Notes on orthogonal series II.

by

S. KACZMARZ (Lwów).

1. Let $\sum_1^\infty a_n \varphi_n(t)$ denote an orthogonal series and let $\sum_1^\infty a_n^2 < \infty$; then the following theorem was proved by D. MENCHOFF[1]):

If the increasing sequence of numbers $W(n) = o(\log^2 n)$, *then there exists an orthogonal normal system* $\{\varphi_n(t)\}$ *and a sequence of numbers* $\{a_n\}$ *such that:*

a) $\sum_1^\infty a_n^2 W(n) < \infty$; b) *the series* $\sum_1^\infty a_n \varphi_n(t)$ *is almost everywhere divergent.*

The object of this paper is to give another proof of the theorem, the idea being the same, only the following lemma is proved simpler.

Lemma. *Given an integer p, there exists a sequence of orthogonal functions* $f_{p,m}(t)$, $m = 1, 2 .., 2p$, *defined in the interval* $(0, 5)$, *such that:*

1) $\int_0^5 f_{p,m}^2(t)\, dt \leqslant \frac{C}{p}$; 2) *for any* $t \in (1, 2)$ *there is an m, dependent from t,* $m \leqslant p$, *for which*

$$|f_{p,m}(t) + f_{p,m+1}(t) + \ldots + f_{p,2p-1}(t)| > C_1 \log p,$$

where C and C_1 *denote absolute constants.*

[1]) D. Menchoff, Sur les séries de fonctions orthogonales, Fund. Math. 4 (1923) p. 82—105, th. 2.

We define the sequence $\{f_{p,m}(t)\}$ in the interval $(0, 4)$ by

$$f_{p,m}(t)=\frac{1}{k-p-m-\frac{1}{2}} \quad \text{for } t\in\left(\frac{k-1}{p},\ \frac{k}{p}\right),\ k=1,2,..,4p.$$

Then

$$\int_0^4 f^2_{p,m}(t)\,dt=\sum_1^{4p}\frac{1}{(k-p-m-\frac{1}{2})^2}\,\frac{1}{p}\leqslant\frac{2}{p}\sum_1^{\infty}\frac{1}{(i-\frac{1}{2})^2}=\frac{A}{p}.$$

We have further

$$\int_0^4 f_{p,m}(t)\,f_{p,n}(t)\,dt=\frac{1}{p}\sum_1^{4p}\frac{1}{(k-p-m-\frac{1}{2})\,(k-p-n-\frac{1}{2})}$$

$$=\frac{1}{p(m-n)}\sum_1^{4p}\left[\frac{1}{k-p-m-\frac{1}{2}}-\frac{1}{k-p-n-\frac{1}{2}}\right]$$

$$=\frac{1}{p(m-n)}\left[\sum_{1-p-m}^{3p-m}\frac{1}{l-\frac{1}{2}}-\sum_{1-p-n}^{3p-n}\frac{1}{l-\frac{1}{2}}\right].$$

Cancelling equal terms in both sums we get for $n<m\leqslant 2p$

$$\int_0^4 f_{p,m}(t)\,f_{p,n}(t)\,dt=\frac{1}{p(m-n)}\left[\sum_{1-p-m}^{-p-n}\frac{1}{l-\frac{1}{2}}-\sum_{3p-m+1}^{3p-n}\frac{1}{l-\frac{1}{2}}\right].$$

Hence

$$(1.1)\qquad \left|\int_0^4 f_{p,m}(t)\,f_{p,n}(t)\,dt\right|\leqslant\frac{1}{p(m-n)}\left[\frac{m-n}{p+n-\frac{1}{2}}+\frac{m-n}{3p-m+\frac{1}{2}}\right]$$
$$\leqslant\frac{1}{p}\cdot\frac{2}{p+\frac{1}{2}}<\frac{2}{p^2}.$$

Take now t, $1\leqslant t\leqslant 2$; then $t\in\left(1+\frac{m-1}{p},1+\frac{m}{p}\right)$, $m\leqslant p$ and

$$\sum_{k=m}^{2p-1} f_{p,k}(t)=-\sum_0^{2p-m-1}\frac{1}{k+\frac{1}{2}},$$

that is

$$(1.2)\qquad \left|\sum_{k=m}^{2p-1} f_{p,k}(t)\right|>C_1\log(2p-m)\geqslant C_1\log p.$$

To obtain an orthogonal system, required in the lemma, we define the functions $f_{p,m}(t)$ in the interval $(4, 5)$ as follows.

