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On Fourier coefficients and transforms of functions of two variables

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Abstract. Let $f(x_1, x_2)$ be a function of two variables, of period 1 in each, and let $c_\mu = c_{m,n}$ be the Fourier coefficients of f. Then, if $1 \leqslant p < \frac{4}{3}$ and $q = \frac{1}{2}p' = \frac{1}{2}p/(p-1)$, we have

$$\Big\{\sum_{|\mu|=r} |c_\mu|^q\Big\}^{1/q} \leqslant A_p \|f\|_p \qquad (A_p = 5^{1/p'})$$

for all $r > 0$. There is a corresponding result for Fourier transforms of functions $f \in L^p(\boldsymbol{R}^2)$, $1 \leqslant p < \frac{4}{3}$, but the previous $q = \frac{1}{2}p'$ has to be replaced by $q = \frac{1}{3}p'$. Moreover, the result fails in the extreme case $p = \frac{4}{3}$. The results are strictly two-dimensional.

1. Let $\xi = (x_1, x_2)$ denote points on the two-dimensional torus

$$(Q) \qquad 0 \leqslant x_1 < 1, \quad 0 \leqslant x_2 < 1,$$

and $\mu = (m_1, m_2)$ – lattice points in $\boldsymbol{R}^2$ (m_j – integers). Given any integrable function $f(\xi)$ on Q consider its Fourier series

$$\sum c_\mu e^{2\pi i(\mu\cdot\xi)},$$

where

$$c_\mu = \int_Q f(\xi) e^{-2\pi i(\mu\cdot\xi)} d\xi,$$

with $\mu\cdot\xi = m_1x_1 + m_2x_2$, $d\xi = dx_1dx_2$.

The origin of this Note is the following question which Charles Fefferman proposed some time ago. Does there exist a positive number p strictly less than 2 such that

$$\Big(\sum_{|\mu|=r} |c_\mu|^2\Big)^{\frac{1}{2}} \leqslant A \|f\|_p,$$

where A is independent of r. The following theorem gives an answer to the problem.

THEOREM 1. *For any $r > 0$, we have*

$$(1.1) \qquad \Big(\sum_{|\mu|=r} |c_\mu|^2\Big)^{\frac{1}{2}} \leqslant A \|f\|_{4/3},$$

where $A = 5^{1/4}$.

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Proof. Let us consider the set $S = S_r$ of lattice points $\mu = (m_1, m_2)$ with $|\mu| = r$ (we assume that S is not empty, since otherwise there is nothing to prove). We then have, for a suitable sequence $\{\gamma_\mu\}$ with

$$\sum_{\mu\in S} |\gamma_\mu|^2 = 1,$$

the equation

$$\Big(\sum |c_\mu|^2\Big)^{\frac{1}{2}} = \sum c_\mu \gamma_\mu = \sum \gamma_\mu \int_Q f(\xi) e^{-2\pi i(\mu\cdot\xi)} d\xi$$
$$= \int_Q f(\xi) \Big[\sum_\mu \gamma_\mu e^{-2\pi i(\mu\cdot\xi)}\Big] d\xi,$$

so that, by Hölder's inequality with exponents 4/3 and 4,

$$\Big(\sum_{|\mu|=r} |c_\mu|^2\Big)^{\frac{1}{2}} \leqslant \|f\|_{4/3} \Big\| \sum_{\mu\in S} \gamma_\mu e^{-2\pi i(\mu\cdot\xi)} \Big\|_4, \tag{1.2}$$

and it is enough to show that the last factor is $\leqslant A$.

Write

$$J = \int_Q \Big| \sum \gamma_\mu e^{-2\pi i(\mu\cdot\xi)} \Big|^4 d\xi = \int_Q \Big| \sum_{\mu,\nu\in S} \gamma_\mu \bar{\gamma}_\nu e^{2\pi i(\nu-\mu)\cdot\xi} \Big|^2 d\xi. \tag{1.3}$$

We have

$$\sum \gamma_\mu \bar{\gamma}_\nu e^{2\pi i(\nu-\mu)\cdot\xi} = \sum \Gamma_\varrho e^{2\pi i(\varrho\cdot\xi)},$$

with

$$\Gamma_\varrho = \sum_{\nu-\mu=\varrho} \gamma_\mu \bar{\gamma}_\nu. \tag{1.4}$$

