On Fourier coefficients and transforms of functions of two variables

by

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Abstract. Let \( f(x_1, x_2) \) be a function of two variables, of period 1 in each, and let \( c_\mu = c_{\mu_1, \mu_2} \) be the Fourier coefficients of \( f \). Then, if \( 1 < p < \frac{3}{2} \) and \( q = \frac{3}{2} p' = \frac{3}{2} p/(p-1) \), we have

\[
\left( \sum_{|\mu| < r} |c_\mu|^q \right)^{1/q} \leq A_p \|f\|_p \quad (A_p = 5^q)
\]

for all \( r > 0 \). There is a corresponding result for Fourier transforms of functions \( f \in L^p(\mathbb{R}) \), \( 1 < p < \frac{4}{3} \), but the previous \( q = \frac{3}{2} p' \) has to be replaced by \( q = \frac{3}{2} p' \). Moreover, the result fails in the extreme case \( p = \frac{4}{3} \). The results are strictly two-dimensional.

I. Let \( \xi = (x_1, x_2) \) denote points on the two-dimensional torus \( \mathbb{Q} \)

\[
0 \leq x_1 < 1, \quad 0 \leq x_2 < 1,
\]

and \( \mu = (m_1, m_2) \) - lattice points in \( \mathbb{R}^2 \) (\( m_1, m_2 \) - integers). Given any integrable function \( f(\xi) \) on \( \mathbb{Q} \) consider its Fourier series

\[
\sum_{\mu} c_\mu e^{2\pi i \mu \cdot \xi},
\]

where

\[
c_\mu = \int_{\mathbb{Q}} f(\xi) e^{-2\pi i \mu \cdot \xi} \, d\xi,
\]

with \( \mu \cdot \xi = m_1 x_1 + m_2 x_2 \), \( d\xi = dx_1 dx_2 \).

The origin of this Note is the following question which Charles Fefferman proposed some time ago. Does there exist a positive number \( p \) strictly less than 2 such that

\[
\left( \sum_{|\mu| < r} |c_\mu|^q \right)^{1/q} \leq A \|f\|_p,
\]

where \( A \) is independent of \( r \). The following theorem gives an answer to the problem.

**Theorem 1.** For any \( r > 0 \), we have

\begin{equation}
\left( \sum_{|\mu| < r} |c_\mu|^q \right)^{1/q} \leq A \|f\|_p,
\end{equation}

where \( A = 5^{1/4} \).
Proof. Let us consider the set \( S = S_\mu \) of lattice points \( \mu = (m_1, m_2) \) with \( |\mu| = r \) (we assume that \( S \) is not empty, since otherwise there is nothing to prove). We then have, for a suitable sequence \( \{g_\xi\} \) with
\[
\sum_{\xi \in \mathbb{Z}} |g_\xi|^2 = 1,
\]
the equation
\[
\left( \sum_{\xi \in \mathbb{Z}} |g_\xi|^2 \right)^{1/2} = \sum_{\xi \in \mathbb{Z}} g_\xi = \sum_{\xi \in \mathbb{Z}} f(\xi) e^{-2\pi i \langle \mu, \xi \rangle} d\xi = \int f(\xi) \left[ \sum_{\mu \in \mathbb{Z}} g_\mu e^{-2\pi i \langle \mu, \xi \rangle} \right] d\xi,
\]
so that, by Hölder's inequality with exponents \( 4/3 \) and 4,
\[
\left( \sum_{\xi \in \mathbb{Z}} |g_\xi|^2 \right)^{1/2} \leq \|f\|_{L^4} \left( \sum_{\mu \in \mathbb{Z}} |g_\mu|^4 \right)^{1/4},
\]
and it is enough to show that the last factor is \( \ll A \).

