there exists an o.c. submeasure λ such that $\Gamma = \Gamma(\lambda)$.*

Proof. (a)$\Rightarrow$(b): Evidently $A \in \mathscr{N}(\Gamma)$ iff $A \in \mathscr{N}(M)$ iff $\gamma(A) = \sup\{\eta(A)\colon \eta \in M\} = 0$. Let $\mathscr{E}$ be the family of all sets $E \in \mathscr{R} \setminus \mathscr{N}(M)$ for which there is a submeasure $\eta_E \in M$ such that

$$(*) \qquad \text{if } A \subset E \text{ and } \eta_E(A) = 0, \quad \text{then } \gamma(A) = 0.$$

We first prove that every set $F \in \mathscr{R} \setminus \mathscr{N}(M)$ contains a set $E \in \mathscr{E}$. Indeed, choose $\eta \in M$ so that $\eta(F) > 0$. Then, applying the Kuratowski–Zorn Principle, we find a maximal disjoint family $\mathscr{D}$ of subsets D of F such that $\eta(D) = 0$ but $\gamma(D) > 0$. $\mathscr{D}$ is countable by (ccc) and it is readily seen that $E = F \setminus \bigcup \mathscr{D}$ and $\eta_E = \eta$ are as needed.

Now, again by the Kuratowski–Zorn Principle, we find a maximal disjoint family $\mathscr{C} \subset \mathscr{E}$; $\mathscr{C}$ is countable by (ccc). By choosing for each $E \in \mathscr{C}$ a submeasure $\eta_E \in M$ satisfying $(*)$ we form a family H, which is as required in (b). In fact, otherwise we find a set $F \in \mathscr{N}(H) \setminus \mathscr{N}(M)$ which can be made disjoint with $\bigcup \mathscr{C}$, and then application of what was proved in the preceding paragraph yields a contradiction with the maximality of $\mathscr{C}$.

(b)$\Rightarrow$(c): Let H be as in (b). Then the FN-topology $\Gamma(H)$ is semimetrizable and $H \subset M$ implies $\Gamma(H) \subset \Gamma$. Since $\mathscr{N}(\Gamma(H)) \subset \mathscr{N}(\Gamma)$, we have $\Gamma \subset \Gamma(H)$ by 1.1. Let $H = \{\eta_1, \eta_2, \ldots\}$. Then a submeasure λ for which $\Gamma = \Gamma(\lambda)$ can be defined for example thus:

$$(+) \qquad \lambda(E) = \sum_n 2^{-n} \min\{1, \eta_n(E)\}$$

or

$$(++) \qquad \lambda(E) = \sum_n 2^{-n} \frac{\eta_n(E)}{1+m_n},$$

where $m_n = \sup\{\eta_n(A)\colon A \in \mathscr{R} \text{ and } \eta_n(A) < \infty\}$; $m_n < \infty$ by [4]; 4.10.

Note that if all η_n are additive then so is λ defined by the second equality.

(c)$\Rightarrow$(a) is trivial.

1.3. Corollary. *If Γ_1 and Γ_2 are o.c., Γ_1 satisfies* (ccc) *and $\mathscr{N}(\Gamma_1) = \mathscr{N}(\Gamma_2)$, then these topologies are semimetrizable and $\Gamma_1 = \Gamma_2$.*

In the implication (c)$\Rightarrow$(b) above only semimetrizability of Γ is essential. Actually the following more general statement holds:

1.4. Proposition. *If a family $(\Gamma_i)_{i \in I}$ of FN-topologies on a ring $\mathscr{R}$ is such that $\Gamma = \sup\{\Gamma_i\colon i \in I\}$ is semimetrizable, then there exists a countable subset J of I such that $\Gamma = \sup\{\Gamma_j\colon j \in J\}$.*

Proof. Let $(\mathscr{U}_n)$ be a countable base of Γ-neighbourhoods of $\emptyset$. Since $\Gamma = \sup_i \Gamma_i$, every $\mathscr{U}_n$ contains a Γ-neighbourhood $\mathscr{V}_n$ of $\emptyset$ of the size $\mathscr{V}_n = \mathscr{V}_{i(n,1)} \cap \ldots \cap \mathscr{V}_{i(n,k_n)}$, where $\mathscr{V}_{i(n,k)}$ is a $\Gamma_{i(n,k)}$-neighbourhood of $\emptyset$ ($k = 1, \ldots, k_n$). It suffices to set $J = \bigcup_n \{i(n,k)\colon 1 \leqslant k \leqslant k_n\}$.

1.5. Corollary. *Let M be a family of σ-measures on a σ-ring $\mathscr{R}$, taking values in (possibly different) topological abelian groups. If M satisfies* (ccc) *then $\Gamma(M)$ is semimetrizable and, moreover, there exists a countable $M_0 \subset M$ such that $\Gamma(M) = \Gamma(M_0)$.*

Indeed, $\Gamma(M)$ is o.c., satisfies (ccc) and $\Gamma(M) = \sup\{\Gamma(\mu)\colon \mu \in M\}$. We can apply 1.2 and 1.4.

1.6. Corollary. *If a submeasure μ on $\mathscr{R}$ satisfies* (ccc), *then there exists an o.c. λ (resp., a finite positive σ-measure ν) on $\mathscr{R}$ such that*

$$\eta \ll \lambda \ll \mu \qquad (\eta \ll \nu \ll \mu)$$

for every o.c. submeasure (resp., finite positive σ-measure) η on $\mathscr{R}$.

Obviously, $\nu \ll \lambda \ll \mu$.

Proof. Set $\Gamma = \Gamma(M)$, where M is the family of o.c. submeasures η on $\mathscr{R}$ with $\eta \ll \mu$. Then apply Theorem 1.2 and define λ by the formula $(+)$ or $(++)$. The alternative statement is treated similarly.

1.7. COROLLARY. *Let μ be an o.c. submeasure on $\mathscr{R}$. Then there exists a finite positive σ-measure ν and an o.c. submeasure γ such that*

$$\mu \sim \nu + \gamma$$

and the only finite positive σ-measure $\varphi \ll \gamma$ is $\varphi = 0$. Moreover, if ν_1, γ_1 is another pair possessing all these properties, then $\nu_1 \sim \nu$ and $\gamma_1 \sim \gamma$.

Proof. Choose as ν any of the finite positive σ-measures whose existence is guaranteed by 1.6. Let Z be the union of a maximal disjoint family of sets $A \in \mathcal{N}(\nu) \setminus \mathcal{N}(\mu)$. Then ν is equivalent with the submeasure $E \to \mu(E \setminus Z)$, so that if γ denotes the submeasure $E \to \mu(E \cap Z)$, we have $\mu \sim \nu + \gamma$. The "uniqueness" of this decomposition is easy to be shown. Let us note that the above decomposition of μ is nothing else but its Lebesgue decomposition with respect to ν (see [6]; Section 4).

There is an old problem [19] whether it is possible to find an o.c. submeasure μ on a σ-ring $\mathscr{R}$, which is not equivalent with any finite positive σ-measure. In view of Theorem 1.7 the question reduces to the following one: Does there exist an o.c. submeasure μ on a σ-ring $\mathscr{R}$ such that the only finite positive σ-measure ν on $\mathscr{R}$ with $\nu \ll \mu$ is $\nu = 0$?

We are going to show that such μ could not be strongly subadditive: $\mu(A \cup B) + \mu(A \cap B) \leqslant \mu(A) + \mu(B)$, $A, B \in \mathscr{R}$ (see e.g. [20]).

