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Received October 17, 1972 (599)

STUDIA MATHEMATICA, T. XLIX. (1973)

On the continuity property of Gaussian random fields

by

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Abstract. The conditions for sample paths to be continuous are considered for Gaussian random fields. Especially, the necessary conditions are described.

§ 1. Introduction. Let $X = \{X(\bar{t}),\ \bar{t}\in \boldsymbol{R}^d\}$ be a zero mean, real, stationary, separable, mean continuous, Gaussian random field with a d-dimensional Euclidean parameter space. Then, the covariance function $\varrho(\bar{t}) = E\big(X(\bar{t}+\bar{x})X(\bar{s})\big)$ is expressed by $\int_{\boldsymbol{R}^d} \cos(\bar{t},\ \bar{\lambda})\, dF(\bar{\lambda})$, where $(\,,\,)$ denotes the inner product, $\bar{t}, \bar{s}\in \boldsymbol{R}^d$ and $F(\cdot)$ [1] is a bounded positive measure.

The purpose of this paper is to describe the continuity condtitions of path functions (which are known for the 1-dimensional parameter case) for random fields. Most sufficient conditions for sample functions to be continuous are already described for random fields. Thus, we shall be concerned mainly with sufficient conditions for sample functions to be discontinuous.

In the case of the 1-dimensional parameter space, the conditions in terms of the spectral measure $F(\cdot)$ were given by Kahane [4] and Nisio [7]. The corresponding results for random fields are the following. Let $s_n = F(\bar{\lambda}\in S_{2^{n+1}}) - F(\bar{\lambda}\in S_{2^n})$, where $S_{2^{n+1}} = \{\lambda;\ |\bar{\lambda}|\leqslant 2^{n+1}\}$, $n = 0, 1, 2, \ldots$

THEOREM 1. *If $X(\bar{t})$ is continuous, then* $\sum_{n=1}^{\infty} s_n^{\frac{1}{2}} < \infty$.

THEOREM 2. *If there exists a decreasing sequence $\{M_n\}$ such that $s_n \leqslant M_n$ and $\sum_{n=1}^{\infty} M_n^{\frac{1}{2}} < \infty$, then X has continuous paths.*

As is shown by Marcus [5] and Marcus and Shepp [6], these conditions are neither too strong, nor necessary and sufficient. However, they give a simple criterion for some cases. In § 2 and § 3, we shall give the proof of the above theorems.

A result corresponding to theorem of Marcus and Shepp ([6], p. 380) is as follows.

(1) F is occasionally used as a measure or as a point function.

THEOREM 3. *Let X be the above-mentioned Gaussian process and let $\{Y(r), r\in R\}$ be a stationary Gaussian process which is a restriction to a 1-dimensional parameter subspace of X. Let $\sigma^2(h) = E\big((Y(r+h)-Y(r))^2\big)$. Let ψ be any nondecreasing local minorant of σ, that is, for some $\varepsilon > 0$,*

$$\sigma(h) \geqslant \psi(h) \geqslant 0, \quad 0 \leqslant h \leqslant \varepsilon,$$

$$\psi(h)\uparrow, \quad 0 \leqslant h \leqslant \varepsilon.$$

If

$$\int^{\infty} \psi(e^{-x^2})\,dx = \infty,$$

then the paths of X are not continuous.

Proof. The proof is easy. In § 4, we shall deal with applications and comments.

A necessary and sufficient condition for Gaussian sample paths to be continuous is given by Sudakov [9], but it seems to be difficult to express his condition by any explicit formula involving the spectral function or the covariance function.