Write
\[
J = \int |\sum \gamma_\mu e^{-2\pi i \langle \mu, \xi \rangle}|^2 d\xi = \int \left| \sum_{\mu \in \mathbb{Z}} \gamma_\mu e^{-2\pi i \langle \mu, \xi \rangle} \right|^2 d\xi.
\]
We have
\[
\sum_{\mu \in \mathbb{Z}} \gamma_\mu e^{-2\pi i \langle \mu, \xi \rangle} = \sum_{\mu \in \mathbb{Z}} \Gamma_\mu e^{2\pi i \langle \mu, \xi \rangle},
\]
with
\[
\Gamma_\mu = \sum_{\nu \in \mathbb{Z}} \gamma_{\mu + \nu}.
\]
Here \( \mu \) and \( \nu \) are in \( S \) and \( \gamma \) takes all admissible values. Thus \( \gamma \) designates lattice points that are differences of two lattice points on \( S \). By Parseval's formula,
\[
J = \sum_{\xi} |\Gamma_\mu|^2.
\]
It is immediate that
\[
\Gamma_\mu = \sum_{\mu} |\gamma_\mu|^2 = 1.
\]
If \( g \neq 0 \), the sum (1.4) consists of one or two terms (the former if \( \nu = -\mu \)) and in any case, in view of the inequality \( (a+b)^2 \leq 2a^2 + 2b^2 \),
\[
|\Gamma_\mu|^2 \leq 2 \sum_{\nu \in \mathbb{Z}} |\gamma_\mu|^2 |\gamma_\nu|^2 \quad (g \neq 0).
\]
Hence
\[
\sum_{\xi \in \mathbb{Z}} |\Gamma_\mu|^2 \leq 2 \sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}} |\gamma_\mu|^2 |\gamma_\nu|^2.
\]
A moment's consideration shows that the part of the right-hand side that contains a given \( |\gamma_\mu|^2 \) (\( \mu \) fixed) is
\[
\sum_{\xi \in \mathbb{Z}} |\gamma_\xi|^2 \sum_{\nu \in \mathbb{Z}} |\gamma_\nu|^2 = 4 |\gamma_\mu|^2 \sum_{\nu \in \mathbb{Z}} |\gamma_\nu|^2 = 4 |\gamma_\mu|^2 (1 - |\gamma_\mu|^2) \ll 4 |\gamma_\mu|^2,
\]
so that
\[
\sum_{\xi \in \mathbb{Z}} |\Gamma_\mu|^2 \leq 4 \sum_{\nu \in \mathbb{Z}} |\gamma_\nu|^2 = 4.
\]
This together with \( |\Gamma_0|^2 = 1 \) gives \( J \leq 5 \) and so also (1.1) with \( A = 5^{1/4} \).

2. Theorem 2. Suppose that
\[
f \sim f^p, \quad f \sim \sum_{\nu \in \mathbb{Z}} e^{2\pi i \langle \nu, \xi \rangle},
\]
where \( 1 < p < 4/3 \), so that \( p' = p/(p-1) > 4 \). Then, for \( q = 2p' \) (thus \( 2 < q < \infty \)) we have
\[
\left( \sum_{\mu \in \mathbb{Z}} |\gamma_\mu|^2 \right)^{1/2} \ll A_p \|f\|_p
\]
with \( A_p = 5^{1/4} \).

This is a corollary of Theorem 1 and M. Riesz' theorem on the interpolation of linear operations (see, e.g., [22], p. 95). For the inequality (2.1) holds for \( p = 4 \), \( q = 2 \), \( A_{4,4} = 5^{1/4} \), and also clearly if \( p = 1 \), \( q = \infty \), \( A_1 = 1 \). Hence given \( p \), \( 1 < p < 4/3 \), if first we determine \( t \) from the equation
\[
1/p = (1-t)\cdot 1 + t \cdot t
\]
(thus \( t = (4/p) - 3 \), \( 1-t = 4/p' \)) and then \( q \) from the equation
\[
1/q = (1-t)\cdot 1 + t \cdot 0
\]
(so that \( q = 2/(1-t) = 4/p' \)), we obtain (2.1) with
\[
A_p \ll (5^{1/4})^{-1} \cdot t^t = 5^{1/4}.
\]
3. Remarks. a) In Theorems 1 and 2 we consider lattice points situated on a circle. But the only property we used of the circle was that it has no more than two chords of identical length and direction, and it is clear that if \( S \) is any curve (or merely a point set in the plane) with
the property that it has no more than \( k \) chords of identical length and direction, then

\[
\left( \sum_{x \in S} |e_x|^2 \right)^{1/2} \leq A_k \| f \|_{L^1(S)},
\]

where \( A_k \) depends only on \( k \) (as the proof of Theorem 1 shows we may take \( A_k = (2k+1)^k \)). This is an extension of (3.1) and it leads to an obvious extension of (2.1). In this form the theorem is valid for any number of dimensions \( n = 1, 2, 3, \ldots \) However, already for \( n = 3 \) the sphere does not have the required property and the problem of analogues of (3.1) and (2.1) in this case remains open.

b) Perhaps a simple example pertaining to the case \( n = 1 \) deserves mention.