1.9. THEOREM. *For every strongly subadditive o.c. submeasure μ on a σ-ring $\mathscr{R}$ there exists a finite positive σ-measure ν_0 such that $\nu_0 \sim \mu$.*

Proof. We can assume that $\mathscr{R}$ is a σ-algebra on a set $R \in \mathscr{R}$ and also that $\mu(R) < \infty$ ([4]; 4.8). By a result of Kelley ([16]; Theorem 14) we easily see that for each $A \in \mathscr{R}$ there exists a finite positive σ-measure ν_A on $\mathscr{R}$ such that $\nu_A \ll \mu$ and $\nu_A(A) = \mu(A)$. Let M be the set of all finite positive σ-measures ν on $\mathscr{R}$ with $\nu \leqslant \mu$. By 1.5 there is a sequence $(\nu_n) \subset M$ such that the σ-measure $\nu_0 = \sum_n 2^{-2} \nu_n$ is equivalent with μ and $\nu_0 \leqslant \mu$.

We close this section with a result whose "σ-additive" part is due to Dubrovskiĭ [9] (cf. also [10]; IV. 9.2 and [12]; 3.10), and that concerning submeasures to Aleksiuk [1] (another proof can be found in [4]).

1.10. THEOREM. *Let M be a family of uniformly σ-additive scalar valued σ-measures (resp., a family of uniformly o.c. submeasures) on $\mathscr{R}$. Then $\Gamma(M) = \Gamma(\mu_M)$, where μ_M is the o.c. submeasure defined by $\mu_M(E) = \sup\{v(\mu, E)\colon \mu \in M\}$ (resp., $\mu_M(E) = \sup\{\eta(E)\colon \eta \in M\}$), and if α is a submeasure on $\mathscr{R}$ such that $\mu \ll \alpha$ for every $\mu \in M$, then all the functions $\mu \in M$ are equi-continuous with respect to α. (Hence $\mu_M \ll \alpha$.) Moreover, there exists a countable subfamily H of M such that $\Gamma(M) = \Gamma(H) = \Gamma(\mu_H) = \Gamma(\lambda)$, where λ is the finite σ-measure (resp., the o.c. submeasure) defined by the formula $(++)$.*

(Apply suitable results from this section and from [4]; §§ 4 and 5.)

Remark. In an earlier draft of this paper the proof of the implication (a)$\Rightarrow$(b) in Theorem 1.2 was modelled on that of Theorem 3.3 in [23] and used a version of the Lebesgue decomposition for submeasures (cf. [23]; 1.5, 1.6, [11] and [4]). The present proof seems to be more elegant. It is essentially identical with the proofs of [28]; Theorem III. 4.5 and [21]; Theorem 1 (this result corresponds to our 1.5), and has many common points also with those of [9]; Theorem 3 and [13]; Lemma 7.

2. Control submeasures and control measures for σ-measures on σ-rings.

Let $\mu\colon \mathscr{R} \to G$ be a measure, finitely additive in general. We say that a submeasure (resp., a nonnegative measure) ν on $\mathscr{R}$ is a *control submeasure* (resp., a *control measure*) for μ iff $\nu \sim \mu$, that is $\Gamma(\nu) = \Gamma(\mu)$. (The term control is borrowed from [3].) Thus the existence of a control submeasure for μ means exactly that $\Gamma(\mu)$ is semimetrizable. If μ is σ-additive or exhaustive then $\Gamma(\mu)$ is o.c. or exh. and therefore a control submeasure (measure) ν for μ, if exists, must be necessarily o.c. or exh., respectively [4]. Further, let us observe that such a control measure ν can be assumed bounded if a) ν is o.c. and $\mathscr{R}$ is a σ-ring, or b) ν is exh. ($\mathscr{R}$ arbitrary). Indeed, in the case a) there is a set $R_0 \in \mathscr{R}$ such that $\nu(R_0) < \infty$ and $\nu(A) = 0$ or ∞ for each set $A \in \mathscr{R}$ which is disjoint with R_0 ([4]; 5.5) and, moreover, as easily seen, there is only a finite number of disjoint atoms $A_1, \ldots, A_k$ with $\nu(A_i) = \infty$. Then a finite σ-measure ν' equivalent with ν can be define as follows: set $\nu'(E) = \nu(E)$ if $\nu(E) < \infty$, $= 1$ if $E \subset A_i$ for some i and $\nu(E) = \infty$, and then extend ν' by additivity on the whole σ-ring $\mathscr{R}$. The case b) can be reduced to a) via the Stone representation theorem (see Section 4).

From now on we shall assume in the present section that $\mathscr{R}$ is a σ-ring of sets, G a Hausdorff topological abelian group, X a locally convex topological vector space; $\mathrm{ca}(\mathscr{R}, G)$ denotes the set of all σ-measures $\mu\colon \mathscr{R} \to G$.

Our first results follow immediately from 1.2 and 1.3.

2.1. THEOREM. *If $\mu \in \mathrm{ca}(\mathscr{R}, G)$ then a control submeasure for μ exists iff μ satisfies* (ccc).

2.2. THEOREM. *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two Hausdorff group topologies on G. Suppose that $\mu\colon \mathscr{R} \to G$ is σ-additive under $\mathscr{T}_1$ and $\mathscr{T}_2$. Then $\Gamma(\mu; \mathscr{T}_1) = \Gamma(\mu; \mathscr{T}_2)$ provided μ satisfies* (ccc). *Hence if ν is a control submeasure (resp., measure) for $\mu\colon \mathscr{R} \to (G, \mathscr{T}_1)$ then ν is also a control submeasure (resp., measure) for $\mu\colon \mathscr{R} \to (G, \mathscr{T}_2)$.*

Thus the existence of a control submeasure (or measure) for μ does not depend on the particular choice of a Hausdorff group topology on G under which μ is σ-additive.

2.3. THEOREM. *If $\mu \in \mathrm{ca}(\mathscr{R}, X)$ then a control measure for μ exists iff μ satisfies* (ccc).

Proof. Suppose that μ satisfies (ccc). Then, by the preceding theorem, $\Gamma(\mu) = \Gamma(\mu;\ \sigma(X, X'))$. Further, since $\Gamma(\mu) = \Gamma(x'\mu\colon x' \in X') = \Gamma(v(x'\mu)\colon x' \in X')$, by Theorem 1.2 or Corollary 1.5 we can find a sequence $(x_n') \subset X'$ such that $\Gamma(\mu) = \Gamma(v(x_n'\mu)\colon n \in N)$. Then

$$\nu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{v(x_n'\mu, E)}{1+m_n}, \qquad E \in \mathscr{R},$$

where $m_n = \sup\{v(x_n'\mu, E)\colon E \in \mathscr{R}\}$, is a control ($\sigma$-additive) measure for μ.

DEFINITION. We say that a Hausdorff topological abelian group G (or its topology) satisfies *countable summability condition*, (csc), iff every family $(x_i)_{i \in I}$ of non-zero elements of G such that every its countable subfamily $(x_i)_{i \in J}$ ($J \subset I$) is summable, is countable, that is, card $I \leqslant \aleph_0$.

This property was considered by I. Kluvánek [17] under the name of "property (Σ)". Kluvánek has shown ([17]; Theorems 3.1, 3.2) that (csc) is necessary and sufficient in order that any σ-measure defined on any σ-ring and taking on values in G be concentrated on a set belonging to this σ-ring. The role (csc) plays in this statement becomes more clear if one observes that (csc) can be equivalently formulated as follows: If $\sigma(I)$ is the σ-ring of all countable subsets of a set I and $\mu\colon \sigma(I) \to G$ is a σ-measure, then $\{i \in I\colon \mu(\{i\}) \neq 0\} \in \sigma(I)$.

Evidently, if G is metrizable then (csc) is fulfilled. By the Orlicz–Pettis theorem (see [8] or [27]) all locally convex topologies lying between $\sigma(X, X')$ and $\tau(X, X')$ satisfy (csc) or none of them does. This shows in particular that there are nonmetrizable locally convex topologies which satisfy (csc).

Let us note also that if Γ is an o.c. FN-topology on a σ-ring $\mathscr{R}$ then Γ satisfies (ccc) iff the quotient topological group $\mathscr{R}/\mathscr{N}(\Gamma)$ satisfies (csc).

