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The moduli of smoothness and convexity and the Rademacher averages of trace classes S_p $(1 \leqslant p < \infty)$*

by

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Abstract. It is proved that the moduli of smoothness and convexity of the trace classes S_p have the same order as the corresponding moduli of L_p $(1 < p < \infty)$ and the Rademacher averages of S_p behave in the same manner as the corresponding averages of L_p $(1 \leqslant p < \infty)$. As a corollary some results on p-absolutely summing operators are obtained.

Let $1 \leqslant p < \infty$. By S_p we denote the Banach space of compact operators on a Hilbert space H such that

$$\|A\|_p = \left(\operatorname{tr}(A^*A)^{p/2}\right)^{1/p} < \infty.$$

In the present paper we investigate some geometric properties of these spaces. It is shown that several properties are similar to the corresponding properties of L_p spaces, despite of the fact that for $p \neq 2$ and the infinite-dimensional Hilbert space H, S_p is not isomorphic to any subspace of L_p (cf. [16]). In particular the moduli of smoothness and convexity of S_p have the same order as the corresponding moduli of L_p $(1 < p < \infty)$. This fact in the case of modulus of convexity and $p \geqslant 2$ was proved by Dixmier [1].

Furthermore the Rademacher averages of S_p behave in the same manner as the corresponding averages of L_p. Namely we prove the following inequalities: There exist constants C_p such that for arbitrary $A_0, \ldots, A_n$ in S_p $(n = 0, 1, \ldots)$ we have (1)

$$\int_0^1 \Big\| \sum_{j=0}^n A_j r_j(t) \Big\|_p dt \leqslant C_p \Big(\sum_{j=0}^n \|A_j\|_p^2 \Big)^{1/2} \quad \text{for } p \geqslant 2, \tag{0.1}$$

$$\int_0^1 \Big\| \sum_{j=0}^n A_j r_j(t) \Big\|_p dt \geqslant C_p \Big(\sum_{j=0}^n \|A_j\|_p^2 \Big)^{1/2} \quad \text{for } p \leqslant 2. \tag{0.2}$$

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(1) Further $\|\cdot\|_a^b$ denotes $(\|\cdot\|_a)^b$.

That means that S_p for $p \geqslant 2$ is of the type 2 in the terminology of [6] and for $p \leqslant 2$ is of the cotype 2 in the terminology of [14]. The inequality (0.2) holds also in every predual of a C^*-algebra.

It follows from (0.1) and (0.2) that every bounded operator from an $\mathscr{L}_\infty$-space in the sense of [13] into S_p is q-absolutely summing for $q > p > 2$ and is 2-absolutely summing for $1 \leqslant p \leqslant 2$.

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§ 1. Notation and preliminaries. We begin with some notation. By $r_m(\cdot)$ $(m = 0, 1, \ldots)$ we denote the mth Rademacher function, i.e.

$$r_m(t) = \operatorname{sgn}(\sin 2^{m+1}\pi t) \quad \text{for } 0 \leqslant t \leqslant 1.$$

By $w_n(\cdot)$ $(n = 0, 1, \ldots)$ we denote the nth Walsh function, i.e.

$$w_0(t) = 1,$$
$$w_n(t) = r_{m_1}(t)\cdot r_{m_2}(t)\ldots r_{m_k}(t) \quad \text{for } n = 1, \ldots, \text{ and } 0 \leqslant t \leqslant 1,$$

where $n = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_k}$ is a binary expansion of n.

Let X, Y be Banach spaces. We denote by $L(X, Y)$ the space of all the operators from X into Y with the usual operator norm

$$\|A\| = \sup_{\|x\|\leqslant 1} \|Ax\|.$$

Let H be a Hilbert space. We write $L(H)$ instead of $L(H, H)$. By $K(H)$ we denote the space of all the compact operators from H into H with the operator norm $\|\cdot\|$.

If $A \in L(H)$, then A^* denotes the adjoint of A. We define the sequence $(s_j(A))_{j=1}^\infty$ of s-numbers of the operator A by

$$s_j(A) = \lambda_j, \quad j = 1, 2, \ldots,$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ is a decreasing sequence of non-zero eigenvalues of the operator $(A^*A)^{1/2}$, each repeated a number of times equal to its multiplicity.

Let $1 \leqslant p < \infty$. We put

$$S_p = \Big\{A \in K(H)\colon \sum_{j=1}^\infty |s_j(A)|^p < \infty\Big\}.$$

It is well known (cf. [4]) that S_p is a Banach space under the norm $\|A\|_p = \big(\sum_{j=1}^\infty |s_j(A)|^p\big)^{1/p}$ with the usual operations of addition of operators and multiplication by scalars. By S_∞ we mean $K(H)$.

Let $A \in S_1$. We define the trace of A by

$$\operatorname{tr} A = \sum_{j=1}^\infty \lambda_j(A) \tag{1.1}$$

where $(\lambda_j(A))$ is a sequence of eigenvalues of A, each repeated a number of times equal to its multiplicity. It is well known (cf. [4]) that if $A \in S_1$ then the series on the right-hand side of (1.1) is absolutely convergent and

$$|\operatorname{tr} A| \leqslant \|A\|_1 = \operatorname{tr}(A^*A)^{1/2}.$$

The trace is linear functional on S_1. Furthermore $A \in S_p$ if and only if $(A^*A^2)^{p/2} \in S_1$ and we have

$$\|A\|_p = \big(\operatorname{tr}(A^*A)^{p/2}\big)^{1/p} \quad (1 \leqslant p < \infty).$$

In the sequel we need the following

Lemma 1.1 (general Horn inequality). *Let $A_m \in K(H)$ for $m = 1, \ldots, N$; $N = 2, 3, \ldots$ Then*

$$\sum_{j=1}^n s_j\Big(\prod_{m=1}^N A_m\Big) \leqslant \sum_{j=1}^n \prod_{m=1}^N s_j(A_m) \quad (n = 1, 2, \ldots). \tag{1.2}$$

Proof. For $N = 2$ this is the classical Horn inequality (cf. [4]). Assume that (1.2) holds for some $N = r \geqslant 2$ and for every r operators in $K(H)$. Consider any $(r+1)$ operators in $K(H)$, say $A_1, \ldots, A_r, A_{r+1}$ and put $s_j = s_j(A_{r+1})$ and $t_j = s_j\big(\prod_{m=1}^r A_m\big)$ for $j = 1, 2, \ldots$. Applying the Horn inequality for the operators $\prod_{m=1}^r A_m$ and A_{r+1} we get

$$\sum_{j=1}^n s_j\Big(\prod_{m=1}^{r+1} A_m\Big) \leqslant \sum_{j=1}^n s_j t_j \quad (n = 1, 2, \ldots). \tag{1.3}$$

Now, applying the Abel transform to the right-hand side of (1.3), we obtain

$$\sum_{j=1}^n s_j t_j = s_1 t_1 + \sum_{i=2}^n s_i\Big(\sum_{j=1}^i t_j - \sum_{j=1}^{i-1} t_j\Big) = s_n \sum_{j=1}^n t_j + \sum_{i=1}^{n-1} (s_i - s_{i+1}) \sum_{j=1}^i t_j.$$

On the other hand by the inductive hypothesis we have

$$\sum_{j=1}^i t_j \leqslant \sum_{j=1}^i \prod_{m=1}^r s_j(A_m), \quad \text{for } i = 1, 2, \ldots$$

Hence, remembering that $s_j \geqslant s_{j+1}$ for $j = 1, 2, \ldots$, we get

$$\begin{aligned}\sum_{j=1}^n s_j t_j &\leqslant s_n \sum_{j=1}^n \prod_{m=1}^r s_j(A_m) + \sum_{i=2}^{n-1} (s_i - s_{i+1}) \sum_{j=1}^i \prod_{m=1}^r s_j(A_m) \\ &= \sum_{j=1}^n s_j \prod_{m=1}^r s_j(A_m) = \sum_{j=1}^n \sum_{m=1}^{r+1} s_j(A_m) \quad (n = 1, 2, \ldots).\end{aligned} \tag{1.4}$$

Combining (1.3) with (1.4) we get

$$\sum_{j=1}^{n} s_j\Big(\prod_{m=1}^{r+1} A_m\Big) \leqslant \sum_{j=1}^{n}\prod_{m=1}^{r+1} s_j(A_m) \qquad (n = 1, 2, \ldots).$$

This completes the induction.