Let \( S \) be the set of non-negative integers whose ternary developments contain only the digits 0 and 1. It is easy to see that any integer \( r \neq 0 \) can be represented at most once as a difference of two numbers from \( S \). For such a difference is a number \( \sum e_x 3^x \) where all the \( e_x \) are 0, ±1, and if we had \( \sum e_x 3^x = \sum e_y 3^y \), i.e. \( \sum \eta_x 3^x = 0 \), where \( \eta_x = e_x - e_y \), then all the \( \eta_x \) must be equal to 0. For otherwise, assuming \( \eta_0 \neq 0 \) and \( \eta_1 = 0 \) for \( j > k \), we would have the inequality

\[
1 \cdot 3^k - 2(1 + 3 + \ldots + 3^{k-1}) \leq 0,
\]

which is impossible. (The same property has the set of non-negative integers \( \sum e_x 3^x \) where all the \( e_x \) are 0, ±1, provided \( \eta_x = \eta_0 \).

It follows by the argument that gave Theorem 1 that if \( f(\omega) \), \( 0 < x < 1 \), is in \( L^1(\mathbb{R}^n) \) and \( c \) are the Fourier coefficients of \( f \), then

\[
\left( \sum_{x \in S} |c_x|^2 \right)^{1/2} \geq A \| f \|_{L^2(S)}.
\]

\( A = 3^{4n} \). The same argument and conclusion hold if \( S \) is replaced by the set \( S' \) of non-negative integers whose ternary development contains only digits 0 and 1. The set \( S' \) has some formal resemblance to Cantor's set of numbers \( z = \sum x \langle 3^{-x} \rangle \) (\( \langle r \rangle = 0, 2 \).

c) Since the right-hand side of (1.1) can be made arbitrarily small by subtracting from \( f \) a polynomial, it follows that if \( f \in L^1(\mathbb{R}) \), then

\[
\lim_{\| f \|_{L^1(\mathbb{R})} \to 0} \sum_{x \in S} |c_x|^2 = 0.
\]

Theorem 2 admits of a similar corollary.

d) The proof of Theorem 1 was based on the dual result: If

\[
y = \sum c_{n,n} e^{2\pi i \omega_n \cdot \xi},
\]

then \( \| y \|_1 \leq \| y \|_1 \). Since \( \| y \|_1 \leq \| y \|_1 \), interpolation of operations shows that if \( 1 \leq p \leq 2 \), then

\[
\| y \|_p \leq \| y \|_1 \| y \|_p \quad (q = 2p').
\]

A similar conclusion holds for functions \( \sum y e^{2\pi i \omega_n \cdot \xi} \) of a single variable, where \( \omega \) belongs to sets \( S \) or \( S' \) considered in b) above.

4. We shall now consider analogues of Theorems 1 and 2 for Fourier transforms. Though the arguments are modelled on those for Fourier series they are somewhat less simple. It is also curious that quantitively the results are somewhat different.

Let \( f \in L^p(\mathbb{R}^n) \) and let

\[
\hat{f}(\omega) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i \omega \cdot y} dy,
\]

be the Fourier transform of \( f \). We would like to estimate

\[
\left( \int_{\mathbb{R}^n} |\hat{f}(\omega)|^q d\omega \right)^{1/q},
\]

\( d\omega \) denoting the element of length, in terms of

\[
\| f \|_p = \left( \int_{\mathbb{R}^n} |\hat{f}(\omega)|^p d\omega \right)^{1/p},
\]

for suitable \( p \) and \( q \). The main result here is as follows.

