On the compactness of the function-set by the convergence in mean of general type

by

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1. *Introduction.* Let $L_p\,(p \geqslant 1)$ be the function space consisting of all functions $f(x)$ defined and measurable in $(-\infty, \infty)$ and such that

$$\int_{-\infty}^{\infty} |f(x)|^p\, dx < \infty .$$

A set F of functions which are elements of L_p, is called compact, if every subset of F contains at least one sequence which is convergent in mean.

KOLMOGOROFF[1]) has derived necessary and sufficient conditions in order that F is compact under the restrictions that $p > 1$ and elements of F are defined in a mesurable bounded set. TAMARKIN[2]) has found the necessary and sufficient conditions in the case $p > 1$ and the region of definition is a measurable set which may be bounded or not. TULAJKOV[3]) has proved that if $p = 1$, the TAMARKIN's conditions are also necessary and sufficient for the compactness of F. M. RIESZ[4]) has derived the necessary and

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1) A. Kolmogoroff, Über Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel, Göttinger Nachrichten (1931) p. 60—63.

2) J. D. Tamarkin, On the compactness of the space L_p, Bull. Amer. Math. Soc. 38 (1932) p. 79—84.

3) A. Tulajkov, Zur Kompaktheit im Raum L_p für $p = 1$, Göttinger Nachrichten (1933) p. 167—170.

4) M. Riesz, Sur les ensembles compacts de fonctions sommables, Acta Szeged 6 (1933) p. 136—142.

sufficient conditions from the well known theorem of HAUSDORFF[5]) concerning the compactness of a set in abstract metric space.

Let $f(x)$ be defined in $(-\infty, \infty)$ and let

$$(1.1)\quad f_\delta(x) = \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(y)\,dy$$

$$(1.2)\quad f^N(x) = \begin{cases} f(x) & \text{in } -N \leqslant x \leqslant N \\ 0 & \text{elsewhere.} \end{cases}$$

TAMARKIN and TULAJKOV's theorem is the following.

Theorem A. *In order that a set $F \subset L_p (p \geqslant 1)$ be compact, it is necessary and sufficient that there exist a constant $M = M(F)$ and, for given positive ε, two constants $\delta = \delta(\varepsilon, F)$ and $N_0 = N_0(\varepsilon, F)$ depending only on F and ε, F respectively but not on $f(x) \in F$, such that the following conditions are satisfied:*

$$\text{(i)}\quad \int_{-\infty}^{\infty} |f(x)|^p\,dx \leqslant M,$$

$$\text{(ii)}\quad \int_{-\infty}^{\infty} |f(x) - f_h(x)|^p\,dx \leqslant \varepsilon, \quad 0 < h \leqslant \delta,$$

$$\text{(iii)}\quad \int_{-\infty}^{\infty} |f(x) - f^N(x)|^p\,dx \leqslant \varepsilon, \quad N \geqslant N_0$$

for all elements of $f \in F$.

2. Let $M(u)$ be defined in $(-\infty, \infty)$ and such that

(2.1) $M(u)$ is continuous,

(2.2) $M(0) = 0$ and $M(u) > 0$ for $u > 0$,

(2.3) $M(u) = M(|u|)$ for negative u,

(2.4) there exist two positive numbers α, β such that $M(u) > \beta$ holds always for $u > \alpha$,

$$(2.5)\quad \lim_{|u| = 0} \frac{M(u)}{|u|} = 0, \quad \lim_{|u| = \infty} \frac{M(u)}{|u|} = \infty.$$

The function $M(u)$ conditioned as above is called N'-function[6]).

[5]) F. Hausdorff, Grundzüge der Mengenlehre, Leipzig (1930), p. 107.

[6]) Concerning the properties of N'-function, see: Z. W. Birnbaum und W. Orlicz, Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen, Studia Math. 3 (1931) p. 1—67.