We shall apply several times the interpolation technique for the scale of S_p-spaces. We need for this the following concepts: if $D\subset C$ is a subset of the complex plane, then the operator-function $\varphi\colon D\to L(H)$ is said to be *w-continuous* (resp. *w-analytic in the interior of* D) if for arbitrary $x, y\in H$ the function $(\varphi x, y)\colon D\to C$ is continuous (resp. analytic in the interior of D). We shall use the following well-known fact:

PROPOSITION 1.2. *Let N be a positive integer. Let $1\leqslant p_0 < p_1\leqslant\infty$. Let $\mathscr{D} = \{z\in C\colon 0\leqslant \operatorname{Re} z\leqslant 1\}$ and let for $n = 0, \ldots, 2^N-1$ $\varphi_n\colon \mathscr{D}\to K(H)$ be a w-continuous function w-analytic in the interior of $\mathscr{D}$. If*

$$(1.5)\qquad \int_0^1 \Big\|\sum_{n=0}^{2^N-1}\varphi_n(\mu+iy)\,w_n(t)\Big\|_{p_\mu}^{p_\mu} dt \leqslant M_\mu^{p_\mu}$$

for $-\infty < y < +\infty$ and $\mu = 0, 1$, then

$$(1.6)\qquad \Big(\int_0^1 \Big\|\sum_{n=0}^{2^N-1}\varphi_n(z)w_n(t)\Big\|_{r(z)}^{r(z)} dt\Big)^{1/r(z)} \leqslant M_0^{\operatorname{Re}(1-z)} M_1^{\operatorname{Re} z} \quad \textit{for } z\in\mathscr{D}$$

where $r(z) = \big[\operatorname{Re}\big((1-z)p_0^{-1}+zp_1^{-1}\big)\big]^{-1}$.

Proof. Define a new Hilbert space H_N as the l_2-product of 2^N copies of H. Define $\Psi\colon \mathscr{D}\to K(H_N)$ by

$$\Psi(z) = \bigotimes_{j=1}^{2^N} 2^{-N/p(z)}\sum_{n=0}^{2^N-1}\varphi_n(z)\,w_n\big(2^{-(N+1)}(2j-1)\big)$$

where $p(z) = \big((1-z)p_0^{-1}+zp_1^{-1}\big)^{-1}$ and we employ the following notation:

If $A_j\in K(H)$ for $1\leqslant j\leqslant 2^N$, then $\bigotimes_{j=1}^{2^N} A_j\in K(H_N)$ is defined by $\bigotimes_{j=1}^{2^N} A_j(x_1, x_2, \ldots, x_{2^N}) = (A_1x_1, A_2x_2, \ldots, A_{2^N}x_{2^N})$.

It is evident that Ψ is w-continuous in $\mathscr{D}$ and w-analytic in the interior of $\mathscr{D}$. Since the Walsh functions $(0\leqslant n\leqslant 2^N-1)$ are constant on the intervals $\big(2^{-N}(j-1), 2^{-N}j\big)$ for $1\leqslant j\leqslant 2^N$, we have

$$\int_0^1\Big\|\sum_{n=0}^{2^N-1}\varphi_n(z)w_n(t)\Big\|_{r(z)}^{r(z)} dt = \sum_{j=1}^{2^N}\Big\|2^{-N/p(z)}\sum_{n=0}^{2^N-1}\varphi_n(z)\,w_n\big(2^{-(N+1)}(2j-1)\big)\Big\|_{r(z)}^{r(z)}.$$

It is easy to see that for every $A_j\in K(H)$ $(1\leqslant j\leqslant 2^N)$

$$\Big\|\bigotimes_{j=1}^{2^N} A_j\Big\|_p = \Big(\sum_{j=1}^{2^N}\|A_j\|_p^p\Big)^{1/p} \qquad \text{for } 1\leqslant p\leqslant\infty.$$

Hence

$$\Big(\int_0^1\Big\|\sum_{n=0}^{2^N-1}\varphi_n(z)w_n(t)\Big\|_{r(z)}^{r(z)} dt\Big)^{1/r(z)} = \|\Psi(z)\|_{r(z)}.$$

Now the desired conclusion is an immediate consequence of the following

THEOREM ([4], § 13). *Let $1\leqslant p_0 < p_1\leqslant\infty$. Let $\Psi\colon \mathscr{D}\to K(H)$ be an operator-function w-continuous in $\mathscr{D}$ and w-analytic in the interior of $\mathscr{D}$. If*

$$\|\Psi(\mu+iy)\|_{p_\mu}\leqslant M_\mu$$

for $-\infty < y < +\infty$ and $\mu = 0, 1$, then

$$\|\Psi(z)\|_{r(z)}\leqslant M_0^{\operatorname{Re}(1-z)}M_1^{\operatorname{Re} z} \qquad \textit{for } z\in\mathscr{D}$$

where $r(z) = \big[\operatorname{Re}\big((1-z)p_0^{-1}+zp_1^{-1}\big)\big]^{-1}$.

§ 2. Uniform smoothness and convexity of S_p $(1 < p < \infty)$.

We begin with the following lemma

LEMMA 2.1. *If p is an even positive integer then for $A, B\in S_p$*

$$(2.1)\qquad \|A+B\|_p^p+\|A-B\|_p^p\leqslant\sum_{j=1}^{\infty}\big[|s_j(A)+s_j(B)|^p+|s_j(A)-s_j(B)|^p\big].$$

Proof. Let $p = 2k$ where k is a positive integer. By the definition of the norm $\|\cdot\|_p$ we have

$$\|A+B\|_p^p+\|A-B\|_p^p = \operatorname{tr}[(A^*+B^*)(A+B)]^k+\operatorname{tr}[(A^*-B^*)(A-B)]^k.$$

Let us observe that

$$[(A^*+B^*)(A+B)]^k = \textstyle\sum_{(+)} C$$

where $\sum_{(+)}$ is extended over all the operators

$$C = \prod_{\nu=1}^{2k} C_\nu$$

such that

(+) C_ν equals either A^* or B^* for odd ν's and C_ν equals either A or B for even ν's.