The following proposition gathers a few rather simple cases where (csc) appears.

2.4. PROPOSITION. *A Hausdorff locally convex vector space $(X, \mathscr{T}) = X$ satisfies* (csc) *in each of the following cases:*

(a) *There exists a metrizable* (*not necessarily linear or group*) *topology α on X such that for every sequence $(x_n) \subset X$ if $x_n \to 0$ in $\mathscr{T}$ then $x_n \to 0$ in α.*

(b) *There exists a linear Hausdorff topology α on X satisfying* (csc) *and such that every subseriesly $\mathscr{T}$-convergent series $\sum x_n$ of elements of X is α-convergent.*

(c) *X is the strict inductive limit of a sequence (X_n) of locally convex spaces such that*

(c′) *each X_n is metrizable, or*

(c″) *each X_n satisfies* (csc) *and is closed in X_{n+1}.*

(d) *X is the locally convex direct sum of locally convex spaces satisfying* (csc).

(e) *X is the product of a sequence (X_n) of spaces satisfying* (csc).

Proof. We prove for example (c). Let $(x_i)_{i \in I}$ be as required in the definition of (csc). Denote $I_n = \{i \in I\colon x_i \in X_n\}$. Then for every sequence (i_k) of distinct elements from I_n we have $x_{i_k} \to 0$ in X_n in the case (c′), and (x_{i_k}) is summable in X_n in the case (c″). It follows that I_n is countable, hence also $I = \bigcup_n I_n$. In the case (c″) we can reason in another way also: Consider the σ-measure $\mu\colon \sigma(I) \to X$ defined by $\mu(E) = \sum_{i \in E} x_i$. Since μ is bounded (see e.g. [4]; 4.12), by Theorem II. 6. 5 in [26] there is $n \in N$ such that $\mu\colon \sigma(I) \to X_n$. Since X_n satisfies (csc), I is countable.

2.5. THEOREM. *Let X be a Hausdorff locally convex space. Any σ-measure μ defined on any σ-ring $\mathscr{R}$ and taking on values in X has a control measure iff X satisfies* (csc).

Proof. Sufficiency: (csc) implies that μ satisfies (ccc), so we can apply 2.3.

Necessity: Let I be a set and let $\mu\colon \sigma(I) \to X$ be a σ-measure. Since by assumption there exists a control measure ν for μ, the set $\{i\colon \mu(\{i\}) \neq 0\} = \{i\colon \nu(\{i\}) > 0\}$ is countable. Thus X satisfies (csc).

An immediate consequence of 2.4 and 2.5 is the following

2.6. COROLLARY. *If there exists a metrizable topology on X which is coarser than the original topology, then every σ-measure $\mu\colon \mathscr{R} \to X$ has a control measure.*

In the case X is a normed linear space this result is due to Bartle, Dunford and Schwartz ([2]; 2.4, [10]; IV. 10. 5) (cf. also [12]); extension to metrizable locally convex spaces is rather obvious. In a less general form 2.6 has been obtained by Labuda ([18]; 2.2). A little earlier result of Hoffmann–Jørgensen ([15]; 4.4) is very close to 2.6, namely it asserts the existence of a finite positive σ-measure ν with $\mu \ll \nu$ and, as the author observes ([15]; p. 7), this ν can be chosen so that $\mathscr{N}(\nu) = \mathscr{N}(\mu)$. In the form: $\mu \ll \nu$ and $\mathscr{N}(\mu) = \mathscr{N}(\nu)$ also our general Theorem 2.3 was obtained (independently and simultaneously) by Musiał [21] (cf. also [22]). The reader should however note that neither the result of Hoffmann–Jørgensen nor that of Musiał contains the relation $\nu \ll \mu$.

Labuda derives his version of 2.6 from a lemma ([18]; 2.1) which says that a control measure exists if X satisfies the following condition:

($\varkappa$) X' is the union of a sequence (K_n) of $\sigma(X', X)$-compact sets.

(This condition is used also by Tweddle [27].)

In view of 2.5, if X has property $(\varkappa)$ then X satisfies (csc). This follows also directly from that the topology (on X) of uniform convergence on the sets K_n is metrizable and lies between $\sigma(X, X')$ and $\varkappa(X, X')$, and from the Orlicz–Pettis theorem ([8]; 1.6).

3. On the Rybakov and Walsh theorems. We assume everywhere below that $\mathscr{R}$ is a σ-ring. An excellent refinement of the Bartle, Dunford and Schwartz theorem was obtained recently by Rybakov [25]:

3.1. THEOREM. *If X is a normed linear space and $\mu \in \mathrm{ca}(\mathscr{R}, X)$, then there exists $x_0' \in X'$ such that $x_0'\mu \sim \mu$, i.e. $v(x_0'\mu)$ is a control measure for μ.*

Proofs of this result, which simplify that of Rybakov, were given also in [4] and [29]. In particular, in [4] Theorem 3.1 was derived from a special case of the following theorem (namely, M was supposed in [4]; 10.7 uniformly σ-additive).

3.2. THEOREM. *Let X be a Banach space and let $M \subset \mathrm{ca}(\mathscr{R}, X)$. If M satisfies* (ccc) *then there exists a sequence of numbers (c_n) and a sequence $(\mu_n) \subset M$ such that*

$$(+) \qquad \sum_{n=1}^{\infty} |c_n| < \infty, \qquad \sum_{n=1}^{\infty} \|c_n \mu_n\| < \infty$$

and the σ-measure μ_0: $\mathscr{R} \to X$ defined by the equality

$$(++) \qquad \mu_0(E) = \sum_{n=1}^{\infty} c_n \mu_n(E)$$

is equivalent with M, i.e. $\Gamma(M) = \Gamma(\mu_0)$ or, what is the same here, $\mathscr{N}(M) = \mathscr{N}(\mu_0)$.

One obtains the previous theorem by setting $M = \{x'\mu\colon \|x'\| \leqslant 1\}$; then $\mu_n = x_n'\mu$ and the x_0' required in 3.1 can be defined as $x_0' = \sum_{n=1}^{\infty} c_n x_n'$.

Proof of 3.2 (sketch). Since M satisfies (ccc), by 1.5 we can find a sequence $(\mu_n) \subset M$ such that $\Gamma(M) = \Gamma(\mu_n\colon n \in N)$. Then the family $P = \{(n\|\mu_n\|)^{-1}\mu_n\colon n \in N\}$ is uniformly σ-additive and $\Gamma(M) = \Gamma(P)$; here and in the sequel $\|\mu\| = \sup_{E \in \mathscr{R}} \|\mu(E)\|$ is the usual norm in $\mathrm{ca}(\mathscr{R}, X)$. It is easy to check that the coefficients which occur in the proof of theorem 10.7 in [4] can be selected so that the resulting c_n satisfy $(+)$.

We show below that Theorem 3.2 is a consequence of the generalized Walsh theorem 3.6.

Applications of 3.2 give the following two generalizations of the Rybakov theorem:

3.3. THEOREM. *Let X, Y be Banach spaces, $L(X, Y)$ the Banach space (usual norm) of continuous linear maps of X into Y, and let μ: $\mathscr{R} \to L(X, Y)$ be a σ-measure. Then there exists $x_0 \in X$ and $y_0' \in Y'$ such that*

$$y_0' \circ \mu(:)x_0 \sim \mu(:)x_0 \sim \mu.$$

Proof. Since μ is σ-additive, the family $M = \{\mu(:)x\colon x \in X, \|x\| \leqslant 1\}$ is uniformly σ-additive $(\mu(:)x\colon \mathscr{R} \to Y)$. Let c_n and $\mu_n = \mu(:)x_n$ $(\|x_n\| \leqslant 1)$ be chosen according to the assertion of 3.2. Then for $x_0 = \sum_{n=1}^{\infty} c_n x_n$ we have $\mu(:)x_0 \sim \mu$. Now it suffices to apply 3.1 to $\mu(:)x_0$: $\mathscr{R} \to Y$ in order to complete the proof.