**Theorem 3.** If \( f \in L^p(\mathbb{R}^n) \), where

\[
1 \leq p \leq 4/3,
\]

then, for each \( p > 0 \), \( \hat{f}(\omega) \) exists almost everywhere on \( |\omega| = q \), and for

\[
q = \frac{1}{3}, \quad p' = \frac{3}{p - 1},
\]

we have

\[
\left( \int_{\mathbb{R}^n} |\hat{f}(\omega)|^q d\omega \right)^{1/q} \leq A_p \| f \|_{L^p(\mathbb{R}^n)},
\]

where \( A_p \) is a constant depending on \( p \) only.
The result being obvious for \( p = 1 \), we may assume that \( 1 < p < 4/3 \).

This implies that
\[
4/3 < q < \infty.
\]

Since, in any case, \( 1 \leq p \leq 2 \), the existence of \( \tilde{f}(\sigma) \) almost everywhere is a classical result; the novelty here is that if \( p < 4/3 \) the transform \( \tilde{f} \) exists almost everywhere on every circle \( |\sigma| = \varrho \).

Also observe that Theorem 3 is an analogue of Theorem 2. The latter was obtained from the limiting case \( p = 2 \) (Theorem 1) by interpolating operations. We cannot follow this path here since Theorem 3 is false in the limiting case \( p = 4/3 \) and we must prove the general case directly, which complicates the proof (see Section 7 below).

We shall initially argue purely formally, and also assume for the sake of simplicity that \( \varrho = 1 \).

5. The left-hand side of (4.1) is then
\[
\int_{|\varrho| = 1} \tilde{f}(\sigma)\varphi(\sigma) d\sigma = 1,
\]

and
\[
\left( \int_{|\varrho| = 1} |\tilde{f}(\sigma)|^q d\sigma \right)^{1/q} = \left( \int_{|\varrho| = 1} \varphi(\sigma) \left| \int_{|\varrho| = 1} \tilde{f}(u) e^{-2\pi i u \cdot \sigma} du \right|^q d\sigma \right)^{1/q}.
\]

Thus the problem reduces to estimating the last integral. We shall denote it by \( I^{p'} \), and it is enough to show that \( I \leq A_p \).

We can then write (the dot \( \cdot \) denoting, as before, scalar multiplication of vectors)
\[
I^{p'} = \int_{|\varrho| = 1} \left| \int_{|\varrho| = 1} \varphi(u) e^{-2\pi i u \cdot \sigma} d\lambda \right|^{p'} \left( \int_{|\varrho| = 1} \varphi(u) e^{-2\pi i u \cdot \sigma} d\sigma \right)^{1/p'}
\]

or, with \( u = \xi + i\eta \),
\[
I^{p'} = \int_{|\varrho| = 1} d\xi d\eta \left| \int_{|\varrho| = 1} \varphi(u) e^{-2\pi i u \cdot \sigma} d\lambda \right|^{p'} \left( \int_{|\varrho| = 1} \varphi(u) e^{-2\pi i u \cdot \sigma} d\sigma \right)^{1/p'}.
\]

Let us introduce new variables
\[
\cos \lambda - \cos \mu = \nu, \quad \sin \lambda - \sin \mu = \omega,
\]

and consider the Jacobian of the transformation. We have
\[
|\begin{vmatrix}
\cos \lambda - \cos \mu \\
\sin \lambda - \sin \mu
\end{vmatrix}|
\]

Since the complex numbers \( e^{i\nu} - e^{i\omega} \) can take admissible values distinct from \( 0 \) at most twice, we can split the domain of integration \( 0 \leq \lambda \leq 2\pi, \quad 0 \leq \mu \leq 2\pi \) into two disjoint sets \( D_1 \) and \( D_2 \) in whose interiors the mapping is one-one (take, e.g., for \( D_1 \) the set \( 0 \leq \lambda < 2\pi, \quad 0 \leq \mu - \lambda < \pi \) (mod \( 2\pi \)) and for \( D_2 \) the set \( 0 \leq \lambda < 2\pi, \quad \pi < \mu - \lambda < 0 \) (mod \( 2\pi \)). Correspondingly, the inner integral in (5.2) is split into two integrals, and, by the triangle inequality (observe that the hypothesis \( p < 2 \) implies \( \frac{1}{p'} > 1 \))
\[
I \leq I_1 + I_2,
\]

where, for \( j = 1, 2 \),

\[
I_j = \left| \int_{D_j} \varphi(\nu) e^{-2\pi i \nu \cdot \mu} d\nu \right|^{1/p'} \left| \int_{|\varrho| = 1} \varphi(\nu) e^{-2\pi i \nu \cdot \sigma} d\sigma \right|^{1/p'}.
\]