For $C = \prod_{\nu=1}^{2k} C_\nu$ denote by $b(C)$ the number of the indices such that C_ν equals either B or B^*. Then

$$[(A^*-B^*)(A-B)]^k = \textstyle\sum_{(+)}(-1)^{b(C)}C.$$

Thus, by the additivity of the trace

$$\|A+B\|_p^p+\|A-B\|_p^p = 2\sum\nolimits_{(++)}\operatorname{tr} C \tag{2.2}$$

where $\sum_{(++)}$ is extended over all C satisfying $(+)$ and the condition

$$b(C) \text{ is an even number.} \tag{++}$$

Clearly, for every operator C we have $|\operatorname{tr} C| \leqslant \sum_{j=1}^{\infty} s_j(C)$. Now, using Lemma 1.1 and the observation that the s-numbers of the operator and its adjoint are equal we get

$$\sum_{j=1}^{\infty} s_j(C) = \sum_{j=1}^{\infty} s_j\Big(\prod_{r=1}^{2k} C_r\Big) \leqslant \sum_{j=1}^{\infty}\prod_{r=1}^{2k} s_j(C_r) = \sum_{j=1}^{\infty}[s_j(A)]^{-b(C)+2k}[s_j(B)]^{b(C)}.$$

It follows from (2.2) that

$$\sum\nolimits_{(++)}\operatorname{tr} C \geqslant 0.$$

Hence

$$\begin{aligned} 2\sum\nolimits_{(++)}\operatorname{tr} C &\leqslant 2\sum\nolimits_{(++)}\sum_{j=1}^{\infty}[s_j(A)]^{2k-b(C)}[s_j(B)]^{b(C)} \\ &= 2\sum_{j=1}^{\infty}\sum\nolimits_{(++)}[s_j(A)]^{2k-b(C)}[s_j(B)]^{b(C)} \\ &= 2\sum_{j=1}^{\infty}\sum_{\mu=0}^{k}\binom{2k}{2\mu}[s_j(A)]^{2k-2\mu}[s_j(B)]^{2\mu} \\ &= \sum_{j=1}^{\infty}\big[(s_j(A)+s_j(B))^{2k}+(s_j(A)-s_j(B))^{2k}\big]. \end{aligned}$$

This completes the proof.

We recall that the modulus of convexity of a Banach space Y, in symbols δ_Y, is a non-negative function defined for $\varepsilon > 0$ by

$$\delta_Y(\varepsilon) = \inf\left\{1-\left\|\frac{x+y}{2}\right\|:\ x,\ y \in Y,\ \|x\| = \|y\| = 1,\ \|x-y\| \geqslant \varepsilon\right\}.$$

The modulus of smoothness of Y, in symbols ϱ_Y, is defined for $\tau>0$ by

$$\varrho_Y(\tau) = \tfrac{1}{2}\sup\{\|x+y\|+\|x-y\|-2:\ x,\ y\in Y,\ \|x\| = 1,\ \|y\| = \tau\}.$$

THEOREM 2.2. *Let $1<p<\infty$. Then there exist constants C_p and C_p' such that*

$$C_p'\,\delta_{l_p} \leqslant \delta_{S_p} \leqslant \delta_{l_p}, \tag{2.3}$$

$$\varrho_{l_p} \leqslant \varrho_{S_p} \leqslant C_p\,\varrho_{l_p}. \tag{2.4}$$

Proof. For the proof of this theorem we shall need the following lemma. The idea of applying this lemma has its origin in Ph. D. thesis of T. Figiel [3].

LEMMA 2.3. *Let $p \geqslant 2$; then there exists a constant K_p such that for arbitrary real sequences (α_j), (β_j) with $\sum_{j=1}^{\infty}|\alpha_j|^p = 1$ and $\sum_{j=1}^{\infty}|\beta_j|^p \leqslant 1$ we have*

$$\sum_{j=1}^{\infty}|\alpha_j+\beta_j|^p+\sum_{j=1}^{\infty}|\alpha_j-\beta_j|^p-2 \leqslant K_p\Big(\sum_{j=1}^{\infty}|\beta_j|^p\Big)^{2/p}. \tag{2.5}$$

Proof of Lemma 2.3. It is easy to show that if $p \geqslant 2$, then there exist constants K_p' and K_p'' such that for arbitrary real number γ we have

$$|1+\gamma|^p+|1-\gamma|^p-2 \leqslant K_p'|\gamma|^2+K_p''|\gamma|^p. \tag{2.6}$$

The assumption $\sum_{j=1}^{\infty}|\alpha_j|^p = 1$ and (2.6) imply that

$$\begin{aligned} \sum_{j=1}^{\infty}|\alpha_j+\beta_j|^p+\sum_{j=1}^{\infty}|\alpha_j-\beta_j|^p-2 &= \sum_{j=1}^{\infty}|\alpha_j|^p\left(\left|\frac{\alpha_j+\beta_j}{\alpha_j}\right|^p+\left|\frac{\alpha_j-\beta_j}{\alpha_j}\right|^p-2\right) \\ &\leqslant K_p'\sum_{j=1}^{\infty}|\alpha_j|^p\left|\frac{\beta_j}{\alpha_j}\right|^2+K_p''\sum_{j=1}^{\infty}|\alpha_j|^p\left|\frac{\beta_j}{\alpha_j}\right|^p \\ &= K_p'\sum_{j=1}^{\infty}|\alpha_j|^p\left|\frac{\beta_j}{\alpha_j}\right|^2+K_p''\sum_{j=1}^{\infty}|\beta_j|^p. \end{aligned}$$

It follows from the Hölder inequality that

$$\sum_{j=1}^{\infty}|\alpha_j|^{p-2}|\beta_j|^2 \leqslant \Big(\sum_{j=1}^{\infty}|\alpha_j|^p\Big)^{(p-2)/p}\Big(\sum_{j=1}^{\infty}|\beta_j|^p\Big)^{2/p} = \Big(\sum_{j=1}^{\infty}|\beta_j|^p\Big)^{2/p}. \tag{2.7}$$

Finally, since $\big(\sum_{j=1}^{\infty}|\beta_j|^p\big)^{1/p} \leqslant 1$ and $p \geqslant 2$,

$$\begin{aligned} \sum_{j=1}^{\infty}|\alpha_j+\beta_j|^p+\sum_{j=1}^{\infty}|\alpha_j-\beta_j|^p-2 &\leqslant K_p'\Big(\sum_{j=1}^{\infty}|\beta_j|^p\Big)^{2/p}+K_p''\Big(\sum_{j=1}^{\infty}|\beta_j|^p\Big)^{1/p} \\ &\leqslant K_p\Big(\sum_{j=1}^{\infty}|\beta_j|^p\Big)^{2/p} \end{aligned}$$

with $K_p = K_p'+K_p''$. This completes the proof of Lemma 2.3.

Now, let us observe that the inequalities $\delta_{S_p} \leqslant \delta_{l_p}$ and $\varrho_{l_p} \leqslant \varrho_{S_p}$ follow from the fact that l_p is a subspace of S_p. We shall prove the remaining inequalities.

Case 1. Estimation of the modulus of convexity for $p \geqslant 2$. The fact that $\delta_{S_p} \geqslant \delta_{l_p}$ is due to Dixmier [1]. We shall briefly indicate his argument. First we establish the inequality

$$(\|A+B\|_p^p + \|A-B\|_p^p)^{1/p} \leqslant 2^{p-1/p}(\|A\|_p^p + \|B\|_p^p)^{1/p}$$

for $A, B \in S_p$ and $2 \leqslant p \leqslant \infty$, which is checked directly for $p = 2$ (from the parallegram identity) and for $p = \infty$ (from the tringle inequality), and follows by interpolation for $2 < p < \infty$. Next we repeat the classical Clarkson's argument (cf. [1]).

Case 2. Estimation of the modulus of smoothness for $p \geqslant 2$. The proof is an easy consequence of the following fact: for every $p \geqslant 2$ there exists a constant K_p such that if $A, B \in S_p$ with $\|B\|_p \leqslant \|A\|_p = 1$, then

$$\|A+B\|_p^p + \|A-B\|_p^p \leqslant K_p \|B\|_p^2 + 2. \tag{2.8}$$

Indeed, combining (2.8) with

$$\|A+B\|_p + \|A-B\|_p - 2 \leqslant p^{-1}(\|A+B\|_p^p + \|A-B\|_p^p - 2) \tag{2.9}$$

we get

$$\|A+B\|_p + \|A-B\|_p - 2 \leqslant p^{-1} K_p \|B\|_p^2. \tag{2.10}$$

(To prove (2.9) observe that $p^{-1}(a^p - 1) \geqslant a - 1$ for $a \geqslant 0$ and for $p \geqslant 1$, substitute in the above inequality a by $\|A+B\|$ and $\|A-B\|$ respectively and add the resulting inequalities together).