Divide the interval in $N=p\,(2p-1)$ equal subintervals. N being the number of pairs (n, m), if $1\leqslant n<m\leqslant 2p$, every pair (n, m) corresponds to one and only one subinterval $I_{n,m}$. Denote

$$\alpha_{m,l}=\int_0^4 f_{p,m}(t)\,f_{p,l}(t)\,dt.$$

Put in the interval $I_{m,l}$, $f_{p,m}(t)=\sqrt{N|\alpha_{m,l}|}$ if $l=1,2..m-1$, $=-\sqrt{N|\alpha_{m,l}|}\operatorname{sign}\alpha_{m,l}$ if $l=m+1,\dots.2p$ and $=0$ in other intervals. Then the system $\{f_{p,m}(t)\}$, $m=1,2\dots2p$ is orthogonal in $(0, 5)$ and

$$\int_0^5 f^2_{p,m}(t)\,dt<\frac{A}{p}+\sum_1^{m-1}|\alpha_{m,l}|+\sum_{m+1}^{2p}|\alpha_{m,l}|\leqslant\frac{A}{p}+\frac{4p}{p^2}=\frac{C}{p}$$

by (1.1). The property 1) is also proved and the same is with the property 2) by the inequality (1.2).

2. The proof of the theorem runs now on the same lines as in Menchoff's paper. Take an increasing sequence of integers $\{p_k\}$ with properties:

$$p_1=0,\quad 1+2\,(p_1+p_2+..p_k)<p_{k+1}\quad (k=1,2,\dots),$$
$$\frac{W(n)}{(\log n)^2}<\frac{1}{k^2}\ \text{for } n\geqslant p_k.$$

Put $N_0=0$, $N_k=1+2\,(p_1+p_2+\,..+p_k)$ and denote by G_k the set of indices n such that $N_{k-1}<n\leqslant N_k$. We put $a_1\varphi_1(t)=1$ in $(0,1)$. If n belongs to G_{k+1}, we divide $(0,1)$ in intervals I_j such that any $a_i\varphi_i(t)$, $i\in(G_1+G_2+..+G_k)$, is constant in I_j. Denote by I'_j and I''_j the left and right half of I_j and define, for $m=n-N_k$ and an interval P of measure d, the function $h_{p,m}(t, P)$ as the function $f_{p,m}(t)$, compressed from $(0, 5)$ to the interval $(0, d)$ and translated in the interval P. Now we put

$$a_n\,\varphi_n(t)=h_{p_{k+1},m}(t, I'_j)\,/\log p_{k+1}\ \text{ for } t\in I'_j$$
$$=-h_{p_{k+1},m}(t, I''_j)\,/\log p_{k+1}\ \text{ for } t\in I''_j.$$

It is easy to prove, that the series $\sum_1^{\infty} a_n\,\varphi_n(t)$ is orthogonal and $\sum_1^{\infty} a_n^2\,W(n)<\infty$ on account of the property 1) of the lemma.

The proof of the divergence is based on the property 2). Consider the indices belonging to G_k. Then, by the said property and relations $N_k - N_{k-1} = 2p_k$, $N_k < 3p_k$, we have in the interval I_j'

$$\left| \sum_{N_{k-1}+m}^{N_{k-1}+2p_k-1} a_n \varphi_n(t) \right| \geqslant C_1$$

in a set of measure $\frac{1}{5}|I_j'|$ and similarly in I_j''.

Therefore is in a set E_k, of measure $\frac{1}{5}$, for suitable indices n and n'

$$\left| \sum_{n}^{n'} a_k \varphi_k(t) \right| \geqslant C_1. \tag{2.1}$$

The set $E = \limsup_{k\to\infty} E_k$, $|E| = \lim_{k\to\infty} |E_k + E_{k+1} + \dots|$, is of measure 1 and the series is in E divergent, as shown by (2.1).

(Reçu par la Rédaction le 20 11. 1934).