Here μ and ν are in S and ϱ takes all admissible values. Thus ϱ designates lattice points that are differences of two lattice points on S. By Parseval's formula,

$$J = \sum_\varrho |\Gamma_\varrho|^2.$$

It is immediate that

$$\Gamma_0 = \sum_\varrho |\gamma_\mu|^2 = 1.$$

If $\varrho \neq 0$, the sum (1.4) consists of one or two terms (the former if $\nu = -\mu$) and in any case, in view of the inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$,

$$|\Gamma_\varrho|^2 \leqslant 2 \sum_{\nu-\mu=\varrho} |\gamma_\mu|^2 |\gamma_\nu|^2 \quad (\varrho \neq 0).$$

Hence

$$\sum_{\varrho\neq 0} |\Gamma_\varrho|^2 \leqslant 2 \sum_{\varrho\neq 0} \sum_{\nu-\mu=\varrho} |\gamma_\mu|^2 |\gamma_\nu|^2.$$

A moment's consideration shows that the part of the right-hand side that contains a given $|\gamma_\mu|^2$ (μ — fixed) is

$$\sum_{\varrho\neq 0} 2 |\gamma_\mu|^2 \sum_{\nu-\mu=\pm\varrho} |\gamma_\nu|^2 = 4 |\gamma_\mu|^2 \sum_{\nu\neq\mu} |\gamma_\nu|^2 = 4 |\gamma_\mu|^2 (1 - |\gamma_\mu|^2) \leqslant 4 |\gamma_\mu|^2,$$

so that

$$\sum_{\varrho\neq 0} |\Gamma_\varrho|^2 \leqslant 4 \sum_\mu |\gamma_\mu|^2 = 4.$$

This together with $|\Gamma_0|^2 = 1$ gives $J \leqslant 5$ and so also (1.1) with $A = 5^{1/4}$.

2. THEOREM 2. *Suppose that*

$$f \in L^p, \quad f \sim \sum c_\mu e^{2\pi i(\mu\cdot\xi)},$$

where $1 < p < 4/3$, *so that* $p' = p/(p-1) > 4$. *Then, for* $q = \frac{1}{2}p'$ (*thus* $2 < q < \infty$) *we have*

$$\Big(\sum_{|\mu|=r} |c_\mu|^q\Big)^{1/q} \leqslant A_p \|f\|_p \tag{2.1}$$

with $A_p = 5^{1/p'}$.

This is a corollary of Theorem 1 and M. Riesz' theorem on the interpolation of linear operations (see, e.g. [2_{II}], p. 95). For the inequality (2.1) holds for $p = \frac{4}{3}$, $q = 2$, $A_{4/3} = 5^{1/4}$, and also clearly if $p = 1$, $q = \infty$, $A_1 = 1$. Hence given p, $1 < p < 4/3$, if first we determine t from the equation

$$1/p = (1-t)\cdot\tfrac{3}{4} + t\cdot 1$$

(thus $t = (4/p) - 3$, $1 - t = 4/p'$) and then q from the equation

$$1/q = (1-t)\cdot\tfrac{1}{2} + t\cdot 0$$

(so that $q = 2/(1-t) = \frac{1}{2}p'$), we obtain (2.1) with

$$A_p \leqslant (5^{1/4})^{1-t}\cdot 1^t = 5^{1/p'}.$$

3. Remarks. a) In Theorems 1 and 2 we consider lattice points situated on a circle. But the only property we used of the circle was that it has no more than two chords of identical length and direction, and it is clear that if S is any curve (or merely a point set in the plane) with the property that it has no more than k chords of identical length and direction, then

$$\Big(\sum_{\mu\in S}|c_\mu|^2\Big)^{\frac12}\leqslant A_k\|f\|_{4/3}, \tag{3.1}$$

where A_k depends only on k (as the proof of Theorem 1 shows we may take $A_k=(2k+1)^{\frac14}$). This is an extension of (1.1) and it leads to an obvious extension of (2.1). In this form the theorem is valid for any number of dimensions $n=1,2,3,\ldots$ However, already for $n=3$ the sphere does not have the required property and the problem of analogues of (1.1) and (2.1) in this case remains open.

b) Perhaps a simple example pertaining to the case $n=1$ deserves mention.