It follows from (2.10) and the definition of the modulus of smoothness that

$$\varrho_{S_p}(\tau) \leqslant p^{-1} K_p \tau^2 \quad \text{for } 0 < \tau \leqslant 1.$$

On the other hand we have (cf. [5])

$$\varrho_{l_p}(\tau) \geqslant K_p' \tau^2 \quad \text{for } 0 < \tau \leqslant 1.$$

Hence

$$\varrho_{S_p}(\tau) \leqslant C_p \varrho_{l_p}(\tau) \quad \text{for } 0 < \tau \leqslant 1 \text{ and } C_p = \frac{K_p}{K_p' p}.$$

To complete the proof of Case 2 we shall prove (2.8). If $p = 2k$ ($k = 1, 2, \ldots$) then (2.8) follows from Lemma 2.1 and Lemma 2.3. If $p > 2$, $p \neq 2k$, then (2.8) follows from the case of even numbers by interpolation.

Fix A, B in S_p with $\|B\|_p \leqslant \|A\|_p = 1$. Put $k = [p/2]$ and $p_0 = 2k$, $p_1 = 2k+2$. Define the strip $\mathscr{D}$ and the function $r\colon \mathscr{D} \to \langle p_0, p_1 \rangle$ as in Proposition 1.2. Next define $p\colon \mathscr{D} \to C$ by

$$p(z) = \big((1-z)p_0^{-1} + z p_1^{-1}\big)^{-1}$$

and observe that $\big(r(z)\big)^{-1} = \operatorname{Re}\big(p(z)\big)^{-1}$.

Let (a_m), (a_m'), (b_m), (b_m') be the orthonormal systems in H such that

$$A a_m = \alpha_m a_m', \quad B b_m = \beta_m b_m' \quad \text{for } m = 1, 2, \ldots$$

where $\alpha_m = s_m(A)$ and $\beta_m = s_m(B)$ $(m = 1, 2, \ldots)$.

Define the functions $\varphi_0, \varphi_1\colon \mathscr{D} \to K(H)$ by

$$\varphi_0(z) a_m = \alpha_m^{p/p(z)} 2^{1/p(z)} a_m',$$

$$\varphi_1(z) b_m = \beta_m^{p/p(z)} 2^{1/p(z)} \|B\|_p^{1-p/p(z)} b_m' \quad (m = 1, 2, \ldots).$$

It is obvious that φ_0 and φ_1 are w-continuous in $\mathscr{D}$ and w-analytic in the interior of $\mathscr{D}$. It is easy to check that for all $-\infty < y < +\infty$ and $\mu = 0, 1$

$$\|\varphi_0(\mu + iy)\|_{p_\mu} = 2^{1/p_\mu},$$
$$\|\varphi_1(\mu + iy)\|_{p_\mu} = 2^{1/p_\mu}\|B\|_p.$$

Since we already establish (2.8) for all even integers, in particular for p_0 and p_1, we have

$$2^{-1}\|\varphi_0(\mu+iy) + \varphi_1(\mu+iy)\|_{p_\mu}^{p_\mu} + 2^{-1}\|\varphi_0(\mu+iy) - \varphi_1(\mu+iy)\|_{p_\mu}^{p_\mu} \leqslant M_\mu^{p_\mu} = (2 + K_{p_\mu} \cdot \|B\|_p^2)^{p_\mu}$$

for all $-\infty < y < +\infty$ and $\mu = 0, 1$.

Hence by Proposition 1.2 we obtain for $\theta = p_1(p - p_0)/p(p_1 - p_0)$

$$\begin{aligned}(\|A+B\|_p^p + \|A-B\|_p^p)^{1/p} &= \big(2^{-1}\|\varphi_0(\theta) + \varphi_1(\theta)\|_p^p + 2^{-1}\|\varphi_0(\theta) - \varphi_1(\theta)\|_p^p\big)^{1/p} \\ &= \Big(\int_0^1 \|\varphi_0(\theta) w_0(t) + \varphi_1(\theta) w_1(t)\|_p^p \, dt\Big)^{1/p} \\ &\leqslant M_0^{1-\theta} M_1^\theta \leqslant (2 + K_p \|B\|_p^2)^{1/p}\end{aligned}$$

where $K_p = \max(K_{p_0}, K_{p_1})$. Thus

$$\|A+B\|_p^p + \|A-B\|_p^p \leqslant 2 + K_p \|B\|_p^2$$

i.e. the inequality (2.8).

The estimations for $1 < p < 2$ follow from the case of $p \geqslant 2$ by the duality beetwen the moduli of convexity and smoothness due to Lindenstrauss [12].

§ 3. The Khinchin inequality for $S_p (1 \leqslant p < \infty)$. We say that a Banach space X has a subquadratic Rademacher average (cf. [2] Remark 1 after Corollary 4.9), if there exists a constant C such that for all finite sequences (x_j) in X

$$\int_0^1 \Big\| \sum_{j=1}^{\infty} x_j r_j(t) \Big\| dt \leqslant C \Big(\sum_{j=1}^{\infty} \|x_j\|^2 \Big)^{1/2} \tag{3.1}$$

and X has a superquadratic Rademacher average if there exists a constant C' such that for all finite sequences (x_j) in X

$$\int_0^1 \Big\| \sum_{j=1}^{\infty} x_j r_j(t) \Big\| dt \geqslant C' \Big(\sum_{j=1}^{\infty} \|x_j\|^2 \Big)^{1/2}. \tag{3.2}$$

Remark. It follows from the Kahane theorem [8] that X has a subquadratic (resp. superquadratic) Rademacher average if for every $1 \leqslant q < \infty$ there exists a constant C_q (resp. C'_q) such that for all finite sequences (x_j) in X

$$\Big(\int_0^1 \Big\| \sum_{j=1}^{\infty} x_j r_j(t) \Big\|^q dt \Big)^{1/q} \leqslant C_q \Big(\sum_{j=1}^{\infty} \|x_j\|^2 \Big)^{1/2} \tag{3.3}$$

(resp.

$$\Big(\int_0^1 \Big\| \sum_{j=1}^{\infty} x_j r_j(t) \Big\|^q dt \Big)^{1/q} \geqslant C'_q \Big(\sum_{j=1}^{\infty} \|x_j\|^2 \Big)^{1/2}). \tag{3.4}$$

THEOREM 3.1. (i) *If $1 \leqslant p \leqslant 2$ then S_p has a superquadratic Rademacher average.*

(ii) *If $p \geqslant 2$ then S_p has a subquadratic Rademacher average.*

Proof. (i) Let $1 \leqslant p \leqslant 2$. Let $A_0, A_1, \ldots, A_{N-1}$ be arbitrary operators in S_p. We shall prove that

$$\Big(\int_0^1 \Big\| \sum_{m=0}^{N-1} A_m r_m(t) \Big\|_p^p dt \Big)^{1/p} \geqslant (2\sqrt{e})^{-1} \Big(\sum_{m=0}^{N-1} \|A_m\|_p^2 \Big)^{1/2}. \tag{3.5}$$

First let us observe that it is enough to prove that

$$\Big(\int_0^1 \Big\| \sum_{m=0}^{N-1} A_m r_m(t) \Big\|_p^p dt \Big)^{1/p} \geqslant (\sqrt{e})^{-1} \Big(\sum_{m=0}^{N-1} \|A_m\|_p^2 \Big)^{1/2} \tag{3.6}$$

for arbitrary self-adjoint operators $A_0, A_1, \ldots, A_{N-1}$ in S_p.

