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Exponential integrability of certain singular integral transforms

by

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Abstract. It is shown that singular integral operators with odd kernels map bounded functions with support of finite measure to locally exponentially integrable functions. In particular it is shown that the periodic Riesz transform of a function of supremum norm one is exponentially integrable of order a for $a < \pi/2$ and $\pi/2$ is the best possible constant. This extends and gives a new proof of the known result for the periodic Hilbert transform.

The following well-known result concerning exponential integrability of conjugate harmonic functions can be proved relatively easily using methods from complex function theory (e.g. [3], p. 254).

THEOREM. *Let f be a bounded measurable function on $\{e^{i\theta}; -\pi \leqslant \Theta < \pi\}$. Let Hf be the periodic Hilbert transform of f. If $\|f\|_\infty \leqslant 1$ then*

$$\int_0^{2\pi} \exp a|Hf(e^{i\theta})|\,d\theta < \infty \quad \textit{for} \quad 0 \leqslant a < \frac{\pi}{2}.$$

Examples show that the constant $\pi/2$ in this result is the best possible.

In this paper we will show that a result analogous to the above theorem holds for linear transformations defined on $L^\infty(E^n)$ by singular integral operators with odd kernels. The proof is a straightforward application of results of O'Neil and Weiss [1] on rearrangements of functions.

It will follow that the above theorem is true, with the constant $\pi/2$, for periodic Riesz transforms and that the constant is again the best possible. (In particular, this gives a new, and strictly real variable, proof of the theorem stated above.)

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For a measurable function f defined on the non-atomic measure space (M, μ), define f^*, the *non-increasing rearrangement* of f, to be the

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non-negative real valued function defined on $(0, \infty)$ by $f^*(t) = \inf\{y > 0;\ \mu(\{x \in M;\ |f(x)| > y\}) \leqslant t\}$. f^* is readily seen to be non-increasing and equimeasurable with f. f^* is, up to minor modification, the inverse function of the distribution function of $|f|$. (For a discussion of the properties of f^* see [1] or [3]; I, 13). We need the following fact about f^* ([3]; I, 13.9). If φ is an increasing function and S is a measurable subset of M then

$$\int_S \varphi(|f|)\,d\mu \leqslant \int_0^{\mu(S)} \varphi(f^*(t))\,dt. \tag{1}$$

We will also be interested in the integral means of f^*. We define $f^{**}(s) = \frac{1}{s}\int_0^s f^*(t)\,dt$. Since f^* is non-increasing,

$$f^*(t) \leqslant f^{**}(t) \tag{2}$$

for all t.

The class of transforms to be considered is defined as follows. For a point $X = (x_1, \ldots, x_n)$ in Euclidean n-space, E^n, let $|X| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$. Let $S_{n-1} = \{X \text{ in } E^n;\ |X| = 1\}$. Let dm denote Lebesgue measure on E^n and $d\sigma$ denote area measure on S_{n-1}. Condsider a function Ω on E^n such that:

1) Ω is odd, i.e. $\Omega(X) = -\Omega(-X)$ for all X in E^n,

2) Ω is homogenous of degree zero, i.e.

$$\Omega(X/|X|) = \Omega(X) \quad \text{for all } X \neq 0, \text{ and}$$

3) $\|\Omega\| = \int_{S_{n-1}} |\Omega(X)|\,d\sigma(X) < \infty$.

Associated with such an Ω there is a linear map T, defined on $L^p(E^n)$, $p > 1$, by

$$(Tf)(X) = \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_{\substack{\delta \geqslant |Y| \geqslant \varepsilon\\ Y \in E^n}} \frac{\Omega(Y)}{|Y|^n} f(X - Y)\,dm(Y).$$

If f is in L^p, $p > 1$, then Tf is defined a.e. In fact T is a bounded linear map of $L^p(E^n)$ into itself for $1 < p < \infty$ ([2], Chapter VI), such a T is called a *singular integral operator with odd kernel*. Ω is called the *kernel* of T.

A set of examples of such operators of particular interest are the Riesz transforms, $R_{n,j}$. For a positive integer n, and an integer j, $1 \leqslant j \leqslant n$, the jth Riesz transform R_{nj} is the singular integral operator on $L^p(E^n)$ defined by the kernel $\Omega_{nj}(X) = \Omega_{nj}(x_1, \ldots, x_n) = C_n x_j/|X|$ where $C_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}$. In particular R_{11} is the Hilbert transform. Curiously, an elementary computation shows $\|\Omega\| = 2/\pi$ for all n, j.