Let S be the set of non-negative integers whose ternary developments contain only the digits 0 and 1. It is easy to see that any integer $\nu\neq 0$ can be represented at most once as a difference of two numbers from S. For such a difference is a number $\sum\varepsilon_j 3^j$ where all the ε_j are $0,\pm1$, and if we had $\sum\varepsilon_j' 3^j=\sum\varepsilon_j 3^j$, i.e. $\sum\eta_j 3^j=0$, where $\eta_j=\varepsilon_j-\varepsilon_j'$, then all the η_j must be equal to 0. For otherwise, assuming $\eta_k\neq0$ and $\eta_j=0$ for $j>k$, we would have the inequality

$$1\cdot3^k-2(1+3+\ldots+3^{k-1})\leqslant 0,$$

which is impossible. (The same property has the set of non-negative integers $\sum\varepsilon_j n_j$, $\varepsilon_j=\pm1$, provided $n_{j+1}/n_j\geqslant 3$.)

It follows by the argument that gave Theorem 1 that if $f(x)$, $0\leqslant x<1$, is in $L^{4/3}$ and c_ν are the Fourier coefficients of f, then

$$\Big(\sum_{\nu\in S}|c_\nu|^2\Big)^{\frac12}\leqslant A\|f\|_{4/3},$$

$A=3^{1/4}$. The same argument and conclusion hold if S is replaced by the set S' of non-negative integers whose ternary development contains only digits 0 and 2. The set S' has some formal resemblance to Cantor's set of numbers $x=\sum\limits_1^\infty\varepsilon_j 3^{-j}$ $(\varepsilon_j=0,2)$.

c) Since the right-hand side of (1.1) can be made arbitrarily small by subtracting from f a polynomial, it follows that if $f\in L^{4/3}$, then

$$\lim_{r\to\infty}\sum_{|\mu|=r}|c_\mu|^2=0.$$

Theorem 2 admits of a similar corollary.

d) The proof of Theorem 1 was based on the dual result: If

$$g=\sum_{|\mu|=r}\gamma_\mu e^{2\pi i(\mu\cdot\xi)},$$

then $\|g\|_4\leqslant 5^{1/4}\|\gamma\|_2$. Since $\|g\|_\infty\leqslant\|\gamma\|_1$, interpolation of operations shows that if $1\leqslant p\leqslant 2$, then

$$\|g\|_q\leqslant 5^{1/2p'}\|\gamma\|_p\qquad (q=2p').$$

A similar conclusion holds for functions $\sum\gamma_\nu e^{2\pi i\nu x}$ of a single variable, where ν belongs to sets S or S' considered in b) above.

4. We shall now consider analogues of Theorems 1 and 2 for Fourier transforms. Though the arguments are modelled on those for Fourier series they are somewhat less simple. It is also curious that quantitively the results are somewhat different.

Let $x\in\boldsymbol{R}^2$ and let

$$\hat f(x)=\int_{\boldsymbol{R}^2}f(y)e^{-2\pi i(x\cdot y)}dy$$

be the Fourier transform of f. We would like to estimate

$$\Big(\int_{|x|=\varrho}|\hat f(x)|^q d\sigma\Big)^{1/q},$$

$d\sigma$ denoting the element of length, in terms of

$$\|f\|_p=\Big\{\int_{\boldsymbol{R}^2}|\hat f(x)|^p dx\Big\}^{1/p},$$

for suitable p and q. The main result here is as follows.

THEOREM 3. *If $f\in L^p(\boldsymbol{R}^2)$, where*

$$1\leqslant p<4/3,$$

then, for each $\varrho>0$, $\hat f(x)$ exists almost everywhere on $|x|=\varrho$, and for

$$q=\frac13 p'=\frac13\,\frac{p}{p-1}$$

we have

$$\Big(\int_{|x|=\varrho}|\hat f(x)|^q d\sigma\Big)^{1/q}\leqslant A_p\varrho^{1/p'}\|f\|_p, \tag{4.1}$$

where A_p is a constant depending on p only.

The result being obvious for $p=1$, we may assume that $1<p<4/3$. This implies that

$$4/3 < q < \infty.$$

Since, in any case, $1 \leqslant p \leqslant 2$, the existence of $\hat{f}(x)$ almost everywhere is a classical result; the novelty here is that if $p<4/3$ the transform $\hat{f}$ exists almost everywhere on *every* circle $|x|=\varrho$.

Also observe that Theorem 3 is an analogue of Theorem 2. The latter was obtained from the limiting case $p=2$ (Theorem 1) by interpolating operations. We cannot follow this path here since Theorem 3 is false in the limiting case $p=4/3$ and we must prove the general case directly, which complicates the proof (see Section 7 below).

We shall initially argue purely formally, and also assume for the sake of simplicity that $\varrho=1$.