Now for every N'-function $M(u)$, we define for $v \geqslant 0$ a function $N(v)$ such that

$$(2.6)\quad N(v) = \operatorname{Max}_{u \geqq 0} [uv - M(u)],$$

and for $v < 0$, $N(v) = N(|v|)$. Thus defined function $N(v)$ is said the complementary function of $M(u)$. Then the complementary function $N(v)$ of an N'-function is a convex N'-function and such that

$$(2.7)\quad uv \leqslant M(u) + N(v)\ ^{7)}.$$

Now we consider a convex N'-function $M(u)$ which is necessarily nondecreasing for $u > 0$. We say that a measurable function defined in $(-\infty, \infty)$ is integrable with respect to $M(u)$ if

$$\int_{-\infty}^{\infty} M[f(x)]\,dx < \infty.$$

Let $\{f_n(x)\}$ be the sequence of functions such that $f_m(x) - f_n(x)$ are integrable with respect to $M(u)$ $(m, n = 0, 1, 2\ldots)$. If $\lim_{m,n=\infty} \int_{-\infty}^{\infty} M[f_m(x) - f_n(x)]\,dx = 0$, then we say that $\{f_n(x)\}$ is convergent in mean with respect to $M(u)$. Throughout the paper we suppose that $M(u)$ is a convex N'-function such that

$$M(2u) \leqslant LM(u) \qquad \text{for } u > 0,$$

L being a constant independent of u.

Let E be a function-set consisting of all functions which are integrable with respect to $M(u)$. The object of the present paper is to derive the necessary and sufficient conditions that a subset of E should be compact[8]).

3. We will prove lemmas which are usefull in the sequel.

Lemma 1. *A function $f(x)$ which is integrable with respect to $M(u)$ is also integrable in ordinary sense in every finite interval.*

[7]) The concept of a complementary function is due to Birnbaum and Orlicz, loc. cit.

[8]) The space E is complete in the sense of mean convergence above defined. A set $F \subset E$ is said to be compact if any subset of F contains at least a sequence of functions which is convergent in mean.

For, let the complementary function of $M(u)$ be $N(v)$. Then by (2.7)

$$|f(x)| \leqslant M[f(x)] + N(1).$$

Thus the lemma is immediate.

Lemma 2.

$$M[f(x)+g(x)] \leqslant L\{M[f(x)]+M[g(x)]\}.$$

This is easy. From lemma 2 we see immediately that if $f(x)$ and $g(x)$ are integrable with respect to $M(u)$, then $f(x)+g(x)$ is also.

Lemma 3.

$$\int_{-\infty}^{\infty} M[f_\delta(x)]\,dx \leqslant \int_{-\infty}^{\infty} M[f(x)]\,dx.$$

For,

$$\int_{-\infty}^{\infty} M[f_\delta(x)]\,dx = \int_{-\infty}^{\infty} M[\frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(y)\,dy]\,dx \leqslant \int_{-\infty}^{\infty}\frac{dx}{2\delta}\int_{x-\delta}^{x+\delta} M[f(y)]\,dy^{9)} =$$

$$\frac{1}{2\delta}\int_{-\infty}^{\infty} dx \int_{-\delta}^{\delta} M[f(x+y)]\,dy = \frac{1}{2\delta}\int_{-\delta}^{\delta} dy \int_{-\infty}^{\infty} M[f(x+y)]\,dx = \int_{-\infty}^{\infty} M[f(x)]\,dx.$$

Lemma 4[10].

$$\lim_{\delta=0}\int_{-\infty}^{\infty} M[f_\delta(x)-f(x)]\,dx = 0.$$

Proof. We can find N independent of δ (<1) such that

$$(3.1)\quad \int_{N}^{\infty} M[f_\delta(x)-f(x)]\,dx < \varepsilon,$$

$$(3.2)\quad \int_{-\infty}^{-N} M[f_\delta(x)-f(x)]\,dx < \varepsilon,$$

for given ε. For

$$\int_{N}^{\infty} M[f_\delta(x)-f(x)]\,dx \leqslant L\{\int_{N}^{\infty} M[f_\delta(x)]\,dx + \int_{N}^{\infty} M[f(x)]\,dx\}$$

[9]) By the Jensen inequality.

[10]) We can also prove that $\lim_{h=0}\int_{-\infty}^{\infty} M[f(x+h)-f(x)]\,dx = 0$.

and

$$\int_{N}^{\infty} M[f_\delta(x)]\,dx \leqslant \frac{1}{2\delta}\int_{-\delta}^{\delta} dy \int_{N}^{\infty} M[f(x+y)]\,dx =$$

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} dy \int_{N+y}^{\infty} M[f(x)]\,dx \leqslant \int_{N-1}^{\infty} M[f(x)]\,dx.$$

Also for $\int_{-\infty}^{-N}$. Thus (3.1), (3.2) are immediate.