Pick and fix T, a singular integral operator with odd kernel, defined for functions in $L^p(E^n)$. Let Ω be the kernel of T. Our main result is the following:

THEOREM. *Let f be a function in $L^\infty(E^n)$ such that the support of f has finite measure. Let S be a subset of E^n of finite measure, then*

$$\int_S \exp\alpha|Tf(Y)|\,dm(Y) < \infty \quad \text{if } 0 \leqslant \alpha < \frac{1}{\|\Omega\|\,\|f\|_\infty}.$$

Proof. It suffices to consider the case $\|f\|_\infty = 1$. Pick and fix α, $0 < \alpha < \frac{1}{\|\Omega\|}$. By (1) and (2)

$$\int_S \exp\alpha|Tf(Y)|\,dm(Y) \leqslant \int_0^{m(S)} \exp\alpha(Tf)^*(t)\,dt \leqslant \int_0^{m(S)} \exp\alpha(Tf)^{**}(t)\,dt.$$

Hence it suffices to show that for any positive ε,

$$(Tf)^{**}(t) \leqslant \|\Omega\|(1+\varepsilon)\left(\log\frac{1}{t} + O(1)\right) \quad \text{as } t \to 0. \tag{3}$$

However, O'Neil and Weiss have shown ([1], Theorem 3) that

$$(Tf)^{**}(s) \leqslant \|\Omega\|\frac{1}{s}\int_0^\infty f^*(t)\sinh^{-1}\left(\frac{s}{t}\right)dt.$$

But $f^*(t) = 0$ for $t > K = m\{\text{support of } f\}$ and $|f^*(t)| \leqslant \|f\|_\infty$ for all t. Thus

$$\begin{aligned}(Tf)^{**}(s) &\leqslant \|\Omega\|\frac{1}{s}\int_0^K \sinh^{-1}\left(\frac{s}{t}\right)dt\\ &\leqslant \|\Omega\|\int_{s/K}^\infty \frac{1}{x^2}\sinh^{-1}(x)\,dx\\ &\leqslant \|\Omega\|(1+\varepsilon)\left(\log\frac{1}{s} + O(1)\right) \quad \text{as } s \to 0\end{aligned}$$

the last inequality, which follows from $\sinh^{-1}(y) = \log(y + \sqrt{1+y^2})$ and elementary estimates, holding for any positive ε. Thus (3) is established and the proof is complete.

We now show that this result implies the corresponding result for the periodic Riesz transforms and in that case (and also for the ordinary Riesz transform) the constant obtained is the best possible.

Let T^n be the n-torus $\{(\theta_1, \ldots, \theta_n);\ -\pi \leqslant \theta_i < \pi,\ i = 1, \ldots, n\}$. It is convenient to regard T^n as a subset of E^n. Given f in $L^p(E^n)$, $p > 1$, denote also by f the periodic extension of f to E^n. It is known that $R_{nj}(f)$

is defined a.e. and is periodic. Hence $R_{nj}(f)$ can be regarded as a function on T^n. Call this restriction of $R_{nj}(f)$ the periodic Riesz transform, $\bar{R}_{nj}(f)$. $\bar{R}_{nj}$ is a bounded linear operator of $L^p(T^n)$ into itself $1 < p < \infty$.

COROLLARY. *Given f in $L^\infty(T^n)$, then for each $j, 1 \leqslant j \leqslant n$,*

$$\int_{T^n} \exp\alpha|\bar{R}_{nj}f| < \infty \quad \text{for } 0 \leqslant \alpha < \frac{\pi}{2}\cdot\frac{1}{\|f\|_\infty}.$$

Proof. As before we may assume $\|f\|_\infty = 1$. Pick and fix $\alpha < \frac{\pi}{2}$. It suffices to show

$$\int_{T^n} \exp\alpha|R_{nj}f| < \infty. \tag{4}$$

If f had support on a set of finite measure this would follow directly from the previous theorem and the observation that $\|\Omega\| = \frac{2}{\pi}$ for the Riesz transform. Although this is not the case it is almost true in that the size of $R_{nj}f$ on T^n is determined by the behavior of f near T^n. Let W be an open neighborhood of $\overline{T^n}$ with characteristic function χ_W. $R_{nj}f = R_{nj}(\chi_W\cdot f)+R_{nj}\big((1-\chi_W)f\big)$. Since $\overline{T^n}$ is a compact subset of the interior of a set on which $(1-\chi_W)f$ vanishes, there is a constant K such that $\big|R_{nj}\big((1-\chi_W)f\big)\big| < K$ on T^n. Hence it suffices to verify (4) for $\chi_W f$ instead of f. This follows from the previous theorem.

OBSERVATION. *The constant $\frac{\pi}{2}$ of the previous corollary is the best possible.*

To show this it suffices to exhibit for each n, j a function f_j on T^m such that $\|f_j\|_\infty = 1$ and $\int_{T^n} \exp\frac{\pi}{2}|\bar{R}_{nj}f_j| = \infty$. Let f_j be the function defined on T^n by $f_j(X) = \operatorname{sgn}(x_j)$. That f_j is the required function can be seen by making elementary estimates on the appropriate integrals or by noting that since $f_j(X)$ depends only on x_j, $\bar{R}_{nj}(f)(X)$ will depend only on x_j and as a function of x_j will be the periodic Hilbert transform of $\operatorname{sgn}(x)$. However $\bar{R}_{11}\big(\operatorname{sgn}(x)\big)(t) = \frac{2}{\pi}\log\frac{1}{t} + O(1)$ for t small. Hence the integral diverges.

PROBLEM. *The above example shows that the constant $\frac{1}{\|\Omega\|}$ in the theorem is the best possible constant if $\Omega = \Omega_{n,j}$. It would be interesting to know what the best possible constant is in this result for general Ω.*

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