Let \( D_j \) be the image of \( D_j \) in the plane of the variables \( \nu, \omega \). Then
\[
I_j^{p'} = \left( \int_{|\varrho| = 1} d\sigma \left| \int_{|\varrho| = 1} \varphi(\nu, \omega) e^{-2\pi i \nu \cdot \sigma} d\nu \right|^{p'} \right)^{1/p'}
\]

where (see (5.2))
\[
\varphi(\nu, \omega) = \frac{1}{A} \varphi(e^{i\nu} e^{i\omega}).
\]

The inner integral being the Fourier transform of the function equal to \( \varphi(\nu, \omega) \) in \( D_j \) and to 0 elsewhere, we may apply the Hausdorff-Young inequality, provided \( \frac{1}{p'} > 1 \), i.e., \( p' > \frac{4}{3} \), or
\[
1 < p < \frac{4}{3}
\]

and since the exponent conjugate to \( \frac{1}{p'} \) is \( p/(3-p) \), we have
\[
I_j \leq \left( \int_{|\varrho| = 1} \left| \varphi(\nu, \omega) e^{-2\pi i \nu \cdot \sigma} d\nu \right|^{-p/(3-p)} d\sigma \right)^{p/(3-p)}
\]

or
\[
I_j \leq \left( \int_{|\varrho| = 1} \left| \varphi(\nu, \omega) e^{-2\pi i \nu \cdot \sigma} d\nu \right|^{-p/(3-p)} d\sigma \right)^{p/(3-p)}
\]

and
\[
I \leq \left( \int_{|\varrho| = 1} \left| \varphi(\nu, \omega) e^{-2\pi i \nu \cdot \sigma} d\nu \right|^{-p/(3-p)} d\sigma \right)^{p/(3-p)}.
\]
The exponent in the last denominator is positive. It is also strictly less than 1 provided \( p < \frac{1}{2} \) (see (5.5)).

Let us set
\[
|\psi(\omega^2)|^{2(2p-\rho)} = \psi(\lambda), \quad \chi(\lambda) = \int_0^\infty \frac{\psi(\mu)}{|\sin(\lambda-\mu)|^{\rho-\frac{3p(2p-\rho-1)}{4}} d\mu.
\]

Then
\[
(5.6) \quad I_1 \leq \left( \int_0^\infty \psi(\lambda) \chi(\lambda) d\lambda \right)^{2-2p}.
\]

By hypothesis,
\[
(5.7) \quad \|\psi\|_{L^{2p-\rho-1}} = 1,
\]
and since \( \chi \) is, effectively, a fractional (Riemann–Liouville) integral of \( \psi \) of order
\[
(5.8) \quad 1 - \frac{2(2p-1)}{2-p} = 4 - 3p,
\]
\( \chi \) belongs to \( \mathcal{L}^r \) where \( r \) is defined by the equation
\[
(5.9) \quad \frac{1}{q'} = \frac{1}{2-p} - \frac{1}{r} = \frac{4-3p}{2-p}.
\]

More precisely,
\[
(5.10) \quad \|z\|_r \leq A_{p,r} \|\psi\|_{L^{2p-r-1}} = A_{p,r}.
\]

The exponent \( q \) has so far been indetermined. If we select it in such a way that \( r \) is conjugate to \( q'(2-p)/p \) (see (5.6), (5.7), and (5.10)), Hölder’s inequality applied to the integral in (5.8) will show that
\[
(5.11) \quad I_1 \leq A_p \quad (j = 1, 2).
\]

Thus we must have
\[
(5.12) \quad \frac{1}{q'} = \frac{p}{2-p} + \frac{1}{r} = 1,
\]

together with (5.9). Adding (5.9) and (5.12) we obtain successively
\[
\frac{2}{q'} = \frac{2}{2-p} - \frac{6-4p}{2-p} = \frac{3}{3-2p}, \quad q' = \frac{3}{3-2p} q = \frac{p}{3(p-1)} = \frac{1}{3} p'.
\]

Hence we have (5.11) and so also \( I \leq I_1 + I_2 \leq A_p \).