5. The left-hand side of (4.1) is then $\int\limits_{|x|=1} \hat{f}(x)\varphi(x)\,d\sigma$ for a suitable φ with

$$\int\limits_{|x|=1} |\varphi(x)|^{q'}\,d\sigma = 1,$$

and

$$\text{(5.1)}\qquad \Big\{\int\limits_{|x|=1} |\hat{f}(x)|^q\,d\sigma\Big\}^{1/q} = \int\limits_{|x|=1} \varphi(x)\Big\{\int\limits_{R^2} f(u)\,e^{-2\pi i(u\cdot x)}\,du\Big\}\,d\sigma$$
$$= \int\limits_{R^2} f(u)\Big\{\int\limits_{|x|=1} \varphi(x)\,e^{-2\pi i(u\cdot x)}\,d\sigma\Big\}\,du$$
$$\leqslant \|f\|_p \Big\{\int\limits_{R^2}\Big|\int\limits_{|x|=1} \varphi(x)\,e^{-2\pi i(u\cdot x)}\,d\sigma\Big|^{p'}du\Big\}^{1/p'}.$$

Thus the problem reduces to estimating the last integral. We shall denote it by $I^{\frac{1}{2}p'}$, and it is enough to show that $I \leqslant A_p$.

We can then write (the dot '·' denoting, as before, scalar multiplication of vectors)

$$I^{\frac{1}{2}p'} = \int\limits_{R^2} du\,\Big|\int\limits_0^{2\pi} \varphi(e^{i\lambda})\,e^{-2\pi i(e^{i\lambda}\cdot u)}\,d\lambda \int\limits_0^{2\pi} \bar{\varphi}(e^{i\mu})\,e^{2\pi i(e^{i\mu}\cdot u)}\,d\mu\Big|^{\frac{1}{2}p'}$$
$$= \int\limits_{R^2} du\,\Big|\int\limits_0^{2\pi}\int\limits_0^{2\pi} \varphi(e^{i\lambda})\bar{\varphi}(e^{i\mu})\,e^{-2\pi i(e^{i\lambda}-e^{i\mu})\cdot u}\,d\lambda\,d\mu\Big|^{\frac{1}{2}p'}$$

or, with $u=\xi+i\eta$,

$$\text{(5.2)}\qquad I^{\frac{1}{2}p'} = \iint\limits_{R^2} d\xi\,d\eta\,\Big|\int\limits_0^{2\pi}\int\limits_0^{2\pi} \varphi(e^{i\lambda})\bar{\varphi}(e^{i\mu})\,e^{-2\pi i[(\cos\lambda-\cos\mu)\xi+(\sin\lambda-\sin\mu)\eta]}\,d\lambda\,d\mu\Big|^{\frac{1}{2}p'}.$$

Let us introduce new variables

$$\cos\lambda-\cos\mu = v,\qquad \sin\lambda-\sin\mu = w,$$

and consider the Jacobian of the transformation. We have

$$\text{(5.3)}\qquad \Delta = \left|\frac{\partial(v,w)}{\partial(\lambda,\mu)}\right| = |\sin(\lambda-\mu)|.$$

Since the complex numbers $e^{i\lambda}-e^{i\mu}$ can take admissible values distinct from 0 at most twice, we can split the domain of integration $0\leqslant\lambda\leqslant 2\pi$, $0\leqslant\mu\leqslant 2\pi$ into two disjoint sets D_1 and D_2 in whose interiors the mapping is one-one (take, e.g. for D_1 the set $0\leqslant\lambda<2\pi$, $0\leqslant\mu-\lambda<\pi \pmod{2\pi}$ and for D_2 the set $0\leqslant\lambda<2\pi$, $-\pi\leqslant\mu-\lambda<0 \pmod{2\pi}$). Correspondingly, the inner integral in (5.2) is split into two integrals, and, by the triangle inequality (observe that the hypothesis $p\leqslant 2$ implies $\frac{1}{2}p'\geqslant 1$)

$$\text{(5.4)}\qquad I \leqslant I_1+I_2,$$

where, for $j=1,2$,

$$I_j = \Big\{\iint\limits_{R^2} d\xi\,d\eta\,\Big|\int\limits_{D_j} \varphi(e^{i\lambda})\bar{\varphi}(e^{i\mu})\,e^{-2\pi i[(\cos\lambda-\cos\mu)\xi+(\sin\lambda-\sin\mu)\eta]}\,d\lambda\,d\mu\Big|^{\frac{1}{2}p'}\Big\}^{2/p'}.$$

