Notes on orthogonal series I.

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1. We prove first a theorem which is a partial generalisation of a theorem of Kolmogoroff [1]).

Suppose that $\sum_{n=1}^{\infty} a_n \varphi_n(t)$ is an orthogonal series, $\sum_{n=1}^{\infty} a_n^2 < +\infty$, the functions $\varphi_n(t)$ orthogonal and normal in $\langle 0, 1\rangle$. Suppose further that the series is almost everywhere summable by a linear method $T(b_{n,k})$; then the theorem is as follows.

Theorem. *There exists a sequence of indices $\{n_i\}$, dependent only of the linear method T, such that the sequence $s_{n_i}(t) = \sum_{k=1}^{n_i} a_k \varphi_k(t)$ is almost ewerywhere convergent.*

In Kolmogoroff's theorem we assume that the series is summable $(C, 1)$ almost ewerywhere and assert, that the sequence $s_{2^i}(t)$ is convergent almost everywhere; our proof gives in that case the sequence $n_i = i^i$.

To prove the theorem we denote by $\sigma_n(t)$ the expression $\sum_{k=1}^{\infty} s_k(t)\, b_{n,k}$; by hypothesis $\sigma_n(t)$ is almost everywhere convergent. The method T being linear and regular we have:

1) $$\lim_{n\to\infty} \sum_{k=1}^{\infty} b_{n,k} = 1,$$

2) $$\lim_{n\to\infty} b_{n,k} = 0 \qquad (k = 1, 2, \ldots),$$

3) $$\sum_{k=1}^{\infty} |b_{n,k}| \leqslant M.$$

1) Fund. Math. V (1924) p. 96—97.

Denote further by I_n the integral $\int_0^1 [s_n(t) - \sigma_n(t)]^2\, dt$, then we have

$$s_n(t) - \sigma_n(t) = \sum_{k=1}^{n} a_k \varphi_k(t)\,[1 - b_{n,k} - b_{n,k+1} - \ldots]\, dt -$$
$$- \sum_{k=n+1}^{\infty} a_k \varphi_k(t)\,[b_{n,k} + b_{n,k+1} + \ldots]\, dt$$

and

$$I_n = \sum_{k=1}^{n} a_k^2\, [1 - b_{n,k} - b_{n,k-1} \ldots]^2 + \sum_{k=n+1}^{\infty} a_n^2\, [b_{n,k} + b_{n,k+1} + \ldots]^2.$$

We write

$$1 - b_{n,k} - b_{n,k+1} - \ldots = b_{n,1} + b_{n,2} + \ldots + b_{n,k-1} + \varepsilon_n,$$

where $\varepsilon_n \to 0$ with $1/n$.

Let now the sequence $\{n_i\}$ be defined as follows:

a) $$|\varepsilon_{n_i}| \leqslant \frac{1}{i},$$

b) $$\Big|\sum_{j=1}^{l} b_{n_i, j}\Big| < \frac{1}{i} \text{ for } l \leqslant n_{i-1}, \quad n_0 = 0,$$

c) $$\Big|\sum_{j=1}^{l} b_{n_i, j}\Big| > 1 - \frac{1}{i} \text{ for } l > n_{i+1}.$$

The inequalites a) and c) are possible on account of the property 1) of the matrix $T(b_{n,k})$ and the inequality b) by the property 2). Consider now the sum

$$\sum_{i=1}^{\infty} I_{n_i} = \sum_{i=1}^{\infty}\Big[\sum_{k=1}^{n_i} a_k^2\,(b_{n_i,1} + \ldots + b_{n_i,k-1} + \varepsilon_{n_i})^2 +$$
$$+ \sum_{k=n_i+1}^{\infty} a_k^2\,(b_{n_i,k} + \ldots)\Big] =$$
$$= \sum_{k=1}^{\infty} a_k^2\,[(b_{n_j,1} + \ldots + b_{n_j,k-1} + \varepsilon_{n_j})^2 + (b_{n_{j+1},1} + \ldots + b_{n_{j+1}\,k-1} +$$
$$+ \varepsilon_{n_{j+1}})^2 + \ldots] + \sum_{k=n_1+1}^{\infty} a_k^2\,[(b_{n_1,k} + \ldots)^2 + (b_{n_2,k} + \ldots)^2 +$$
$$+ \ldots + (b_{n_l,k} + \ldots)^2],$$

where n_j and n_l are chosen so that

$$n_{j-1} < k \leqslant n_j, \quad n_l < k \leqslant n_{l+1}.$$

We see that the first sum on the right hand is less than

$$\sum_{k=1}^{\infty} a_k^2 \left[(M+1)^2 + \frac{1}{(j+1)^2} + \frac{1}{(j+2)^2} + \dots\right] < C \sum_{k=1}^{\infty} a_n^2$$

by the properties 3) and b). The second one is by the properties a) and c) less than

$$\sum_{k=1}^{\infty} a_k^2 \left[\frac{4}{1^2} + \frac{4}{2^2} + \dots + \frac{4}{l^2} + M\right] \leqslant C_1 \sum_{k=1}^{\infty} a_n^2 .$$

It follows that $\sum_{n=1}^{\infty} I_n$ is finite and hence $s_{n_l} - \sigma_{n_l} \to 0$ almost everywhere, that is, the sequence $s_{n_l}(t)$ is almost everywhere convergent. Thus the theorem is proved.