3.4. THEOREM. *Let X be a normed linear space and let, for each $n \in N$, μ_n be a σ-measure from a σ-ring $\mathscr{R}_n$ into the space X. Then there exists $x_0' \in X'$ such that $x_0'\mu_n \sim \mu_n$ for every $n \in N$.*

Proof. As easily seen, we can assume that X is complete as well as that $E \cap F = \emptyset$ whenever $E \in \mathscr{R}_k$, $F \in \mathscr{R}_n$ and $k \neq n$. Let $\mathscr{R}$ be the σ-ring generated by $\bigcup_{n=1}^{\infty} \mathscr{R}_n$. It is clear that every set $E \in \mathscr{R}$ is uniquely represented in the form $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n \in \mathscr{R}_n$. Then the formula $\mu(E) = \sum_{n=1}^{\infty} a_n \mu_n(E_n)$, where $a_n = (2^n\|\mu_n\|)^{-1}$, defines a σ-measure μ: $\mathscr{R} \to X$ (the series is uniformly convergent on $\mathscr{R}$). By 3.1 there exists $x_0' \in X'$ with $x_0'\mu \sim \mu$. Then evidently $x_0'\mu_n \sim \mu_n$ for every $n \in N$.

Recently B. Walsh [29] established a theorem from which follows not only the Rybakov theorem 3.1 but also the fact that the $x_0' \in X'$ with $x_0'\mu \sim \mu$ form a norm-dense G_δ subset of X'. Below we considerably generalize Walsh's theorem; our proof, though exploiting some ideas due to Walsh, seems to be more direct.

We begin with a lemma which is evidently a particular case of our generalized Walsh theorem; a similar lemma was used in [4].

3.5. LEMMA. *Let X be an arbitrary Hausdorff topological vector space, and let $\mu_0, \mu \in \mathrm{ca}(\mathscr{R}, X)$. If μ_0 and μ satisfy* (ccc) *then the set of those scalars t for which the equality $\mathscr{N}(t\mu_0 + (1-t)\mu) = \mathscr{N}(\mu_0) \cap \mathscr{N}(\mu)$ does not hold is at most countable.*

(The equality is equivalent with $\Gamma(t\mu_0 + (1-t)\mu) = \sup\{\Gamma(\mu_0), \Gamma(\mu)\}$.)

Proof. From 2.1 we deduce easily the existence of an o.c. submeasure η on $\mathscr{R}$ such that $\mathscr{N}(\eta) = \mathscr{N}(\mu_0) \cap \mathscr{N}(\mu)$. Denote $\mathscr{N}(t) = \mathscr{N}(t\mu_0 + (1-t)\mu)$. Let us first observe that $t_1 \neq t_2$ implies $\mathscr{N}(t_1) \cap \mathscr{N}(t_2) \subset \mathscr{N}(\eta)$. In fact, if $E \in \mathscr{N}(t_1) \cap \mathscr{N}(t_2)$ then $t_1\mu_0(F) + (1-t_1)\mu(F) = 0 = t_2\mu_0(F) + (1-t_2)\mu(F)$ and hence $\mu_0(F) = \mu(F)$ for every $F \subset E$. It follows that $\mu_0(F) = \mu(F) = t_1\mu(F) + (1-t_1)\mu(F) = t_1\mu_0(F) + (1-t_1)\mu_0(F) = 0$ for every $F \subset E$, that is $E \in \mathscr{N}(\eta)$. Now let us note that the set of those t such that there is a set $E \in \mathscr{N}(t)$ with $\eta(E) > 0$ is at most countable. For, otherwise, we can find an infinite sequence (t_n) of scalars and sets $E_n \in \mathscr{N}(t_n)$ such that $a = \inf_n \eta(E_n)$

>0. Then the sets $F_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k$ are disjoint and $\eta(F_n) = \eta(E_n) \geqslant a$ for every $n \in \boldsymbol{N}$. But this is impossible because η is o.c. Thus $\mathscr{N}(t) \subset \mathscr{N}(\eta)$ holds for all, with exception of at most countably many, scalars t. Since the converse inclusion holds for all t, the proof is completed.

In our next theorem we assume that X and Y are topological vector spaces and the topology of X is metrizable. The space $\mathrm{ca}(\mathscr{R}, X)$ is endowed with the topology of uniform convergence on $\mathscr{R}$; this topology is determined by the paranorm $\|\mu\| = \sup_{E \in \mathscr{R}} |\mu(E)|$, where $|\cdot|$ is a paranorm defining the topology of the space X.

3.6. THEOREM. *Let W be a convex subset of Y and $a\colon W \to \mathrm{ca}(\mathscr{R}, X)$ a continuous affine mapping. If W is of the second Baire category in itself and $M = a(W)$ satisfies* (ccc), *then*

$$W_0 = \{w_0 \in W\colon a(w) \ll a(w_0) \text{ for every } w \in W\} = \{w_0 \in W\colon \Gamma(M) = \Gamma(a(w_0))\}$$

is a dense G_δ subset of W (hence $W \setminus W_0$ is of the first Baire category in W).

Proof. a is affine means that $a\left(\sum_{i=1}^n t_i w_i\right) = \sum_{i=1}^n t_i a(w_i)$ whenever $w_i \in W$, $t_i \geqslant 0$ and $\sum_{i=1}^n t_i = 1$.

Since M satisfies (ccc), there exists a sequence $(w_n) \subset W$ such that, for every $w \in W$, $a(w) \ll \lambda = \sum_{n=1}^\infty 2^{-n} \min\{1, \bar{\mu}_n\}$, where $\mu_n = a(w_n)$ and $\bar{\mu}_n$ is the submeasure majorant for μ_n with respect to the given paranorm in X. Let us take an arbitrary sequence of positive reals (r_n) such that $r_n \searrow 0$, and set

$$M_n = \{\mu \in \mathrm{ca}(\mathscr{R}, X)\colon \exists_{s>0} \forall_{E \in \mathscr{R}} \bar{\mu}(E) < s \Rightarrow \eta_n(E) < r_n\},$$

where $\eta_n = \bar{\mu}_1 + \ldots + \bar{\mu}_n$ $(n \in \boldsymbol{N})$. Consider the set $W_n = a^{-1}(M_n)$.

1° $W_n \neq \emptyset$: In fact, from the preceding lemma we easily deduce the existence of a convex combination v_n of $w_1, \ldots, w_n$ such that $a(v_n) \sim \eta_n$. Evidently $v_n \in W_n$.

2° W_n is open in W: Let $\mu_0 \in M_n$ and let $s > 0$ be such that $\eta_n(E) < r_n$ whenever $\bar{\mu}_0(E) < s$. Then if $\|\mu_0 - \mu\| < s/2$, we see that $\bar{\mu}(E) < s/2$ implies $\eta_n(E) < r_n$, so that $\mu \in M_n$. Thus M_n is open in $\mathrm{ca}(\mathscr{R}, X)$, hence also W_n in W.

3° W_n is dense in W: Let $w \in W$ be arbitrary. Then using again our lemma we see that for every $\varepsilon > 0$ there is $t \in [0, \varepsilon)$ such that $ta(v_n) + (1-t)a(w) \sim \overline{a(v_n)} + \overline{a(w)}$ (²). Since $\eta_n \ll \overline{a(v_n)} + \overline{a(w)}$, it follows that $tv_n + (1-t)w \in W_n$. Thus on the segment joining v_n and w there are points from W_n as close to w as we wish. Therefore W_n is dense in W.

(²) Where $\overline{a(v_n)}$, $\overline{a(v)}$ are, of course, submeasure majorants for $a(v_n)$ and $a(w)$, respectively.