Let $\bar{D}_j$ be the image of D_j in the plane of the variables v, w. Then

$$I_j^{\frac{1}{2}p'} = \Big\{\iint\limits_{R^2} d\xi\,d\eta\,\Big|\iint\limits_{\bar{D}_j} \omega(v,w)\,e^{-2\pi i(v\xi+w\eta)}\,dv\,dw\Big|^{\frac{1}{2}p'}\Big\},$$

where (see (5.2))

$$\omega(v,w) = \frac{1}{\Delta}\,\varphi(e^{i\lambda})\bar{\varphi}(e^{i\mu}).$$

The inner integral being the Fourier transform of the function equal to $\omega(v,w)$ in $\bar{D}_j$ and to 0 elsewhere, we may apply the Hausdorff–Young inequality, provided $\frac{1}{2}p'\geqslant 2$, i.e., $p'\geqslant 4$, or

$$\text{(5.5)}\qquad 1<p\leqslant\frac{4}{3},$$

and since the exponent conjugate to $\frac{1}{2}p'$ is $p/(2-p)$, we have

$$I_j \leqslant \Big\{\iint\limits_{R^2} |\omega(u,v)|^{p/(2-p)}\,du\,dv\Big\}^{(2-p)/p}$$
$$\leqslant \Big\{\int\limits_0^{2\pi}\int\limits_0^{2\pi} |\varphi(e^{i\lambda})\bar{\varphi}(e^{i\mu})|^{p/(2-p)}\,\Delta\,d\lambda\,d\mu\Big\}^{(2-p)/p}$$
$$= \Big\{\int\limits_0^{2\pi}\int\limits_0^{2\pi} \frac{|\varphi(e^{i\lambda})\bar{\varphi}(e^{i\mu})|^{p/(2-p)}}{|\sin(\lambda-\mu)|^{2(p-1)/(2-p)}}\,d\lambda\,d\mu\Big\}^{(2-p)/p}.$$

The exponent in the last denominator is positive. It is also strictly less than 1 provided $p < \frac{4}{3}$ (see (5.5)).

Let us set

$$|\varphi(e^{i\lambda})|^{p/(2-p)} = \psi(\lambda), \qquad \chi(\lambda) = \int_0^{2\pi} \frac{\psi(\mu)}{|\sin(\lambda-\mu)|^{2(p-1)/(p-2)}}\, d\mu.$$

Then

$$I_j \leqslant \Big[\int_0^{2\pi} \psi(\lambda)\chi(\lambda)\, d\lambda\Big]^{(2-p)/p}. \tag{5.6}$$

By hypothesis,

$$\|\psi\|_{q'(2-p)/p} = 1, \tag{5.7}$$

and since χ is, effectively, a fractional (Riemann–Liouville) integral of ψ of order

$$1 - \frac{2(p-1)}{2-p} = \frac{4-3p}{2-p}, \tag{5.8}$$

χ belongs to L^r where r is defined by the equation

$$\frac{1}{q'}\cdot\frac{p}{2-p} - \frac{1}{r} = \frac{4-3p}{2-p}. \tag{5.9}$$

More precisely,

$$\|\chi\|_r \leqslant A_{p,q}\|\psi\|_{q'(2-p)/p} = A_{p,q}. \tag{5.10}$$

The exponent q has so far been indetermined. If we select it in such a way that r is conjugate to $q'(2-p)/p$ (see (5.6), (5.7), and (5.10)), Hölder's inequality applied to the integral in (5.6) will show that

$$I_j \leqslant A_p \qquad (j = 1, 2). \tag{5.11}$$

Thus we must have

$$\frac{1}{q'}\cdot\frac{p}{2-p} + \frac{1}{r} = 1, \tag{5.12}$$

together with (5.9). Adding (5.9) and (5.12) we obtain successively

$$\frac{2}{q'}\cdot\frac{p}{2-p} = \frac{6-4p}{2-p}, \qquad q' = \frac{p}{3-2p}, \; q = \frac{p}{3(p-1)} = \frac{1}{3}p'.$$

Hence we have (5.11) and so also $I \leqslant I_1 + I_2 \leqslant A_p$.

This completes the proof of Theorem 3, though we still have to dispose of the assumption $\varrho = 1$ and justify the formal character of the proof.