Indeed, if $A_0, A_1, \ldots, A_{N-1}$ are not self-adjoint then put

$$A'_m = \operatorname{Re} A_m \quad \text{and} \quad A''_m = \operatorname{Im} A_m \quad \text{for } m = 0, \ldots, N-1.$$

It follows from (3.6) and from the fact that $\|\operatorname{Re} C\|_p \leqslant \|C\|_p$ and $\|\operatorname{Im} C\|_p \leqslant \|C\|_p$ for every operator C in S_p, that

$$\begin{aligned}
(2\sqrt{e})^{-1} \Big(\sum_{m=0}^{N-1} \|A_m\|_p^2 \Big)^{1/2} &\leqslant (2\sqrt{e})^{-1} \Big(\sum_{m=0}^{N-1} (\|A'_m\|_p + \|A''_m\|_p)^2 \Big)^{1/2} \\
&\leqslant (2\sqrt{e})^{-1} \Big(\sum_{m=0}^{N-1} \|A'_m\|_p^2 \Big)^{1/2} + (2\sqrt{e})^{-1} \Big(\sum_{m=0}^{N-1} \|A''_m\|_p^2 \Big)^{1/2} \\
&\leqslant 2^{-1} \Big(\int_0^1 \Big\| \sum_{m=0}^{N-1} A'_m r_m(t) \Big\|_p^p dt \Big)^{1/p} + 2^{-1} \Big(\int_0^1 \Big\| \sum_{m=0}^{N-1} A''_m r_m(t) \Big\|_p^p dt \Big)^{1/p} \\
&\leqslant \Big(\int_0^1 \Big\| \sum_{m=0}^{N-1} A_m r_m(t) \Big\|_p^p dt \Big)^{1/p}
\end{aligned}$$

and we obtain (3.5).

By the homogenity of (3.6) we can assume without loss of generality that $\sum_{m=0}^{N-1} \|A_m\|_p^2 = 1$.

Since the dual of S_p can be identified with S_{p^*} where $p^* = p/(p-1)$ (with $L(H)$ for $p = 1$)(cf. [4]), there exist operators $B_0, \ldots, B_{N-1}$ in S_{p^*} with

$$\sum_{m=0}^{N-1} \|B_m\|_{p^*}^2 = 1 \tag{3.7}$$

and

$$\sum_{m=0}^{N-1} \operatorname{tr}(B_m A_m) = \Big(\sum_{m=0}^{N-1} \|A_m\|_p^2 \Big)^{1/2} = 1. \tag{3.8}$$

Moreover, since $A_0, \ldots, A_{N-1}$ are self-adjoint, we can choose $B_0, \ldots, B_{N-1}$ also to be self-adjoint.

Put

$$\mathcal{N} = \{1 \leqslant n \leqslant 2^N - 1 \colon n = 2^{m_1} + 2^{m_2} + 2^{m_3} + \ldots + 2^{m_{k_n}}, \ k_n \text{ is an odd number}\}.$$

For every $0 \leqslant t \leqslant 1$ let us consider the operator $\Phi(t)$ in $L(H)$ defined by

$$\Phi(t) = \sum_{n \in \mathcal{N}} B_{m_1} B_{m_2} \ldots B_{m_{k_n}} w_n(t) \tag{3.9}$$

where $n = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_{k_n}}$ with $0 \leqslant m_1 < \ldots < m_{k_n} \leqslant N-1$ is the binary expansion of n.

Now our assertion is an easy consequence of the following inequality

$$\Big(\int_0^1 \|\Phi(t)\|_q^q dt \Big)^{1/q} \leqslant \exp\Big[2^{-1} \sum_{m=0}^{N-1} \|B_m\|_q^2 \Big] \quad \text{for } 2 \leqslant q \leqslant \infty. \tag{3.10}$$

Indeed, let us observe that

$$\int_0^1 \operatorname{tr}[\Phi(t) A_m r_m(t)]\,dt = \operatorname{tr} B_m A_m \qquad \text{for } m = 0, \ldots, N-1.$$

Thus, by (3.8),

$$\Big|\int_0^1 \operatorname{tr}\Big[\Phi(t)\Big(\sum_{m=0}^{N-1} A_m r_m(t)\Big)\Big]dt\Big| = \Big|\sum_{m=0}^{N-1} \operatorname{tr} B_m A_m\Big| = 1 \tag{3.11}$$

On the other hand by (3.10) and (3.7) we have

$$\int_0^1 \operatorname{tr}\Big[\Phi(t)\Big(\sum_{m=0}^{N-1} A_m r_m(t)\Big)\Big]dt \leqslant \Big(\int_0^1 \|\Phi(t)\|_{p^*}\Big\|\sum_{m=0}^{N-1} A_m r_m(t)\Big\|_p dt$$

$$\leqslant \Big(\int_0^1 \|\Phi(t)\|_{p^*}^{p^*} dt\Big)^{1/p^*}\Big(\int_0^1 \Big\|\sum_{m=0}^{N-1} A_m r_m(t)\Big\|_p^p dt\Big)^{1/p}$$

$$\leqslant \sqrt{e}\Big(\int_0^1 \Big\|\sum_{m=0}^{N-1} A_m r_m(t)\Big\|_p^p dt\Big)^{1/p}.$$

Combining the above inequality with (3.11) we get

$$\Big(\int_0^1 \Big\|\sum_{m=0}^{N-1} A_m r_m(t)\Big\|_p^p dt\Big)^{1/p} \geqslant e^{-1}.$$

Thus we have only to prove (3.10).

Let us consider arbitrary self-adjoint operators $C_0, \ldots, C_{N-1}$ in $K(H)$ and for the number $n \in \mathcal{N}$ with $n = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_{k_n}}$ denote by D_n the composition $D_n = C_{m_1} \ldots C_{m_{k_n}}$. The inequality (3.10) follows by interpolation from the following two inequalities

$$\Big(\int_0^1 \Big\|\sum_{n\in\mathcal{N}} D_n w_n(t)\Big\|_2^2 dt\Big)^{1/2} \leqslant \exp\Big(2^{-1}\sum_{m=0}^{N-1} \|C_m\|_2^2\Big), \tag{3.12}$$

$$\sup_{0\leqslant t\leqslant 1} \Big\|\sum_{n\in\mathcal{N}} D_n w_n(t)\Big\|_\infty \leqslant \exp\Big(2^{-1}\sum_{m=0}^{N-1} \|C_m\|_\infty^2\Big). \tag{3.13}$$

Indeed, put $q_0 = 2$, $q_1 = \infty$. Let us define the strip $\mathcal{D}$ as in Proposition 1.2, the function $r\colon \mathcal{D} \to \langle 2, \infty\rangle$ by $r(z) = [\operatorname{Re} 2(1-z)]^{-1}$ and the function $q\colon \mathcal{D} \to C$ by $q(z) = [2(1-z)]^{-1}$. Let us observe that

$$[r(z)]^{-1} = \operatorname{Re}\,[q(z)]^{-1}.$$

For every $m = 0, \ldots, N-1$ let $(b_{mj})_{j=1}^\infty$ and $(b'_{mj})_{j=1}^\infty$ be the orthonormal systems in H such that

$$B_m b_{mj} = \beta_{mj} b'_{mj} \qquad \text{for } j = 1, 2, \ldots$$

where $\beta_{mj} = s_j(B_m)$. (Such systems exist — cf. [4].)