4° Now we are going to show that $W_0 = \bigcap_{n=1}^\infty W_n$ satisfies the assertion of the theorem. Since W is of the second category in itself, $W_0 \neq \emptyset$. Let $w_0 \in W_0$. If $\overline{a(w_0)}(E) = 0$, then $\eta_n(E) = 0$ for every $n \in \boldsymbol{N}$. Hence $a(w) \ll a(w_0)$ for each $w \in W$. Let us observe that $W_0 = \{w_1 \in W\colon a(w_1) \sim a(w_0)\}$. Now, again by our lemma, if $w \in W$ then all the points of the segment joining the points w_0 and w, with exception of at most countable many of them, belong to W_0. It follows that W_0 is dense in W.

It is obvious that Theorems 3.1, 3.3 and 3.4 can be derived from the Walsh Theorem. What is less obvious, the same is with Theorem 3.2. This can be shown as follows: Let $(\mu_n) \subset M$ be chosen as in the proof of 3.2, and let $b_n = \max\{1, \|\mu_n\|\}$, $n \in N$. Further, let Y be the linear space of all scalar sequences $y = (c_n)$ such that $\|y\| = \sum_{n=1}^\infty |b_n c_n| < \infty$. It is evident that Y is a Banach space under the norm defined by the last equality (Y is isomorphic with l^1). Now we define a linear mapping $a\colon Y \to \mathrm{ca}(\mathscr{R}, X)$ by the formula $a(y) = \sum_{n=1}^\infty c_n \mu_n$, $y = (c_n) \in Y$; since $\mathrm{ca}(\mathscr{R}, X)$ is a Banach space, the series on the right side of the equality is convergent in $\mathrm{ca}(\mathscr{R}, X)$. Since $\|a((c_n))\| \leqslant \|(c_n)\|$, the mapping a is continuous. Since $a(y) \ll \sum_n 2^{-2} \min\{1, \bar{\mu}_n\}$ for every $y \in Y$, the range $a(Y)$ of a satisfies (ccc). By Theorem 3.6 there exists $y_0 = (c_n^0) \in Y$ such that $a(y) \ll a(y_0) = \mu_0$ for every $y \in Y$, and all those y_0 form a dense G_δ subset of Y. Since $\mu_n \in a(Y)$ for each $n \in \boldsymbol{N}$, we see that $\mu \ll \mu_0$ for all $\mu \in M$, and this completes the present proof of 3.2.

An examination of the way Lemma 3.5 is used in the proof of 3.6 shows that it suffices to impose, instead of convexity, the following condition on W: If $w_0, w \in W$ then there exists $\varepsilon > 0$ such that $tw_0 + (1-t)w \in W$ for every $t \in [0, \varepsilon)$.

As concerns the topology on W, only the continuity of mappings $[0,1] \ni t \mapsto tw_1 + (1-t)w_2$ $(w_1, w_2 \in W)$ was used in the proof of Theorem 3.6 (steps 3° and 4°).

Let us note also that (in the notation of 3.6) if we know a priori that $W_0 \neq \emptyset$, then the assumption that W is of the second category is superfluous in order to prove that W_0 is dense and G_δ in W. (Take $v \in W_0$ and replace η_n by $\overline{a(v)}$; only 2° and 4° are to be verified.)

The Rybakov theorem fails to be true, in general, if X is a locally convex non-normed vector space. Indeed, let X be the space of all scalar sequences $x = (\xi_n)$ endowed with the (metrizable) topology of pointwise convergence on $\boldsymbol{N}$, and let $\mu\colon \mathscr{P}(\boldsymbol{N}) \to X$ be defined by $\mu(E) =$ the characteristic function of E in $\boldsymbol{N}$. Then μ is a σ-measure and for every $x' \in X'$ the set $\{n \in \boldsymbol{N}\colon x'\mu(\{n\}) \neq 0\}$ is finite. Hence there is no $x' \in X'$ with $x'\mu \sim \mu$.

In despite of this example, it is not difficult to find conditions which assure validity of the 'individual' or 'global' Rybakov type theorem in non-normed spaces.

First we show however how the existence of a "Rybakov functional" for μ can be used to obtain a kind of

3.7. HAHN DECOMPOSITION FOR VECTOR MEASURES. Let X be a Hausdorff locally convex space and let $\mu\in ca(\mathscr{R}, X)$. Suppose that there is $x_0'\in X'$ such that $x_0'\mu\sim\mu$. First consider the case X is a real vector space. By the usual Hahn decomposition theorem [10], we can find a set $H\in\mathscr{R}$ such that $x_0'\mu(E)\geqslant 0$ if $E\subset H$, and $x_0'\mu(E)\leqslant 0$ if $E\cap H=\emptyset$, $E\in\mathscr{R}$. (In view of [4]; 4.8, $\mathscr{R}$ can be assumed a σ-algebra.) Hence if $E\subset H$, $\mu(E)=0$ and $F\subset E$, then $\mu(F)=0$, that is E is a μ-zero set. Similarly, if $E\cap H=\emptyset$ and $\mu(E)=0$, then E is a μ-zero set. Thus we see that H has the property that a set $E\in\mathscr{R}$ is μ-zero iff $\mu(E\cap H)=0$ and $\mu(E\setminus H)=0$. In particular, if $\mathscr{R}$ is a σ-algebra on a set R, R can be split into two disjoint sets H and $K=R\setminus H$ so that a set $E\in\mathscr{R}$ is μ-zero iff $\mu(E\cap H)=0$ $=\mu(E\cap K)$. It follows that if E, F are subsets of H (or K) and $\mu(E)\neq 0$ $\neq\mu(F)$, then also $\mu(E\cup F)\neq 0$. For this reason the decomposition $R=H\cup K$ may be treated as an analogue of the classical Hahn decomposition. (Note however that even in the case $X=R$ the classical decomposition and that with the above property can be essentially different.)

Now it is quite evident that if X is complex, there exists a decomposition of R with properties similar to those described above and consisting of at most four disjoint sets.

A general Hahn decomposition theorem for group valued σ-measures was recently obtained by Herer [14]. (Let us notice that the theorem of Herer, as its proof shows, is valid for an arbitrary Hausdorff topological abelian group G provided μ satisfies (ccc).)

3.8. PROPOSITION. *Suppose that X is locally convex and let $\mu\in ca(\mathscr{R}, X)$. Then $x_0'\in X'$ such that $x_0'\mu\sim\mu$ exists iff there is a sequence $(H_n)_{n\in N}$ of disjoint sets in $\mathscr{R}$ and a convex $\sigma(X', X)$-compact subset K of X' such that the following condition is satisfied:*

$$\text{if } E\in\mathscr{R},\ \mu(E)\neq 0 \text{ and } E\subset H_n \text{ for some } n\in N \text{ or } E\cap\bigcup_{n=1}^{\infty}H_n=\emptyset,$$

then $x'\mu(E)\neq 0$ for some $x'\in K$.

Proof. 'Only if': We apply the Hahn decomposition from 3.7 and put $K=\{ax_0': |a|\leqslant 1\}$.

'If': We shall assume, for notational simplicity only, that $\mathscr{R}$ is a σ-algebra on a set R and $\bigcup_{n=1}^{\infty}H_n=R$. For each $n\in N$ let $\mu_n\in ca(\mathscr{R}, X)$ be defined by $\mu_n(E)=\mu(E\cap H_n)$. We can suppose that K is absolutely convex. Let Y' be the linear subspace of X' spanned by K, and let $Y=X/(Y')^{\circ}$. Denoting by i the natural homomorphism of X onto Y we see that each $\nu_n=i\circ\mu_n$: $\mathscr{R}\to Y$ is a σ-measure in the topology $\sigma(Y, Y')$ of the space Y. Now consider Y as a normed space, with K being the closed unit ball in its dual Y'. By the Orlicz–Pettis theorem each ν_n is σ-additive in the norm topology of Y. By Theorem 3.4 we can find $x_0'\in K$ such that $x_0'\nu_n\sim\nu_n$ for every $n\in\boldsymbol{N}$. We claim that $x_0'\mu\sim\mu$. Since $x_0'\mu\ll\mu$ is obvious, we have to prove that $\mu\ll x_0'\mu$. Suppose that a set $E\in\mathscr{R}$ is $x_0'\mu$-zero but $\mu(E)\neq 0$. Let $k\in\boldsymbol{N}$ be such that $\mu(E\cap H_k)\neq 0$. Then also $\nu_k(E\cap H_k)\neq 0$, hence a subset F of $E\cap H_k$ exists such that $0\neq x_0'\nu_k(F)=x_0'\mu(F)$. It follows that E is not $x_0'\mu$-zero, a contradiction.