Begin with the latter. The proof is rigorous if $\varrho = 1$ and if f is, say, bounded and has bounded support, in which case $\hat{f}$ is continuous. If $\{f_n\}$ is a sequence of such functions with $\|f - f_n\|_p \to 0$, then $\|f_m - f_n\|_p \to 0$ and so also $\int_{|x|=1} |f_m - f_n|^q\, d\sigma \to 0$. Hence $\{\hat{f}_n\}$ converges to a limit, call it $\hat{f}$, on $|x| = 1$, in the metric L^q, and $\hat{f}$ satisfies the required inequality.

Let now ϱ be any positive number. If we set $g(x) = f(x/\varrho)$ then $\hat{g}(x) = \varrho^2\hat{f}(\varrho x)$, so that

$$\Big(\int_{|x|=\varrho} |\hat{f}(x)|^q\, d\sigma\Big)^{1/q} = \Big(\int_{|x|=1} |\hat{f}(\varrho x)|^q\, \varrho\, d\sigma\Big)^{1/q} = \Big(\int_{|x|=1} (\varrho^{-2}|\hat{g}(x)|)^q\, \varrho d\sigma\Big)^{1/q}$$

$$= \varrho^{\frac{1}{q}-2}\Big(\int_{|x|=1} |\hat{g}(x)|^q\, d\sigma\Big)^{1/q}$$

$$\leqslant A_p\, \varrho^{\frac{1}{q}-2}\Big(\int_{R^2} |g(x)|^p\, dx\Big)^{1/p} = A_p\, \varrho^{\frac{1}{q}-2}\varrho^{\frac{2}{p}}\Big(\int_{R^2}\Big|f\Big(\frac{x}{\varrho}\Big)\Big|^p \frac{dx}{\varrho^2}\Big)^{1/p}$$

$$= A_p\, \varrho^{\frac{1}{q}-\frac{2}{p'}}\|f\|_p,$$

which for $q = \frac{1}{3}p'$ gives (4.1).

6. Let x denote points and ν lattice points in $\boldsymbol{R}^2$. Let $a = \{a_\nu\} \in l^p$, i.e.,

$$\|a\|_p = \Big(\sum_\nu |a_\nu|^p\Big)^{1/p} < \infty.$$

We shall now prove the following

THEOREM 4. *If $\{a_\nu\} \in l^p$, $1 \leqslant p < 4/3$ and*

$$f(x) \sim \sum a_\nu e^{i(\nu\cdot x)},$$

then for $q = \frac{1}{3}p'$ and any $0 < \varrho \leqslant \pi$ we have

$$\Big(\int_{|x|=\varrho} |f(x)|^q\, d\sigma\Big)^{1/q} \leqslant A_p\, \varrho^{1/p'}\|a\|_p. \tag{6.1}$$

This is an analogue of Theorem 3 though neither is deducible from the other in a simple way. The proof in both cases follows the same pattern but the fact that now, for obvious reasons, we cannot reduce the general case to that of $\varrho = 1$ makes the argument somewhat more cumbersome. It is again enough to argue purely formally and, as a matter of

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fact, it would be enough to consider only the case of $\{a_\nu\}$ finite. The restriction $\varrho \leqslant \pi$ could be relaxed but the point is without much importance. Of course the circle $|x| = \varrho$ in (6.1) can be replaced by $|x - x_0| = \varrho$ for any x_0.

Let C_ϱ denote the circle $|x| = \varrho$ and let us systematically denote the left-hand side of (6.1) by $\|f\|_{q,\varrho}$. Then for a suitable $\varphi(x)$ with $\|\varphi\|_{q',\varrho} = 1$ we have

$$\|f\|_{q,\varrho} = \int_{C_\varrho} f\varphi\, d\sigma = \int_{C_\varrho} \sum a_\nu e^{i(\nu\cdot x)}\varphi\, d\sigma = \sum a_\nu \gamma_\nu,$$

where

$$\gamma_\nu = \int_{C_\varrho} \varphi(x) e^{i(\nu\cdot x)}\, d\sigma,$$

and it is enough to show that

$$\left(\sum |\gamma_\nu|^{p'}\right)^{1/p'} \leqslant A_p \varrho^{1/p'}. \tag{6.2}$$