Let us define the functions $\varphi_n\colon \mathcal{D} \to K(H)$ $(0 \leqslant n \leqslant 2^N - 1)$ by

$$\varphi_n(z) = B_{m_1}(z) B_{m_2}(z) \ldots B_{m_{k_n}}(z)$$

for $n \in \mathcal{N}$ $(n = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_{k_n}}$ with $0 \leqslant m_1 < \ldots < m_{k_n} \leqslant N-1)$

$$\varphi_n(z) = 0 \qquad \text{for } n \notin \mathcal{N}$$

where $B_m(z)$ is a operator defined by

$$B_m(z) b_{mj} = \beta_{mj}^{q/q(z)} \|B_m\|_q^{1-q/q(z)} b'_{mj}$$

for $m = 0, \ldots, N-1$, $j = 1, 2, \ldots$

It is clear that each function φ_n $(n = 0, \ldots, 2^N - 1)$ is w-continuous in $\mathcal{D}$ and w-analytic in the interior of $\mathcal{D}$. Moreover, for $m = 0, \ldots, N-1$

$$\|B_m(iy)\|_2 = \|B_m\|_q,$$

$$\|B_m(1+iy)\|_\infty = \|B_m\|_q \qquad \text{for } -\infty < y < +\infty.$$

It follows from (3.12), (3.13) and Proposition 1.2 that for $\theta = (q-2)q^{-1}$ we have

$$\Big(\int_0^1 \|\Phi(t)\|_q^q dt\Big)^{1/q} = \Big(\int_0^1 \Big\|\sum_{n\in\mathcal{N}} B_{m_1} \ldots B_{m_{k_n}} w_n(t)\Big\|_q^q dt\Big)^{1/q}$$

$$= \Big(\int_0^1 \Big\|\sum_{n=0}^{2^N-1} \varphi_n(\theta) w_n(t)\Big\|_q^q dt\Big)^{1/q} \leqslant \exp\Big[2^{-1}\sum_{m=0}^{N-1} \|B_m\|_q^2\Big].$$

Now, to complete the proof in the case (i), we shall show (3.12) and (3.13).

To prove (3.12) let us observe that from the orthogonality of Walsh system we obtain

$$\int_0^1 \Big\|\sum_{n\in\mathcal{N}} D_n w_n(t)\Big\|_2^2 dt = \int_0^1 \operatorname{tr}\Big(\sum_{n\in\mathcal{N}} D_n^* w_n(t)\Big)\Big(\sum_{n\in\mathcal{N}} D_n w_n(t)\Big)dt$$

$$= \sum_{n\in\mathcal{N}} \operatorname{tr}(D_n^* D_n).$$

Then, by Lemma 1.1 and the definition of D_n we have

$$\sum_{n\in\mathcal{N}} \operatorname{tr}(D_n^* D_n) \leqslant \sum_{n\in\mathcal{N}} \sum_{j=1}^\infty s_j^2(C_{m_1}) \cdot \ldots \cdot s_j^2(C_{m_{k_n}})$$

$$\leqslant \sum_{n\in\mathcal{N}} \Big(\sum_{j=1}^\infty s_j^2(C_{m_1})\Big) \cdot \ldots \cdot \Big(\sum_{j=1}^\infty s_j^2(C_{m_{k_n}})\Big) = \sum_{n\in\mathcal{N}} \|C_{m_1}\|_2^2 \cdot \ldots \cdot \|C_{m_{k_n}}\|_2^2.$$

From the definition of the set $\mathcal{N}$ we get

$$\sum_{n\in\mathcal{N}} \|C_{m_1}\|_2^2 \ldots \|C_{m_{k_n}}\|_2^2 \leqslant \prod_{m=0}^{N-1} (1 + \|C_m\|_2^2) \leqslant \exp\Big[\sum_{m=0}^{N-1} \|C_m\|_2^2\Big].$$

Combining the three inequalities above we obtain (3.12).

To prove (3.13) let us observe that for every $0 \leqslant t \leqslant 1$

$$\Phi(t) = \operatorname{Im}\Psi(t)$$

where

$$\Psi(t) = \big(I + iC_0 r_0(t)\big) \ldots \big(I + iC_{N-1} r_{N-1}(t)\big) \quad \text{for } 0 \leqslant t \leqslant 1$$

(I denote the identity operator).

Hence

$$\|\Phi(t)\|_\infty \leqslant \|\Psi(t)\|_\infty \quad \text{for } 0 \leqslant t \leqslant 1$$

and

$$\|\Psi(t)\|_\infty \leqslant \prod_{m=0}^{N-1} \|I + iC_m r_m(t)\|_\infty = \prod_{m=0}^{N-1} (1 + \|C_m\|_2^2)^{1/2} \leqslant \exp\Big[2^{-1} \sum_{m=0}^{N-1} \|C_m\|_\infty^2\Big].$$

Thus (3.13) holds and this completes the proof of (i).

(ii) Let $p = 2k$ $(k = 1, 2, \ldots)$. Let $A_0, A_1, \ldots, A_{N-1}$ be arbitrary operators in S_p. Let us observe that for $0 \leqslant t \leqslant 1$

$$\Big[\Big(\sum_{m=0}^{N-1} A_m^* r_m(t)\Big)\Big(\sum_{m=0}^{N-1} A_m r_m(t)\Big)\Big]^k = \sum\nolimits_{(+)} C(t)$$

where $\sum_{(+)}$ is extended over all the operators

$$C(t) = C \prod_{m=0}^{N-1} (r_m(t))^{b_m(C)} \quad \text{where } C = \prod_{\nu=1}^{2k} C_\nu$$

satisfying the condition

(+) $C_\nu = A_m^*$ for some $0 \leqslant m \leqslant N-1$ and for odd ν's, $C_\nu = A_m$ for some $0 \leqslant m \leqslant N-1$ and for even ν's. $b_m(C)$ denote the number of the indices ν such that C_ν equals either A_m^* or A_m.

Clearly, $b_m(C) \geqslant 0$ and $\sum_{m=0}^{N-1} b_m(C) = 2k$.

It follows from the additivity of the trace that

$$\int_0^1 \operatorname{tr}\Big[\Big(\sum_{m=0}^{N-1} A_m^* r_m(t)\Big)\Big(\sum_{m=0}^{N-1} A_m r_m(t)\Big)\Big]^k dt = \int_0^1 \operatorname{tr} \sum\nolimits_{(+)} C(t)\, dt$$
$$= \sum\nolimits_{(+)} \operatorname{tr}\Big(\prod_{\nu=1}^{2k} C_\nu\Big) \cdot \int_0^1 \prod_{m=0}^{N-1} (r_m(t))^{b_m(C)}\, dt.$$

Since

$$\int_0^1 \prod_{m=0}^{N-1} (r_m(t))^{b_m(C)}\, dt = \begin{cases} 1 & \text{if all } b_m(C) \text{ are even,} \\ 0 & \text{in the other case,} \end{cases}$$

we have

$$\int_0^1 \operatorname{tr}\Big[\Big(\sum_{m=0}^{N-1} A_m^* r_m(t)\Big)\Big(\sum_{m=0}^{N-1} A_m r_m(t)\Big)\Big]^k dt = \sum\nolimits_{(++)} \operatorname{tr} C \tag{3.14}$$

where $\sum_{(++)}$ is extended over all the operators $C = \prod_{\nu=1}^{2k} C_\nu$ satisfying (+) and the following condition

(++) $b_m(C)$ are even for $m = 0, \ldots, N-1$.

Clearly, for every operator C we have

$$|\operatorname{tr} C| \leqslant \sum_{j=1}^{\infty} s_j(C).$$

Using Lemma 1.1 and the fact that s-numbers of the operator and its adjoint are equal, we get

$$\sum\nolimits_{(++)} |\operatorname{tr} C| \leqslant \sum\nolimits_{(++)} \sum_{j=1}^{\infty} s_j\Big(\prod_{\nu=1}^{2k} C_\nu\Big)$$
$$\leqslant \sum\nolimits_{(++)} \sum_{j=1}^{\infty} \prod_{\nu=1}^{2k} s_j(C_\nu) = \sum_{j=1}^{\infty} \sum\nolimits' \frac{(2k)!}{(2\beta_0)! \ldots (2\beta_{N-1})!} \prod_{m=0}^{N-1} s_j(A_m)^{2\beta_m}$$

where $\sum'$ is extended over all sequences (β_m) of non-negative integers with $\sum_{m=0}^{N-1} \beta_m = k$.