3.9. THEOREM. *Let $\mu\in ca(\mathscr{R}, X)$. Then $x_0'\in X'$ with $x_0'\mu\sim\mu$ exists provided X satisfies one of the following conditions.*

(a) *X is a strict* (LB)*-space or, more generally, X is the strict inductive limit of an increasing sequence (X_n) of locally convex spaces such that X_n is closed in X_{n+1} and the Rybakov theorem holds for functions in* $ca(\mathscr{R}, X_n)$, $n\in\boldsymbol{N}$.

(b) *X is the locally convex direct sum of spaces for which the Rybakov theorem is valid.*

(c) *There exists a continuous (homogeneous!) norm on X.*

Proof. (a) Since the range of μ is bounded, we can find $k\in N$ such that μ maps $\mathscr{R}$ into X_k and is σ-additive in the topology of X_k. By assumption there exists $x_0'\in X_k'$ such that $x_0'\mu\sim\mu$. (b) is similarly treated.

(c) Apply 3.1 and 2.2.

We give below a simple example showing how the Rybakov theorem can be applied in the theory of integration.

3.10. Let X be a normed space and let $\mu\in ca(\mathscr{R}, X)$. Without loss of generality we shall assume that $\mathscr{R}$ is a σ-algebra on a set R. Let us fix an $x_0'\in X'$ with $x_0'\mu\sim\mu$. By $L(\mu)$ we denote the space of all μ-integrable scalar valued functions defined on R; we shall freely use some facts from the theory of vectorial integration presented in [10]. The most natural topology on $L(\mu)$ is that determined by the norm

$$|||f|||=\sup_{E\in\mathscr{R}}\Big\|\int_E f\,d\mu\Big\|.$$

(It is clear that if X is the space of real or complex numbers, the norm $|||\cdot|||$ is equivalent with the standard L_1-norm $\|f\|=\int_R|f|\,dv(\mu)$; in fact, $|||f|||\leqslant\|f\|\leqslant 4\,|||f|||$.)

We are going to prove that

a) if $f, f_n\in L(\mu)$ and $|||f_n-f|||\to 0$, then $f_n\to f$ in μ-measure and

b) if X is a Banach space, then also $L(\mu)$ is a Banach space.

a) Let $|||f_n|||\to 0$. Then $x_0'\big(\int_E f_n\,d\mu\big)=\int_E f_n\,d(x_0'\mu)\to 0$ uniformly for $E\in\mathscr{R}$.

Hence $\int\limits_R |f_n(\cdot)|\,dv(x_0'\mu)\to 0$. It follows that $f_n\to 0$ in the measure $v(x_0'\mu)$, what is equivalent to the convergence in μ-measure because $v(x_0'\mu)\sim\mu$.

b) Let $f_n\in L(\mu)$ and suppose that $|||f_n-f_m|||\to 0$ as $n,m\to\infty$. Then arguing as in a) we see that the sequence (f_n) satisfies the Cauchy condition with respect to the convergence in the measure $v(x_0'\mu)$. Therefore there exists a function f such that $f_n\to f$ in μ-measure. On the other hand, completeness of X assures the existence of a σ-measure $\gamma\colon \mathscr{R}\to X$ such that $\int\limits_E f_n d\mu\to\gamma(E)$ uniformly for $E\in\mathscr{R}$. It follows that $f\in L(\mu)$, $\gamma=\int fd\mu$ and $|||f-f_n|||\to 0$.

Let us still note that if $f\in L(\mu)$ and $\nu=\int fd\mu$, then $\nu\sim x_0'\nu$ (compare with 3.4). Only $\nu\ll x_0'\nu$ needs a proof. So let $v(x_0'\nu,E)=\int\limits_E|f(\cdot)|\,dv(x_0'\mu)=0$. Then $f(r)=0$ for μ-almost all $r\in E$. Hence $\nu(E)=0$. Thus every "Rybakov functional" for μ is also a Rybakov functional for each indefinite integral with respect to μ.

(A theory of integration of the Bartle–Dunford–Schwartz type, where no control measures for vector measures are used, is presented in [5].)

4. Control measures for exhaustive measures on rings. In this section we always assume, unless otherwise is explicitely stated, that $\mathscr{R}$ is a ring of sets, X a Hausdorff locally convex vector space and $\mu\colon\mathscr{R}\to X$ is an exhaustive (finitely additive) measure. As in Section 2, we ask about conditions under which there exists a control measure for μ. To give a satisfactory and complete answer for this question seems to be much more difficult than it was in the setting of Section 2, where $\mathscr{R}$ was a σ-ring and μ a σ-measure. I was unable to find such an answer. The results we are to state below are more or less immediate consequences of suitable results from Section 2. We shall obtain them by application of the Stone representation theorem (see for example [28]) and the following extension theorem (see [4] for this result and further references).

4.1. THEOREM. *Let $\mathscr{R}$ be a ring of sets, $\mathscr{R}_0$ the σ-ring generated by $\mathscr{R}$ and G a sequentially complete Hausdorff topological abelian group. Let an additive function $\mu\colon\mathscr{R}\to G$ be exh. and σ-additive (resp., let μ be an exh. and o.c. submeasure on $\mathscr{R}$). Then there exists a unique σ-measure $\tilde\mu\colon\mathscr{R}_0\to G$ (resp., a unique o.c. submeasure $\tilde\mu$ on $\mathscr{R}_0$) such that $\tilde\mu(E)=\mu(E)$ for every $E\in\mathscr{R}$.*

First let us observe that $\mathscr{R}$ can be supposed a field of sets. Indeed, let $\mathscr{R}$ consists of subsets of a set R, and let μ be an additive set function or submeasure on $\mathscr{R}$. Then $\hat{\mathscr{R}}=\mathscr{R}\cup\{R\setminus E\colon E\in\mathscr{R}\}$ is the field on R generated by $\mathscr{R}$, and we can extend μ in the following way: for $F\in\hat{\mathscr{R}}\setminus\mathscr{R}$ we set $\hat\mu(F)=0$ if μ is additive, and $\hat\mu(F)=\infty$ if μ is a submeasure. Evidently, such an extension of μ preserves exhaustivity.

Thus we shall assume that $\mathscr{R}$ is a field of sets on R. Let S be the Stone space for $\mathscr{R}$ and let s denote an isomorphism from the field $\mathscr{S}$ of closed-open subsets of S onto $\mathscr{R}$. Define the function μs on $\mathscr{S}$ by the formula $\mu s(A)=\mu(s(A))$. It is clear that μs is o.c. (if $\mathscr{S}\ni A_n\searrow\emptyset$ then $A_n=\emptyset$ for large n). Now assume that μ is exh.; then so is μs. Therefore, if X is sequentially complete (or at least that the range of μ is contained in a sequentially complete subset of X), then from 4.1 we get the existence of a unique σ-measure $\tilde\mu\colon\mathscr{B}\to X$ (resp., of a unique o.c. submeasure $\tilde\mu$ on $\mathscr{B}$) such that $\tilde\mu|\mathscr{S}=\mu s$. Here $\mathscr{B}$ denotes the σ-field generated by $\mathscr{S}$, that is the Baire field on S. Similarly, if Γ is an exh. FN-topology on $\mathscr{R}$ and Γ_s denotes its image on $\mathscr{S}$, then by [4]; 8.3 there exists a unique o.c. FN-topology $\tilde\Gamma$ on $\mathscr{B}$ such that $\tilde\Gamma$ induces Γ_s on $\mathscr{S}$. When X is not sequentially complete, use of the extension $\tilde\mu\colon\mathscr{B}\to\tilde X$ of $\tilde\mu$ leads sometimes to desired results also; $\tilde X$ is the completion of X.