2. Some theorems on the $(C, 1)$ summability of orthogonal series make assumptions concerning the order of infinity of the Lebesgue function $\varrho_n(t) = \int_0^1 \left| \sum_{k=1}^{n} \varphi_k(t)\, \varphi_k(u) \left(1 - \frac{k-1}{n}\right) \right| du .$

If for example $w(n)$ is a positive non decreasing function of n, $w(n) \to \infty$, and

$$(2.1) \qquad \Delta \frac{1}{w(n)} = O\left[\frac{1}{n\, w(n)}\right], \quad \sum_{k=1}^{\infty} k \left| \Delta^2 \frac{1}{w(k)} \right| < \infty ,$$

then the orthogonal series is almost everywhere summable $(C, 1)$ under the hypothesis $\sum_{n=1}^{\infty} a_n^2\, w^2(n) < \infty$. [2])

But a stronger result is true, namely the assumptions (2.1) are superfluous.

Theorem. *If* 1) $\sum_{n=1}^{\infty} a_n^2\, w^2(n) < \infty$

2) $\varrho_n(t \leqslant w(n)$,

then the series $\sum_{n=1}^{\infty} a_n \varphi_n(t)$ *is almost everywhere summable* $(C, 1)$.

[2]) S. Kaczmarz, Stud. Math. 1 (1929) p. 112.

This theorem gives an extension of an analogous theorem in the case that $w(n) = O(1)$.

We know that the sequence $s_{n_k}(t)$ is almost everywhere convergent, if we take $\{n_k\}$ such that $k \leqslant w^2(n_k) < k+1$ [3]). Let p be any integer satisfying the inequality $n_k < p < n_{k+1}$, then it is sufficient to prove that $\sigma_p - s_{n_k} \to 0$.

We have

$$(2.2) \qquad \sigma_p - s_{n_k} = \sum_{i=1}^{n_k} a_i \varphi_i(t)(i-1)\frac{1}{p} + \sum_{i=n_k+1}^{p} a_i \varphi_i(t) \left[1 - \frac{i-1}{p}\right].$$

Using Abel's transformation in the second sum on the right we obtain

$$\frac{1}{w(n_k)} \sum_{i=n_k+1}^{p} a_i \varphi_i(t)\, w(n_k) \left[1 - \frac{i-1}{p}\right] =$$

$$= \frac{1}{w(n_k)} \left[\sum_{i=n_k+1}^{p} \bar{s}_i \frac{1}{p} - \bar{s}_{n_k} \left[1 - \frac{n_k}{p}\right] + \bar{s}_p \,.\, 0 \right].$$

In the sum written above $\bar{s}_i$ denote the partial sums of the series obtained by multiplying the terms with indices between n_j and n_{j+1} of the series $\sum_{i=1}^{\infty} a_i \varphi_i(t)$ by $w(n_j)$. The sequence $\{s_{n_k}\}$ is convergent, thus $\bar{s}_{n_k} = o[w(n_k)]$ and further

$$\frac{1}{p} \sum_{i=n_k+1}^{p} \bar{s}_i = \bar{\sigma}_p - n_k \bar{\sigma}_{n_k} \frac{1}{p},$$

where $\bar{\sigma}_i$ denotes the $(C, 1)$ sequence of $\{\bar{s}_i\}$. Therefore the second sum in (2.2) is $o\left(\frac{\bar{\sigma}_p - \bar{\sigma}_{n_k}}{w(n_k)}\right)$. But the expression $\frac{\bar{\sigma}_n}{w(n)}$ converges to zero [4]), hence the second sum is convergent to zero almost everywhere.

The same transformation gives us in the first sum

$$\frac{1}{p} \left[\sum_{l=1}^{k} n_l (\bar{s}_{n_l} - \bar{\sigma}_{n_l}) \left(\frac{1}{w(n_l)} - \frac{1}{w(n_{i+1})} \right) + n_k (\bar{s}_{n_k} - \bar{\sigma}_{n_k}) \frac{1}{w(n_{k+1})} \right].$$

[3]) l. c. [2]), lemme 1, p. 91.
[4]) l. c. [2]), th. 17, remarque p. 111.

Here we have as bevore $\bar{s}_{n_k} - \bar{\sigma}_{n_k} = o\,[w(n_k)]$. It remains to prove that the sum is $o(p)$. It is less indeed than

$$C \sum_{i=1}^{k} n_i \,|\, \bar{s}_{n_i} - \bar{\sigma}_{n_i} \,|\, i^{-\frac{3}{2}}$$

on account of $\sqrt{i} < w(n_i)$, $w(n_{i+1}) \leqslant \sqrt{i+1}$.

On the other hand

$$\int_0^1 \frac{|\bar{s}_{n_i} - \bar{\sigma}_{n_i}|}{i^{\frac{3}{2}}} \leqslant \frac{1}{i^{\frac{3}{2}}} \sqrt{\sum_{k=1}^{\infty} a_k^2\, w^2(k)},$$

and therefore the series $\sum_{i=1}^{\infty} |\, \bar{s}_{n_i} - \bar{\sigma}_{n_i} \,|\, i^{-\frac{3}{2}}$ is almost everywhere convergent. From Kroneckers theorem it follows that

$$\sum_{i=1}^{k} n_i \,|\, \bar{s}_{n_i} - \bar{\sigma}_{n_i} \,|\, i^{-\frac{3}{2}} = o(p)$$

which proves the theorem.

The analoguous theorem for (C, k) summability is also true [5].

[5]) l. c. [2]) Th. 23, p. 119.

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