We have ([20], [19])

$$\frac{(2k)!}{(2\beta_0)! \ldots (2\beta_{N-1})!} \leqslant M_{2k}^{2k} \frac{k!}{\beta_0! \ldots \beta_{N-1}!}$$

where

$$M_{2k} = \Big[\frac{(2k)!}{2^k k!}\Big]^{1/2k}.$$

Thus

$$\sum\nolimits_{(++)} |\operatorname{tr} C| \leqslant M_{2k}^{2k} \sum_{j=1}^{\infty} \sum\nolimits' \frac{k!}{\beta_1! \ldots \beta_{N-1}!} \prod_{m=0}^{N-1} (s_j(A_m))^{2\beta_m}$$
$$= M_{2k}^{2k} \sum_{j=1}^{\infty} \Big(\sum_{m=0}^{N-1} s_j^2(A_m)\Big)^k.$$

Hence, by (3.14)

$$\Big(\int_0^1 \operatorname{tr}\Big[\Big(\sum_{m=0}^{N-1} A_m^* r_m(t)\Big)\Big(\sum_{m=0}^{N-1} A_m r_m(t)\Big)\Big]^k dt\Big)^{1/2k}$$
$$\leqslant \Big(\sum\nolimits_{(++)} |\operatorname{tr} C|\Big)^{1/2k} \leqslant M_{2k}\Big[\sum_{j=1}^{\infty} \Big(\sum_{m=0}^{N-1} s_j^2(A_m)\Big)^k\Big]^{1/2k}.$$

By the triangle inequality in l_k we obtain

$$\Big[\sum_{j=1}^{\infty}\Big(\sum_{m=0}^{N-1} s_j^2(A_m)\Big)^k\Big]^{1/k} \leqslant \sum_{m=0}^{N-1}\Big(\sum_{j=1}^{\infty} s_j^{2k}(A_m)\Big)^{1/k} = \sum_{m=0}^{N-1}\|A_m\|_p^2$$

and consequently

$$(3.15)\qquad \Big(\int_0^1\Big\|\sum_{m=0}^{N-1} A_m r_m(t)\Big\|_{2k}^{2k} dt\Big)^{1/2k}$$

$$= \Big(\int_0^1 \operatorname{tr}\Big[\Big(\sum_{m=0}^{N-1} A_m^* r_m(t)\Big)\Big(\sum_{m=0}^{N-1} A_m r_m(t)\Big)\Big]^k dt\Big)^{1/2k} \leqslant M_{2k}\Big(\sum_{m=0}^{N-1}\|A_m\|_p^2\Big)^{1/2}.$$

This prove that if $p = 2k$ $(k = 1, 2, \ldots)$ then S_p has a subquadratic Rademacher average.

If $p > 2$, $p \neq 2k$ $(k = 1, 2, \ldots)$, then the desired result follows from the case of even numbers by interpolation. To this end fix $A_0, \ldots, A_{N-1}$ in S_p. Put $k = \left[\frac{p}{2}\right]$ and $p_0 = 2k$, $p_1 = 2k+2$. Define the strip $\mathscr{D}$ and the function $r\colon \mathscr{D} \to \langle p_0, p_1\rangle$ as in Proposition 1.2. Next define a function $p\colon \mathscr{D} \to C$ by

$$p(z) = [(1-z)p_0^{-1} + zp_1^{-1}]^{-1}$$

and observe that $[r(z)]^{-1} = \operatorname{Re}[p(z)]^{-1}$.

For every $m = 0, \ldots, N-1$ let $(a_{mj})_{j=1}^{\infty}$ and $(a'_{mj})_{j=1}^{\infty}$ be the orthonormal systems in H such that

$$A_m a_{mj} = \alpha_{mj} a'_{mj}$$

where $\alpha_{mj} = s_j(A_m)$ $\quad (m = 0, \ldots, N-1,\ j = 1, 2, \ldots)$.

Define the functions $\varphi_n\colon \mathscr{D} \to K(H)$ $(n = 0, \ldots, 2^N-1)$ by

$$\varphi_{2^m}(z) a_{mj} = \alpha_{mj}^{p/p(z)} \|A_m\|_p^{1-p/p(z)} a'_{mj}$$

for $m = 0, \ldots, N-1$, $j = 1, 2, \ldots$

$$\varphi_n(z) = 0 \quad \text{for } n \neq 2^m \quad (m = 0, \ldots, N-1).$$

The functions φ_n $(n = 0, \ldots, 2^N-1)$ are w-continuous in $\mathscr{D}$ and w-analytic in the interior of $\mathscr{D}$. It is easy to check that

$$\|\varphi_{2^m}(\mu+iy)\|_{p_\mu} = \|A_m\|_p \quad (m = 0, \ldots, N-1)$$

for $-\infty < y < +\infty$ and $\mu = 0, 1$.

It follows from (3.15) that

$$\Big(\int_0^1\Big\|\sum_{n=0}^{2^N-1}\varphi_n(\mu+iy)w_n(t)\Big\|_{p_\mu}^{p_\mu} dt\Big)^{1/p_\mu} = \Big(\int_0^1\Big\|\sum_{m=0}^{N-1}\varphi_{2^m}(\mu+iy)r_m(t)\Big\|_{p_\mu}^{p_\mu} dt\Big)^{1/p_\mu}$$

$$\leqslant M_{p_\mu}\Big(\sum_{m=0}^{N-1}\|A_m\|_p^2\Big)^{1/2}.$$

Thus, from Proposition 1.2, we have for $\theta = p_1(p-p_0)/p(p_1-p_0)$

$$\Big(\int_0^1\Big\|\sum_{m=0}^{N-1} A_m r_m(t)\Big\|_p^p dt\Big)^{1/p} = \Big(\int_0^1\Big\|\sum_{n=0}^{2^N-1}\varphi_n(\theta)w_n(t)\Big\|_{r(\theta)}^{r(\theta)} dt\Big)^{1/r(\theta)}$$

$$\leqslant M_{p_0}^{1-\theta} M_{p_1}^{\theta}\Big(\sum_{m=0}^{N-1}\|A_m\|_p^2\Big)^{1/2}.$$

This completes the proof of (ii).

Recall that a Banach space X is said to be predual of a Banach space Y, if Y is dual of X. It is well known that if C^*-algebra Y has a predual X, then Y has an identity (for the definition of a C^*-algebra and its properties see [18]). Let us observe that S_1 is a predual of $L(H)$ which is C^*-algebra.

PROPOSITION 3.2. *Let X be a Banach space predual of a C^*-algebra Y. Then X has a superquadratic Rademacher average.*

Proof. This proof is analogous to the proof of Theorem 3.1 in the case (i).

Let $A_0, \ldots, A_{N-1}$ be arbitrary elements in X. Without loss of generality we can assume that $\sum_{m=0}^{N-1}\|A_m\|^2 = 1$. As in the Theorem 3.1 (i) choose the functionals $B_0, \ldots, B_{N-1}$ in Y such that

$$(3.16)\qquad \sum_{m=0}^{N-1}\|B_m\|^2 = 1,$$

$$(3.17)\qquad \sum_{m=0}^{N-1} B_m(A_m) = \Big(\sum_{m=0}^{N-1}\|A_m\|^2\Big)^{1/2} = 1.$$

Denote $B'_m = \operatorname{Re} B_m$, $B''_m = \operatorname{Im} B_m$ and for every $0 \leqslant t \leqslant 1$ define functionals $\Phi'(t)$, $\Phi''(t)$ in Y by substituting in the definition (3.9) B_m by B'_m and B''_m respectively. Define $\Phi(t) = \Phi'(t) + i\Phi''(t)$ and observe that

$$(3.18)\qquad \sup_{0\leqslant t\leqslant 1}\|\Phi(t)\| \leqslant 2\sqrt{e}.$$

Indeed, it follows from the definition of Φ', Φ'' and Φ that

$$\|\Phi(t)\| \leqslant \|\Phi'(t)\| + \|\Phi''(t)\| \leqslant \exp\Big(2^{-1}\sum_{m=0}^{N-1}\|B'_m\|^2\Big) + \exp\Big(2^{-1}\sum_{m=0}^{N-1}\|B''_m\|^2\Big).$$

Since for every B in C^*-algebra Y we have

$$\|\operatorname{Re} B\| \leqslant \|B\| \quad\text{and}\quad \|\operatorname{Im} B\| \leqslant \|B\|,$$

then by (3.16) we obtain

$$\exp\Big(2^{-1}\sum_{m=0}^{N-1}\|B'_m\|^2\Big) + \exp\Big(2^{-1}\sum_{m=0}^{N-1}\|B''_m\|^2\Big) \leqslant 2\sqrt{e}.$$

Thus (3.18).