Now take a function $u(x)$ which is uniformly continuous in $(-\infty,\infty)$ and such that

$$(3.3)\quad \int_{-N-1}^{N+1} M[f(x)-u(x)]\,dx < \varepsilon^{11)}.$$

Then clearly there exists a δ_0, such that

$$(3.4)\quad \int_{-N}^{N} M[u(x)-u_\delta(x)]\,dx < \varepsilon \qquad \text{for } 0<\delta<\delta_0.$$

As in lemma 3, we have

$$(3.5)\quad \int_{-N}^{N} M[u_\delta(x)-f_\delta(x)]\,dx \leqslant \int_{-N-1}^{N+1} M[u(x)-f(x)]\,dx < \varepsilon$$

Hence by lemma 2,

$$\int_{-N}^{N} M[f_\delta(x)-f(x)]\,dx = \int_{-N}^{N} M[f_\delta(x)-u_\delta(x)+u_\delta(x)-u(x)+u(x)-f(x)]\,dx \leqslant$$

$$L\int_{-N}^{N} M[f_\delta(x)-u_\delta(x)]\,dx + L^2\int_{-N}^{N} M[u_\delta(x)-u(x)]\,dx +$$

$$L^2\int_{-N}^{N} M[u(x)-f(x)]\,dx \leqslant L\varepsilon + L^2\varepsilon + L^2\varepsilon, \qquad \text{for } 0<\delta<\delta_0$$

by (3.3), (3.4) and (3.5).

[11]) This is possible; see: Z. W. Birnbaum und W. Orlicz, Über Approximation im Mittel, Studia Math. 2 (1930) p. 197–206.

Combining with (3.1) and (3.2), we have

$$\int_{-\infty}^{\infty} M[f_\delta(x)-f(x)]\,dx \leqslant \varepsilon(1+L+2L^2) \quad \text{for } 0<\delta<\delta_0.$$

Thus the lemma is proved.

4. We will now prove the following theorem.

Theorem. *In order that a set $F\subset E$ be compact, it is necessary and sufficient that there exist a constant $K=K(F)$ and, for a given ε, two constants $\delta=\delta(\varepsilon, F)$ and $N_0=N_0(\varepsilon, F)$ depending only on F and ε, F respectively but not on $f(x)\in F$, such that the following conditions are satisfied:*

$$\text{(4.1)} \quad \int_{-\infty}^{\infty} M[f(x)]\,dx \leqslant K,$$

$$\text{(4.2)} \quad \int_{-\infty}^{\infty} M[f(x)-f_h(x)]\,dx \leqslant \varepsilon, \quad \text{for } 0<h\leqslant\delta,$$

$$\text{(4.3)} \quad \int_{-\infty}^{\infty} M[f(x)-f^N(x)]\,dx \leqslant \varepsilon, \quad \text{for } N\geqslant N_0$$

for all elements $f\in F$.

Proof. *Necessity of* (4.1). Suppose that (4.1) does not hold. Then there exists a sequence $\{f_n(x)\}$ such that

$$\lim_{n=\infty} \int_{-\infty}^{\infty} M[f_n(x)]\,dx=\infty.$$

Let $f(x)$ be any function of F or E.

$$\int_{-\infty}^{\infty} M[f_n(x)]\,dx=\int_{-\infty}^{\infty} M[f_n(x)-f(x)+f(x)]\,dx \leqslant$$

$$L\int_{-\infty}^{\infty} M[f_n(x)-f(x)]\,dx+L\int_{-\infty}^{\infty} M[f(x)]\,dx.$$

Hence

$$\int_{-\infty}^{\infty} M[f_n(x)-f(x)]\,dx \geqslant \frac{1}{L}\int_{-\infty}^{\infty} M[f_n(x)]\,dx-\int_{-\infty}^{\infty} M[f(x)]\,dx \to\infty \quad \text{as } n\to\infty.$$

Thus F is not compact.