The following observation is crucial for us: Suppose Γ is an exh. FN-topology on $\mathscr{R}$ and μ is an exh. submeasure or exh. measure. Then $\mu\ll\Gamma$ iff $\tilde\mu\ll\tilde\Gamma$. It follows that if φ is a control submeasure (or measure) for $\tilde\mu$, then $\nu=\varphi\circ s^{-1}$ is a control submeasure (resp., a control measure) for μ. (It should be however remembered that in general ν is merely exh. though φ is o.c.!) As concerns the existence of such φ, it is clear by the results of Section 2 that we can prove it if we are able to show that $\tilde\mu$ satisfies (ccc).

4.2. EXAMPLES. 1) Let $\mathscr{R}=\mathscr{P}(N)$, $X=$ the space of bounded scalar sequences $x=(\xi_n)$ endowed with the weak topology with respect to the standard norm $\|x\|=\sup|\xi_n|$, and let $\mu\colon\mathscr{R}\to X$ be defined by the formula $\mu(E)=$ the characteristic function of $E\subset N$. Then X has the property (csc), μ is exh. and satisfies (ccc), and yet there is no exh. submeasure η with $\mu\ll\eta$. For, otherwise there exists by ([7]; Lemma) an infinite subset M of N such that η is o.c. on the σ-ring $\mathscr{P}=\mathscr{P}(M)$. It follows that μ is σ-additive on $\mathscr{P}$. Hence, by the Orlicz–Pettis theorem μ is σ-additive on $\mathscr{P}$ in the norm topology of X. This however is evidently false.

2) Let $\mathscr{R}$ be the Borel σ-field on $[0,1]$, X the space of bounded scalar functions on $[0,1]$ endowed with the weak topology with respect to the sup-norm, and let $\mu\colon\mathscr{R}\to X$ be defined similarly as in 1). Then X has the property (csc), μ is exh. and μ does not satisfy (ccc).

Thus even if $\mathscr{R}$ is a σ-field, the condition (ccc) though necessary is not always sufficient for the existence of a control submeasure.

4.3. PROPOSITION. *A control measure for μ exists iff there exists an exh. submeasure η on $\mathscr{R}$ such that $\mu\ll\eta$.*

Proof. "If": The σ-additive extension $\tilde\mu\colon\mathscr{B}\to\tilde X$ of μ satisfies (ccc) because $\tilde\mu\ll\tilde\eta$. It suffices to apply 2.3 and return to $\mathscr{R}$.

4.4. THEOREM. *Suppose that X is a subspace of a sequentially complete locally convex space Y. If Y satisfies* (csc), *then μ has a control measure.*

Indeed, $\tilde{\mu}$: $\mathscr{B}\to Y$ satisfies (ccc). We apply 2.3 or 2.5. From this theorem and 2.4 we get immediately

4.5. COROLLARY. *If X is metrizable or X is a strict (LF)-space, then μ has a control measure ν.*

For metrizable X this result in the form $\mu \ll \nu$ has been obtained by Hoffmann-Jørgensen ([15]; Proposition 3), and in the case X is normed by Brooks [3]. Since these authors do not use σ-additive extensions of vector measures, their reasonings are unnecessarily complicated.

It seems to be worth while to give here a direct proof of Corollary 4.5. Of course, it suffices to consider only the case where X is a normed linear space. We shall need two lemmas on exhaustive submeasures. The first of them was already stated in [4] and can be easily verified by an indirect argument. The proof of the second one is similar in part to that of the Bartle-Dunford-Schwartz theorem [10].

4.6. LEMMA. *Let γ be an exh. submeasure on $\mathscr{R}$. Let (A_n) be an arbitrary sequence of sets from $\mathscr{R}$. Then for every $\varepsilon > 0$ there is $m \in N$ such that $\gamma(A_n \setminus \bigcup_{k=1}^{m} A_k) < \varepsilon$ for every $n \geqslant m$.*

4.7. LEMMA. *Let H be a family of uniformly exhaustive submeasures on $\mathscr{R}$. Then for every $\varepsilon > 0$ there is $\delta > 0$ and a finite sequence $\eta_1, \ldots, \eta_n$ in H such that*

if $E \in \mathscr{R}$ and $\sup_{1\leqslant i\leqslant n} \eta_i(E) \leqslant \delta$ then $\eta(E) \leqslant \varepsilon$ for all $\eta \in H$.

Proof. It is obvious that H can be replaced by the family H' of all submeasures of the form $\sup_{1\leqslant i\leqslant n} \eta_i$, where $\eta_1, \ldots, \eta_n \in H$.

Then our lemma says that:

(*) *For every $\varepsilon > 0$ there is $\delta > 0$ and $\eta_0 \in H'$ such that if $\eta_0(E) \leqslant \delta$ then $\eta(E) \leqslant \varepsilon$ for all $\eta \in H'$.*

Suppose that (*) fails to be true for some $\varepsilon > 0$. Then, as easily seen, there is a sequence $(\eta_n) \subset H'$ and a sequence $(E_n) \subset \mathscr{R}$ such that $\eta_n \leqslant \eta_{n+1}$, $\eta_n(E_n) \leqslant \varepsilon/2^{n+1}$ but

$$(\cdot) \qquad \eta_{n+1}(E_n) > \varepsilon, \qquad n \in \boldsymbol{N}.$$

Let an exh. submeasure γ be defined by the formula $\gamma(E) = \sup\{\eta(E)\colon \eta \in H\}$, $E \in \mathscr{R}$. By the preceding lemma we can define a sequence $0 = m_0 < m_1 < \ldots$ so that

$$(:) \qquad \gamma\Big(E_n \setminus \bigcup_{i=m_{k-1}+1}^{m_k} E_i\Big) < \varepsilon/2^{k+1}, \qquad n \geqslant m_k;\ k \in \boldsymbol{N}.$$

Set

$$F_n = \bigcap_{j=1}^{n} \bigcup_{i=m_{j-1}+1}^{m_j} E_i$$

and $\lambda_n = \eta_{m_n+1}$, $n \in \boldsymbol{N}$. We have $F_1 \supset F_2 \supset \ldots$,

$$\lambda_n(F_{n+1}) \leqslant \sum_{i=m_n+1}^{m_{n+1}} \eta_i(E_i) < \varepsilon/2^n$$

and, by $(\cdot)$ and $(:)$,

$$\lambda_n(F_n) \geqslant \lambda_n(E_{m_n} \cap F_n) = \lambda_n\big(E_{m_n} \setminus (E_{m_n} \setminus F_n)\big)$$
$$> \varepsilon - \sum_{j=1}^{n} \gamma\Big(E_{m_n} \setminus \bigcup_{i=m_{j-1}+1}^{m_j} E_i\Big) > \frac{\varepsilon}{2} + \frac{\varepsilon}{2^n}, \qquad n \in \boldsymbol{N}.$$

Hence $\lambda_n(F_n \setminus F_{n+1}) > \varepsilon/4$ $(n \in N)$ what contradicts the assumption that H (and H') is uniformly exhaustive.