Let us observe that

$$\int_0^1 \Phi(t)\big(A_m r_m(t)\big)\,dt = B_m(A_m) \quad \text{for } m = 0, \ldots, N-1$$

and by (3.17) we get

$$(3.19) \qquad \Big|\int_0^1 \Phi(t)\Big(\sum_{m=0}^{N-1} A_m r_m(t)\Big)\,dt\Big| = \Big|\sum_{m=0}^{N-1} B_m(A_m)\Big| = 1.$$

As in the proof of Theorem 3.1 (i) from (3.18) and (3.19) follows that

$$\int_0^1 \Big\|\sum_{m=0}^{N-1} A_m r_m(t)\Big\|\,dt \geqslant (2\sqrt{e})^{-1}.$$

This completes the proof.

Remark 1. Let us observe that the constant M_{2k} is the best constant in (3.15). This follows from the result of Stechkin [19] which asserts that M_{2k} is the best constant in the case of "real" (scalar) Khinchin inequality for even numbers. The fact that $M_4 \leqslant \sqrt[4]{3}$ in the complex case was observed by Pietsch [17].

Remark 2. Our proof of Proposition 3.2 in the case of real scalars was known before to the specialists in harmonic analysis: Kahane, Katznelson, and Drury. To our best knowledge it does not appear in the literature. It is interesting that in the real case for $p = 1$ it gives better constant, namely $\sqrt{e}$, than the "classical" proof which gives $\sqrt{3}$. In the complex case this proof gives the constant $2\sqrt{e}$ while Pietsch's proof [17] gives $\sqrt{3}$.

§ 4. Applications. In the sequel by $\Pi_p(X, Y)$ we denote the space of p-absolutely summing operators from X into Y (see [13]).

THEOREM 4.1. *For every* $1 < p \leqslant 2$

$$L(c_0, S_p) = \Pi_2(c_0, S_p).$$

Proof. This follows from Theorem 3.1 (i) and Corollary 4.4 of [2].

THEOREM 4.2.

$$L(c_0, S_1) = \Pi_2(c_0, S_1).$$

More generally, if X is a predual of a C^*-algebra then

$$L(c_0, X) = \Pi_2(c_0, X).$$

Proof. This theorem is an immediate consequence of the following general fact due to B. Maurey:

If X has a superquadratic Rademacher average, then

$$L(c_0, X) = \Pi_2(c_0, X).$$

The proof of this fact was communicated to the author by S. Kwapień. For the sake of completness we sketch it here.

Recall that an operator $T\colon X \to Y$ is said to be factored through a Banach space Z if there are operators $\alpha\colon X \to Z$ and $\beta\colon Z \to Y$ such that $T = \beta\alpha$.

The proof bases on the following remarks. (Here E denotes a Banach space.)

(i) $L(c_0, E) = \Pi_2(c_0, E)$ if and only if every operator $T\colon c_0 \to E$ can be factored through a Hilbert space.

(ii) Let $1 < p < 2$. If there is an operator $S\colon E^* \to l_1$ which cannot be factored through l_p then for every natural number n there are $x_1, x_2, \ldots, x_n$ in E with $\|x_1\| = \ldots = \|x_n\| = 1$ such that

$$\Big\|\sum_{i=1}^{n} c_i x_i\Big\| \leqslant 2\Big(\sum_{i=1}^{n} |c_i|^{p^*}\Big)^{1/p^*} \quad \text{for all sequences } (c_i).$$

(iii) Let $1 < p < 2$. If E has a superquadratic Rademacher average then every operator $S\colon E^* \to l_1$ can be factored through l_p.

(iv) If E has a superquadratic Rademacher average and the Banach space F has a subquadratic Rademacher average then every operator $S\colon F \to E$ can be factored through a Hilbert space.

For the proof of (i) see [2], (ii) follows from the fundamental theorem of B. Maurey [15]. (iii) follows from (ii) and the observation that if E has a superquadratic Rademacher average then for every unconditionally convergent sequence (x_i) in E we have $\sum \|x_i\|^2 < \infty$. (iv) is a generalization of a result of S. Kwapień [10], Proposition 1.3 (cf. also [11]). The proof is analogous to that of [10], Proposition 3.1.

COROLLARY 4.3. 1° *Let $1 \leqslant p \leqslant 2$. If a sequence (A_n) in S_p is unconditionally convergent then* $\sum_{n=1}^{\infty} \|A_n\|_p^2 < \infty$.

2° *Let $2 \leqslant p < \infty$. If a sequence (A_n) in S_p is unconditionally convergent then* $\sum_{n=1}^{\infty} \|A_n\|_p^p < \infty$.

Proof. 1° This follows immediately from Theorem 3.1 (i). 2° This follows from Theorem 2.1 and the result of Kadec [7].

COROLLARY 4.4. *For every $2 < p < \infty$ and every $\varepsilon > 0$*

$$L(c_0, S_p) = \Pi_{p+\varepsilon}(c_0, S_p).$$

Moreover, there exists $u \in L(c_0, S_p)$ such that $u \notin \Pi_p(c_0, S_p)$.

Proof. The first assertion follows from Theorem 4.3.2° and Maurey's results in [15]. The second assertion follows from the fact that S_p contains a subspace isometric with l_p (see also [9]).

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Exponential integrability of certain singular integral transforms

by

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Abstract. It is shown that singular integral operators with odd kernels map bounded functions with support of finite measure to locally exponentially integrable functions. In particular it is shown that the periodic Riesz transform of a function of supremum norm one is exponentially integrable of order a for $a < \pi/2$ and $\pi/2$ is the best possible constant. This extends and gives a new proof of the known result for the periodic Hilbert transform.

The following well-known result concerning exponential integrability of conjugate harmonic functions can be proved relatively easily using methods from complex function theory (e.g. [3], p. 254).

THEOREM. *Let f be a bounded measurable function on $\{e^{i\theta}; -\pi \leqslant \Theta < \pi\}$. Let Hf be the periodic Hilbert transform of f. If $\|f\|_\infty \leqslant 1$ then*

$$\int_0^{2\pi} \exp a|Hf(e^{i\theta})|\,d\theta < \infty \quad \textit{for} \quad 0 \leqslant a < \frac{\pi}{2}.$$

Examples show that the constant $\pi/2$ in this result is the best possible.

In this paper we will show that a result analogous to the above theorem holds for linear transformations defined on $L^\infty(E^n)$ by singular integral operators with odd kernels. The proof is a straightforward application of results of O'Neil and Weiss [1] on rearrangements of functions.

It will follow that the above theorem is true, with the constant $\pi/2$, for periodic Riesz transforms and that the constant is again the best possible. (In particular, this gives a new, and strictly real variable, proof of the theorem stated above.)

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For a measurable function f defined on the non-atomic measure space (M, μ), define f^*, the *non-increasing rearrangement* of f, to be the

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