Necessity of (4.2). If (4.2) does not hold, then there exist a positive number ε, a number sequence $\{\delta_n\}$ such that $\delta_n\to 0$ as $n\to\infty$ and a sequence of functions $\{f_n(x)\}$ such that

$$\int_{-\infty}^{\infty} M[f_n(x)-f_{n\,\delta_n}(x)]\,dx>\varepsilon,$$

where $f_{n\,\delta_n}(x)$ means $[f_n(x)]_{\delta_n}=\dfrac{1}{2\delta_n}\displaystyle\int_{x-\delta_n}^{x+\delta_n} f_n(x)\,dx.$

From lemma 2 and 3 we have

$$\varepsilon<\int_{-\infty}^{\infty} M[f_n(x)-f_{n\,\delta_n}(x)]\,dx\leqslant L\int_{-\infty}^{\infty} M[f_n(x)-f(x)]\,dx+$$

$$L^2\int_{-\infty}^{\infty} M[f(x)-f_{\delta_n}(x)]\,dx+L^2\int_{-\infty}^{\infty} M[f_{\delta_n}(x)-f_{n\,\delta_n}(x)]\,dx\leqslant$$

$$(L+L^2)\int_{-\infty}^{\infty} M[f_n(x)-f(x)]\,dx+L^2\int_{-\infty}^{\infty} M[f(x)-f_{\delta_n}(x)]\,dx,$$

where $f(x)$ is any function of F or E.

From lemma 4

$$\frac{\varepsilon}{L+L^2}\leqslant \varliminf_{n=\infty}\int_{-\infty}^{\infty} M[f_n(x)-f(x)]\,dx.$$

Thus F is not compact.

Necessity of (4.3). If (4.3) is not satisfied, then there exist a positive number ε, a number sequence $\{N_n\}$ such that $N_n\to\infty$ as $n\to\infty$ and a sequence of functions $\{f_n(x)\}$ such that

$$\int_{-\infty}^{\infty} M[f_n(x)-f_n^{N_n}(x)]\,dx>\varepsilon.$$

Let $f(x)$ be any function of F or E. Then

$$\varepsilon<\int_{-\infty}^{\infty} M[f_n(x)-f_n^{N_n}(x)]\,dx\leqslant L\int_{-\infty}^{\infty} M[f_n(x)-f(x)]\,dx+$$

$$L^2\int_{-\infty}^{\infty} M[f(x)-f^{N_n}(x)]\,dx+L^2\int_{-\infty}^{\infty} M[f^{N_n}(x)-f_n^{N_n}(x)]\,dx\leqslant$$

$$L\int_{-\infty}^{\infty} M[f_n(x)-f(x)]\,dx+L^2\int_{-\infty}^{\infty} M[f(x)-f^{N_n}(x)]\,dx+$$
$$L^2\int_{-\infty}^{\infty} M[f(x)-f_n(x)]\,dx.$$

Since $f(x)$ is integrable with respect to $M(u)$, we have

$$\lim_{n=\infty}\int_{-\infty}^{\infty} M[f(x)-f^{N_n}(x)]\,dx=\lim_{n=\infty}\Big(\int_{N_n}^{\infty} M[f(x)]\,dx+\int_{-\infty}^{-N_n} M[f(x)]\,dx\Big)=0.$$

Thus

$$\varepsilon\leqslant(L+L^2)\,\underline{\lim}_{n=\infty}\int_{-\infty}^{\infty} M[f(x)-f_n(x)]\,dx.$$

Hence F is not compact.

Next we will prove the sufficiency.

Let F_h^N be the set of $f_h^N(x)$, where $f(x)$ is of F, N and h are fixed and $f_h^N(x)$ denotes $\frac{1}{2h}\int_{x-h}^{x+h} f^N(x)\,dx$. Let $0<h\leqslant\delta$ and $N\geqslant N_0$.

For a given positive η, take an integer n such that $\frac{K}{2^n h}<\eta$ and $d=\frac{h\eta}{2N(2^n)}$; then for any x' and x'' such that

$$|x'-x''|<d,$$

we have $\quad (I=(x'-\delta,\ x''-\delta)+(x'+\delta,\ x''+\delta))$

$$f_h^N(x')-f_h^N(x'')=\frac{1}{2h}\int_I f^N(y)\,dy=\frac{1}{2h}\int_I 2^n\frac{f^N(y)}{2^n}\,dy\leqslant$$
$$\frac{1}{2h}\int_I N(2^n)\,dy+\frac{1}{2h}\int_I M[\frac{f^N(y)}{2^n}]\,dy\leqslant$$
$$\frac{N(2^n)}{h}\,|x'-x''|+\frac{1}{2^{n+1}h}\int_{-\infty}^{\infty} M[f^N(y)]\,dy\leqslant$$
$$\frac{N(2^n)}{h}\,d+\frac{1}{2^{n+1}h}\int_{-\infty}^{\infty} M[f(y)]\,dy\leqslant\frac{\eta}{2}+\frac{\eta}{2}=\eta.$$