We shall write $\sum |\gamma_\nu|^{p'} = \sum |\gamma_\nu|^{2\cdot\frac{1}{2}p'}$ and represent $|\gamma_\nu|^2$ as the Fourier coefficient of a function to which we can apply the Hausdorff–Young inequality (since $\frac{1}{2}p' \geqslant 2$). We have

$$|\gamma_\nu|^2 = \varrho^2 \int_0^{2\pi}\int_0^{2\pi} \varphi(\varrho e^{i\lambda})\overline{\varphi}(\varrho e^{i\mu}) \exp\{\varrho\nu\cdot(e^{i\lambda} - e^{i\mu})\}\, d\lambda\, d\mu = \varrho^2 J_\nu,$$

say. Thus

$$\left(\sum |\gamma_\nu|^{p'}\right)^{1/p'} = \varrho\left(\sum |J_\nu|^{\frac{1}{2}p'}\right)^{1/p'} = \varrho\left\{\left(\sum |J_\nu|^{\frac{1}{2}p'}\right)^{\frac{2}{p'}}\right\}^{\frac{1}{2}}. \tag{6.3}$$

We set

$$\varrho(\cos\lambda - \cos\mu) = v, \qquad \varrho(\sin\lambda - \sin\mu) = w,$$

$$\left|\frac{\partial(v, w)}{\partial(\lambda, \mu)}\right| = \varrho^2 |\sin(\lambda - \mu)| = \Delta,$$

and split the domain of integration in the last integral into two subdomains, D_1 and D_2, in the interior of which the mapping $(\lambda, \mu) \to (v, w)$ is 1–1; thus $\lambda = \lambda(v, w)$, $\mu = \mu(v, w)$. The image of D_j will be denoted by $\overline{D}_j$. Correspondingly, $J_\nu = J_{1,\nu} + J_{2,\nu}$ and, by (6.3),

$$\left(\sum |\gamma_\nu|^{p'}\right)^{1/p'} \leqslant \varrho\left\{\left(\sum |J_{1,\nu}|^{\frac{1}{2}p'}\right)^{2/p'} + \left(\sum |J_{2,\nu}|^{\frac{1}{2}p'}\right)^{2/p'}\right\}^{\frac{1}{2}}. \tag{6.4}$$

Fix j. The projections of $\overline{D}_j$ onto the coordinate axes have length

$2\varrho \leqslant 2\pi$ and so there is a square Q with sides parallel to the coordinate axes and length 2π which comprises $\overline{D}_j$. We can write, with $\nu = (m, n)$,

$$J_{j,\nu} = \iint_{\overline{D}_j} \varphi(\varrho e^{i\lambda})\overline{\varphi}(\varrho e^{i\mu}) \frac{e^{i(mv+nw)}}{\varrho^2 |\sin(\lambda - \mu)|}\, dv\, dw$$
$$= \frac{1}{4\pi^2} \iint_Q \omega(v, w) e^{i(mv+nw)}\, dv\, dw,$$

where $\omega(v, w)$ equals

$$4\pi^2 \frac{\varphi(\varrho e^{i\lambda})\overline{\varphi}(\varrho e^{i\mu})}{\varrho^2 |\sin(\lambda - \mu)|}$$

in $\overline{D}_j$ and is 0 in $Q - \overline{D}_j$. The numbers $J_{j,\nu}$ are then the Fourier coefficients of $\omega(v, w)$, and since the exponent conjugate to $\frac{1}{2}p'$ is $p/(2-p)$, the Hausdorff–Young inequality gives

$$\left(\sum |J_{j,\nu}|^{\frac{1}{2}p'}\right)^{2/p'} \leqslant \left\{\frac{1}{4\pi^2} \iint_Q |\omega(v, w)|^{p/(2-p)}\, dv\, dw\right\}^{(2-p)/p} \tag{6.5}$$
$$\leqslant \left\{\frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} \left(4\pi^2 \frac{|\varphi(\varrho e^{i\lambda})\varphi(\varrho e^{i\mu})|}{\varrho^2 |\sin(\lambda - \mu)|}\right)^{p/(2-p)} \Delta\, d\lambda\, d\mu\right\}^{(2-p)/p}$$
$$\leqslant \varrho^{-4/p'} \left(\int_0^{2\pi} \psi(\lambda)\chi(\lambda)\, d\lambda\right)^{(2-p)/p},$$

where

$$\psi(\lambda) = |\varphi(\varrho e^{i\lambda})|^{p/(2-p)}, \qquad \chi(\lambda) = \int_0^{2\pi} \frac{\psi(\mu)}{|\sin(\lambda - \mu)|^{2(p-1)/(2-p)}}\, d\mu.$$