This completes the proof of Theorem 3, though we still have to dispose of the assumption \( q = 1 \) and justify the formal character of the proof.

Begin with the latter. The proof is rigorous if \( q = 1 \) and if \( \mathcal{L}^1 \) is, say, bounded and has bounded support, in which case \( \mathcal{L}^1 \) is continuous. If \( \{f_n\} \) is a sequence of such functions with \( \|f-f_n\|_{\mathcal{L}^1} \to 0 \), then \( \|f_n-f\|_{\mathcal{L}^1} \to 0 \) and so also \( \|f_n-f\|_{\mathcal{L}^1} \to 0 \) as \( n \to \infty \). Hence \( \{f_n\} \) converges to a limit, call it \( f \), on \( \{x\} = 1 \), in the metric \( \mathcal{L}^1 \), and \( f \) satisfies the required inequality.

Let now \( \varphi \) be any positive number. If we set \( g(x) = f(x) \varphi \) then
\[
\hat{g}(x) = \varphi \hat{f}(x) ,
\]
so that
\[
\left( \int_{|x|=\varphi} |\hat{f}(x)|^q dx \right)^{1/q} = \left( \int_{|x|=\varphi} |\hat{g}(x)|^q dx \right)^{1/q} = \left( \int_{|x|=1} |\hat{g}(x)|^q dx \right)^{1/q} = \varphi^{1-1/q} \left( \int_{|x|=1} |\hat{f}(x)|^q dx \right)^{1/q} = \varphi^{1-1/q} \left( \int_{|x|=\varphi} |\hat{f}(x)|^q dx \right)^{1/q} \leq A_q \varphi^{1-1/q} \left( \int_{|x|=\varphi} |\hat{f}(x)|^q dx \right)^{1/q} = A_q \varphi^{1-1/q} \left( \int_{|x|=\varphi} |\hat{f}(x)|^q dx \right)^{1/q} = A_q \varphi^{1-1/q} \|f\|_{\mathcal{L}^1} ,
\]
which for \( q = \frac{1}{3} p' \) gives (4.1).

6. Let \( x \) denote points and \( \nu \) lattice points in \( \mathbb{R}^1 \). Let \( a = \{a_i\} \mathbb{R}^1 \), i.e.,
\[
|a|_p = \left( \sum_{|\nu| \leq \varphi} |a^\nu|^q \right)^{1/q}.
\]

We shall now prove the following

**Theorem 4.** If \( \{a_i\} \mathbb{R}^1 , 1 < p < 4/3 \) and
\[
f(x) \sim \sum_i a_i e^{i(x-x_i)},
\]
then for \( q = \frac{1}{3} p' \) and any \( 0 < q \leq r \) we have
\[
(6.1) \quad \left( \int_{|x|=q} |f(x)|^q dx \right)^{1/q} \leq A_q \varphi^{1/q} |a|_p.
\]

This is an analogue of Theorem 3 though neither is deducible from the other in a simple way. The proof in both cases follows the same pattern but the fact that now, for obvious reasons, we cannot reduce the general case to that of \( \varphi = 1 \) makes the argument somewhat more cumbersome. It is again enough to argue purely formally and, as a matter of
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\[ J_{\lambda, \mu} = \int_{D_j} \bar{\varphi}(\rho, \theta) \varphi(\rho, \theta) e^{i(\lambda \rho - \mu \theta)} d\rho d\theta. \]

where \( \varphi(\rho, \theta) \) equals

\[ 4 \pi^2 \frac{\varphi(\rho, \theta) \varphi(\rho', \theta')}{\rho^2 \sin(\lambda - \mu)} \]

in \( D_j \) and is 0 in \( Q - D_j \). The numbers \( J_{\lambda, \mu} \), and since the exponent conjugate to \( \lambda \rho - \mu \theta \) is \( p/(2 - p) \), the Hausdorff–Young inequality gives

\[ (\sum |J_{\lambda, \mu}|^{\frac{2}{p'}})^{1/p'} \leq A_p. \]