4.8. THEOREM. *Every uniformly exhaustive family H of submeasures on a ring $\mathscr{R}$ contains a countable subfamily K such that the exhaustive submeasures $\eta_H = \sup_{\eta\in H} \eta$ and $\eta_K = \sup_{\eta \in K}\eta$ are equivalent, hence $\Gamma(H) = \Gamma(K)$.*

Proof. For each $n \in N$ let $\delta_n > 0$ and $K_n = \{\eta_1^n, \ldots, \eta_{k_n}^n\} \subset H$ be chosen so that $\eta_H(E) \leqslant 1/n$ whenever $\eta_i^n(E) \leqslant \delta_n$ for $i = 1, 2, \ldots, k_n$. Set $K = \bigcup_n K_n$. If $\eta_K(E) \leqslant \delta_n$ then $\eta_H(E) \leqslant 1/n$. Hence $\eta_K \leqslant \eta_H \ll \eta_K$, so that $\eta_K \sim \eta_H$.

A submeasure λ which is equivalent with η_H can be defined also by the formula $\lambda = \sum_{n=1}^{\infty} (2^n k_n a)^{-1}(\eta_1^n + \ldots + \eta_{k_n}^n)$, where $a = \sup\{\eta_H(E)\colon E \in \mathscr{R},\ \eta_H(E) < \infty\}$ $(a < \infty$ by [4]; 4.10). In fact, since $\eta_i^n \ll \lambda$, we have $\eta_K \ll \lambda \leqslant \eta_H$, hence $\lambda \sim \eta_H$ ([4]; 6.2 is used here). It follows that $\Gamma(K) = \Gamma(\lambda) = \Gamma(H)$.

4.9. COROLLARY. *If $\mathscr{R}$ is a ring, X a normed linear space and μ: $\mathscr{R}\to X$ an exh. measure, then there exists a bounded additive measure ν on $\mathscr{R}$ such that $\mu \sim \nu$.*

Proof. For every $x' \in X'$ with $\|x'\| \leqslant 1$ let $\eta_{x'} = v(x'\mu)$. Since $\eta_{x'} \leqslant 4\bar{\mu}$ ($\bar{\mu}$ = the submeasure majorant for μ with respect to a fixed norm of X), the family $\{\eta_{x'}\colon \|x'\| \leqslant 1\}$ is uniformly exh. By 4.8 there exists a sequence $(x_n') \subset X'$ such that $\|x_n'\| \leqslant 1$ and $\sup_n \eta_{x_n'} \sim \sup_{\|x'\|\leqslant 1} \eta_{x'} \sim \bar{\mu}$. Then the required ν can be obtained by the formula $\nu = \sum_{n=1}^{\infty} 2^{-n} v(x_n'\mu)$. Indeed, $\bar{\mu} \ll \nu \leqslant \bar{\mu}$.

We formulate now a theorem of the Rybakow type:

4.10. THEOREM. *If X is normed or a strict (LB)-space, or if X is sequentially complete and there is a continuous norm on it, then exists $x' \in X'$ such that $x'\mu \sim \mu$.*

The next results are analogues of Theorem 2.2. Phrases like "$(\mu; \mathcal{T})$ is exh." are used instead of "μ is exh. under the topology $\mathcal{T}$ on X".

4.11. PROPOSITION. *Suppose that $X = (X, \mathcal{T})$ is sequentially complete and let a be another topology on X as in* 2.4 (b), *but* (csc) *being not assumed. If there exists a control measure ν for $(\mu; \mathcal{T})$ then ν is also a control measure for $(\mu; a)$, and conversely.*

In fact, the extension $\tilde{\mu}$ of $(\mu; \mathcal{T})$ is σ-additive under $\mathcal{T}$ and a, so our result follows from 2.2.

4.12. PROPOSITION. *Let a be a locally convex topology on X such that $\sigma(X, X') \subset a \subset \beta(X, X')$. Suppose that not only $\mu = (\mu; \mathcal{T})$ but also $(\mu; a)$ is exh. Then: if there exists a control measure ν for $(\mu; \mathcal{T})$, then ν is also a control measure for $(\mu; a)$, and conversely.*

The assumption that $(\mu; a)$ is exh. can be omitted if X is sequentially complete and $a \subset \tau(X, X')$.

Proof. Let $\tilde{X}$, $\tilde{X}_\sigma (= X^*)$ and $\tilde{X}_a$ denote the completions of the spaces X, $X_\sigma = (X, \sigma(X, X'))$ and $X_a = (X, a)$, respectively. Then by a known theorem on completions we have $\tilde{X} \subset \tilde{X}_\sigma$ and $\tilde{X}_a \subset \tilde{X}_\sigma$, the inclusions being continuous. Therefore the σ-additive extensions $\tilde{\mu}_\sigma: \mathcal{B} \to \tilde{X}_\sigma$, $\tilde{\mu}: \mathcal{B} \to \tilde{X}$ and $\tilde{\mu}_a: \mathcal{B} \to \tilde{X}_a$ of $\mu: \mathcal{R} \to X_\sigma$, $\mu: \mathcal{R} \to X$ and $\mu_a: \mathcal{R} \to X_a$, respectively, exist and coincide. Hence, using the extension $\tilde{\nu}$ of ν, the first assertion follows easily from 2.2. The second assertion is a consequence of the first one and the Orlicz–Pettis theorem.

4.13. PROPOSITION. *Suppose that $\mathcal{R}$ is a σ-ring. If there exists a control measure ν for μ, then $(\mu; \tau(X, X'))$ is exh. and ν is its control measure, and conversely.*

It suffices to prove that $(\mu; \tau(X, X'))$ is exh., but this easily follows from Proposition 1 in [7] and the Orlicz–Pettis theorem.

We close this section with a decomposition theorem being an immediate consequence of [6]; 3.12 (b).

4.14. THEOREM. *Suppose that X is a complete metrizable locally convex vector space, $\mu: \mathcal{R} \to X$ an exh. measure, and let ν be a control measure for μ. If $\mu = \mu_\sigma + \mu_p$ and $\nu = \nu_\sigma + \nu_p$ are the Hewitt–Yosida decompositions of μ and ν, where μ_σ, ν_σ are σ-additive and μ_p, ν_p are purely finitely additive, then ν_σ and ν_p are control measures for μ_σ and μ_p, respectively.*

An analogous statement can be also formulated for the Lebesgue decomposition.

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Nuclear spaces on a locally compact group

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Abstract. This paper is devoted to a construction of two types of nuclear spaces Φ and Ψ consisting of functions on a locally compact group. These spaces resemble the spaces D and S of Schwartz, respectively, although the construction does not depend on any differential structure on G and no approximation by Lie groups is used. The role of differential operators is played by (unbounded) operators which are the inverse operators to convolution operators by appropriately chosen non-negative L_1-functions. Thus both spaces Φ and Ψ consists of infinitely regularized functions.

1. Introduction. The idea of the construction of a nuclear space of functions on a locally compact group by an infinite process of regularization by "good" functions is due to A. Hulanicki. We would also like to express our gratitude to him for many useful suggestions and the help while this paper was written.

The main idea of the construction of the space Φ was published in [5]. The spaces of the type Φ and Ψ are not unique—they depend on the selection of the sequence of regularizing functions which shall be chosen once for all. Therefore we shall say the space Φ or Ψ rather than a space of the type Φ or Ψ. On the few ocasions will be imposed, this will be clearly stated.

Among the main properties of the spaces Φ and Ψ are the following. Both are non-trivial subspaces of $L_2(G)$ and Ψ is dense in $L_2(G)$. Both are invariant under the left regular representation of G which is jointly continuous on Φ and Ψ. Following [5] Aarnes [1] constructed a space which is invariant under left and right reqular representation of G — a simplification of his construction is given here.

There are many questions which should perhaps be asked about the spaces Φ and Ψ which are not answered in this paper. We would rather postpone considering them to the time when these spaces shall prove (or disprove) to be of any use in harmonic analysis on non-Lie non-commutative locally compact groups.

2. The convolution operator and its inverse. Let G be a locally compact group and let μ be a left invariant Haar measure on it. If $f \in L_1(G)$ and