Thus $\{F_h^N\}$ is a equi-continuous set.

$$f_h^N(x)=\frac{1}{2h}\int_{x-h}^{x+h} f^N(y)\,dy\leqslant\frac{1}{2h}\{\int_{x-h}^{x+h} N(1)\,dy+\int_{x-h}^{x+h} M[f^N(y)]\,dy\}\leqslant$$
$$N(1)+\frac{1}{2h}\int_{-\infty}^{\infty} M[f(y)]\,dy\leqslant N(1)+\frac{K}{2h}.$$

Thus $f_h^N(x)$ is uniformly bounded.

Consider any sequence $\{f_n(x)\}$ in F. Let the corresponding sequence of F_h^N be $\{f_{nh}^N(x)\}$. Then since $\{f_{nh}^N(x)\}$ is equi-continuous and uniformly bounded, by the well known Arzela's theorem, it contains a subsequence $\{f_{n_k h}^N(x)\}$ such that

(4.4) $\quad |f_{n_k h}^N(x)-f_{n_l h}^N(x)|<\varepsilon,\qquad$ for $k,\ l>p_0$,

where p_0 does not depend on x.

Let $\{f_{n_k}(x)\}$ be the corresponding sequence to $\{f_{n_k h}^N(x)\}$, $\{f_{n_k}(x)\}$ being a subsequence of $\{f_n(x)\}$.

From (4.4), we have

$$\int_{-N}^{N} M[f_{n_k h}^N(x)-f_{n_l h}^N(x)]\,dx<M(\varepsilon)\cdot 2N,\quad \text{for } n_k,\ n_l>p_0.$$

Now

$$\int_{-\infty}^{\infty} M[f_{n_k}(x)-f_{n_l}(x)]\,dx=\int_{-N}^{N}+\int_{N}^{\infty}+\int_{-\infty}^{-N}=I_1+I_2+I_3,\ \text{say}.$$

Then

$$I_2+I_3=(\int_N^{\infty}+\int_{-\infty}^{-N})\,M[f_{n_k}(x)-f_{n_l}(x)]\,dx=$$
$$\int_{-\infty}^{\infty} M[f_{n_k}(x)-f_{n_k}^N(x)+f_{n_l}(x)-f_{n_l}^N(x)]\,dx\leqslant$$
$$L\int_{-\infty}^{\infty} M[f_{n_k}(x)-f_{n_k}^N(x)]\,dx+L\int_{-\infty}^{\infty} M[f_{n_l}(x)-f_{n_l}^N(x)]\,dx\leqslant 2L\varepsilon,$$

from the assumption (4.3).

$$I_1=\int_{-N}^{N} M[f_{n_k}(x)-f_{n_l}(x)]\,dx=\int_{-N}^{N} M[f_{n_k}^N(x)-f_{n_l}^N(x)]\,dx\leqslant$$

$$L\int_{-N}^{N} M[f_{n_k}^N(x)-f_{n_k h}^N(x)]\,dx + L^2\int_{-N}^{N} M[f_{n_l}^N(x)-f_{n_l h}^N(x)]\,dx +$$

$$L^2\int_{-N}^{N} M[f_{n_k h}^N(x)-f_{n_l h}^N(x)]\,dx \leqslant$$

$$L\int_{-\infty}^{\infty} M[f_{n_k}(x)-f_{n_k h}(x)]\,dx + L^2\int_{-\infty}^{\infty} M[f_{n_l}(x)-f_{n_l h}(x)]\,dx + M(\varepsilon)\cdot 2N.$$

From the assumption (4.2), we have

$$I_1 \leqslant L\varepsilon + L^2\varepsilon + M(\varepsilon)\cdot 2N,$$

for $k,\ l > p_0$.

Thus

$$\overline{\lim_{k,\,l=\infty}}\int_{-\infty}^{\infty} M[f_{n_k}(x)-f_{n_l}(x)]\,dx \leqslant 3L\varepsilon + L^2\varepsilon + M(\varepsilon)\,2N.$$

Since ε is arbitrary, $\{f_{n_k}(x)\}$ converges in mean with respect to $M(u)$. Thus the theorem is proved.

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