We shall write \( \sum |\varphi|^p = \sum |\varphi|^p \) and represent \( |\varphi|^p \) as the Fourier coefficient of a function to which we can apply the Hausdorff–Young inequality (since \( 4p' > 2 \)). We have

\[ |\varphi|^p = \varphi^2 \int \varphi(\rho, \theta) \varphi(\rho', \theta') e^{i(\rho - \rho') \sin(\lambda - \mu)} d\rho d\theta = \psi(\rho) \varphi(\rho, \theta). \]

say. Thus

\[ (\sum |\varphi|^p)^{1/p'} = \varphi \left( \sum |\varphi|^p \right)^{1/p} = \varphi \left( \sum |\varphi|^p \right)^{1/p}. \]

We set

\[ \varphi = (\cos \lambda - \cos \rho, \sin \lambda - \sin \rho) \]

where the condition \( |\psi| = 1 \) imposed on \( \varphi \) can be written

\[ |\psi| = |\varphi(\rho, \theta)|^{1/(p'-2)}. \]

On the other hand, as in the proof of Theorem 3, \( \chi \) is in \( L^r \) with \( r \) defined by (5.9). Moreover, by the first inequality (5.10),

\[ \|\varphi\|_{2p/(3 - 2p)} \leq A_{p', q} e^{-4p' \rho/(3 - 2p)}. \]

If we choose \( \varphi \) in such a way that \( r \) is conjugate to \( q^2/(2 - p) \), which, as we know, leads to \( q = \frac{1}{2} p' \), the right-hand-side of (5.5) is majorized by

\[ A_p e^{-4p' \rho - A_{p', q} e^{-4p' \rho/(3 - 2p)}}. \]
In view of (6.4)
\[
\left( \sum |x|^p \right)^{1/p'} \leq A e^{-q|2p'-1q|} = A e^{q|1-2p'|} = A e^{q|p|',}
\]
since \( q = \frac{3}{2} p' \). This gives (6.2) and so also (6.1).

7. The following example (which I owe to Charles Fefferman) shows that Theorem 3 is false in the extreme case \( p = \frac{3}{2} \).

Let \( f(x) \) be a radial function: \( f(x) = f(|x|) \). Then the Fourier transform
\[
\hat{f}(\xi) = \int \frac{f(y) e^{-i\xi y}}{\xi} dy
\]
(assuming it exists) is also radial. We shall show that there is a radial \( f(x) \in L^p(\mathbb{R}^3) \) such that
\[
\begin{align*}
\hat{f}(1) = & \int \int f(\xi) e^{-i\xi y} d\xi dy = 2\pi \int f(\xi) J_0(2\pi\xi) d\xi \\
is & \geq 0. 
\end{align*}
\]
is \( + \infty \). This, of course, precludes the possibility of (4.1) for \( q = 1 \). We shall show that
\[
f(x) = \frac{\sin 2\pi |x|}{|x|^{1/2}} \frac{1}{\log(2 + |x|)}
\]
has the required properties.

First of all,
\[
|\hat{f}(1)| \leq 2\pi \int_0^\infty \frac{\sin 2\pi \xi}{\xi^{1/2}} \frac{1}{\log(2 + \xi)} d\xi < \infty,
\]
since the integrand is \( O(1) \) for \( 0 < \xi \leq 1 \) and is \( O(\xi^{-1} \log^{-4/3} (2 + \xi)) \) for \( \xi > 1 \).

Next, (see (7.1))
\[
\hat{f}(1) = 2\pi \int_0^\infty \frac{\sin 2\pi \xi}{\xi^{1/2}} J_0(2\pi\xi) d\xi = \int_0^1 + \int_1^\infty = A + B,
\]
say. Since \( J_0(\xi) = O(1) \), the integrand of \( A \) is bounded, and the classical formula
\[
J_0(\xi) = (2/\pi)^{1/2} \xi^{-1/2} \cos \left( \xi - \frac{1}{4} \pi \right) + O(\xi^{-1/2}) \quad (\xi \to + \infty)
\]
shows that
\[
B = O(1) + 2\pi \int_1^\infty \frac{\sin 2\pi \xi}{\xi} [\sin 2\pi \xi + \cos 2\pi \xi] d\xi = O(1) + 2\pi \int_1^\infty \frac{\sin^2 2\pi \xi}{\xi},
\]
so that \( B = + \infty \). Hence \( \hat{f}(1) = + \infty \) and the assertion is established.