The condition $\|\varphi\|_{q'(2-p)/p} = 1$ imposed on φ can be written

$$\|\psi\|_{q'(2-p)/p} = \varrho^{-p/q'(2-p)}. \tag{6.6}$$

On the other hand, as in the proof of Theorem 3, χ is in L^r with r defined by (5.9). Moreover, by the first inequality (5.10),

$$\|\chi\|_r \leqslant A_{p,q} \|\psi\|_{q'(2-p)/p} = A_{p,q}\, \varrho^{-p/q'(2-p)}.$$

If we choose q in such a way that r is conjugate to $q'(2-p)/p$, which, as we know, leads to $q = \frac{1}{3}p'$, the right-hand side of (6.5) is majorized by

$$A\varrho^{-4/p'} \left(\|\psi\|_{q'(2-p)/p} \|\chi\|_r\right)^{(2-p)/p} \leqslant A\varrho^{-4/p'} \left(\varrho^{-p/q'(2-p)} \cdot A_p\, \varrho^{-p/q'(2-p)}\right)^{(2-p)/p}$$
$$= A\varrho^{-4/p'-2/q'}.$$

In view of (6.4)

$$\Big(\sum|\gamma_\nu|^{p'}\Big)^{1/p'} \leqslant A\varrho\cdot\varrho^{-2/p'-1/q'} = A\varrho^{1/q-2/p'} = A\varrho^{1/p'},$$

since $q = \frac{1}{3}p'$. This gives (6.2) and so also (6.1).

7. The following example (which I owe to Charles Fefferman) shows that Theorem 3 is false in the extreme case $p = \frac{4}{3}$.

Let $f(x)$ be a radial function: $f(x) = f(|x|)$. Then the Fourier transform

$$\hat{f}(x) = \int_{R^2} f(y)e^{-2\pi i(x\cdot y)}dy$$

(assuming it exists) is also radial. We shall show that there is a radial $f(x)\in L^{4/3}(\boldsymbol{R}^2)$ such that

$$\hat{f}(1) = \int_0^{2\pi}\int_0^\infty f(\varrho)e^{-2\pi i\varrho\cos\varphi}\varrho d\varrho d\varphi = 2\pi\int_0^\infty f(\varrho)J_0(2\pi\varrho)\varrho d\varrho \tag{7.1}$$

is $+\infty$. This, of course, precludes the possibility of (4.1) for $\varrho = 1$. We shall show that

$$f(x) = \frac{\sin 2\pi|x|}{|x|^{3/2}}\cdot\frac{1}{\log(2+|x|)}$$

has the required properties.

First of all,

$$\|f\|_{4/3}^{4/3} = 2\pi\int_0^\infty\left[\frac{\sin 2\pi\varrho}{\varrho^{3/2}}\,\frac{1}{\log(2+\varrho)}\right]^{4/3}\varrho d\varrho < \infty,$$

since the integrand is $O(1)$ for $0<\varrho\leqslant 1$ and is $O\big(\varrho^{-1}\log^{-4/3}(2+\varrho)\big)$ for $\varrho>1$.

Next, (see (7.1))

$$\hat{f}(1) = 2\pi\int_0^\infty\frac{\sin 2\pi\varrho}{\varrho^{1/2}}\,\frac{J_0(2\pi\varrho)}{\log(2+\varrho)}d\varrho = \int_0^1+\int_1^\infty = A+B,$$

say. Since $J_0(\varrho) = O(1)$, the integrand of A is bounded, and the classical formula

$$J_0(\varrho) = (2/\pi)^{1/2}\varrho^{-1/2}\cos\left(\varrho-\frac{1}{4}\pi\right)+O(\varrho^{-3/2})\quad(\varrho\to+\infty)$$

shows that

$$B = O(1)+2^{1/2}\int_0^\infty\frac{\sin 2\pi\varrho}{\varrho}[\sin 2\pi\varrho+\cos 2\pi\varrho]d\varrho = O(1)+2^{1/2}\int_0^\infty\frac{\sin^2 2\pi\varrho}{\varrho},$$

so that $B = +\infty$. Hence $\hat{f}(1) = +\infty$ and the assertion is established.

8. Remarks. Problems analogous to those discussed here are also considered in Fefferman [1].

Theorem 3 was generalized by P. Sjölin (unpublished) to more general curves.

References

[1] Charles L. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. 124 (1970), pp. 9–36.

[2] A. Zygmund, *Trigonometric Series*, Vols. I, II, pp. 383+364, 1959.

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