§ 2. Proof of Theorem 1. Let $\{T_j, j=1,2,\ldots\}$ be an increasing sequence of positive numbers such that $\sum_{j=1}^{\infty}\frac{1}{\sqrt{T_j}} < \infty$ and $T_j \geqslant 1$. Let

$$\chi(x) = \max(1-|x|, 0),$$

$$\theta_r(\lambda) = \prod_{j=r}^{\infty}\chi\left(\frac{\lambda}{T_j}\right), \quad -\infty<\lambda<\infty,$$

$$\theta_r(\lambda_1,\ldots,\lambda_d) = \theta_r(\lambda_1)\theta_r(\lambda_2)\ldots\theta_r(\lambda_d), \quad -\infty<\lambda_i<\infty\ (i=1,2,\ldots,d),$$

$$K_r(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\lambda}\chi\left(\frac{\lambda}{T_r}\right)d\lambda, \quad -\infty<t<\infty,$$

$$K_r(t_1,\ldots,t_d) = K_r(t_1)\ldots K_r(t_d), \quad -\infty<t_i<\infty\ (i=1,2,\ldots,d),$$

$$l_r(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it\lambda}\theta_r(\lambda)\,d\lambda, \quad -\infty<t<\infty,$$

$$l_r(t_1,t_2,\ldots,t_d) = l_r(t_1)\ldots l_r(t_d), \quad -\infty<t_i<\infty\ (i=1,2,\ldots,d),$$

$$l_r^*(t) = \frac{1}{\sqrt{2\pi}}\int_{-T_{r-1}}^{T_{r-1}} e^{it\lambda}\theta_r(\lambda)\,dx, \quad -\infty<t<\infty,$$

$$l_r^*(t_1,\ldots,t_d) = l_r^*(t_1)\ldots l_r^*(t_d), \quad -\infty<t_i<\infty\ (i=1,2,\ldots,d).$$

Hence we note that

$$l_r(t_1,t_2,\ldots,t_d) \geqslant 0.$$

Also, we remark that

$$l_r(t_1,\ldots,t_d) = \prod_{i=1}^{d}(l_{r+1}*K_r)(t_i).$$

As is well known, $X(t_1\ldots t_d)$ can be written in the form

$$X(t_1,t_2,\ldots,t_d,\omega) = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} e^{i(t_1\lambda_1+t_2\lambda_2+\ldots+t_d\lambda_d)}\,d\Phi(\lambda_1,\lambda_2,\ldots,\lambda_d,\omega),$$

where $\Phi(\cdot)$ is a Gaussian random measure.

Also, we put

$$Y_r(t_1,t_2,\ldots,t_d,\omega) = \left(\frac{1}{\sqrt{2\pi}}\right)^d\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} X(t_1-s_1,t_2-s_2,\ldots,t_d-s_d,\omega)\times$$
$$\times l_r(s_1,s_2,\ldots,s_d)\,ds_1\ldots ds_d,$$

$$Y_r^*(t_1,t_2,\ldots,t_d,\omega) = \left(\frac{1}{\sqrt{2\pi}}\right)^d\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} X(t_1-s_1,t_2-s_2,\ldots,t_d-s_d,\omega)\times$$
$$\times l_r^*(s_1,s_2,\ldots,s_d)\,ds_1\ldots ds_d.$$

We can rewrite Y_r and Y_r^* in the following way:

$$Y_r(t_1,t_2,\ldots,t_d,\omega) = \int_{-T_r}^{T_r}\ldots\int_{-T_r}^{T_r} e^{i(t_1\lambda_1+\ldots+t_d\lambda_d)}\,\theta_r(\lambda_1,\lambda_2,\ldots,\lambda_d)\times$$
$$\times d\Phi(\lambda_1,\lambda_2,\ldots,\lambda_d,\omega),$$

$$Y_r^*(t_1,t_2,\ldots,t_d,\omega) = \int_{-T_{r-1}}^{T_{r-1}}\ldots\int_{-T_{r-1}}^{T_{r-1}} e^{i(t_1\lambda_1+\ldots+t_d\lambda_d)}\,\theta_r(\lambda_1,\lambda_2,\ldots,\lambda_d)\times$$
$$\times d\Phi(\lambda_1,\lambda_2,\ldots,\lambda_d,\omega).$$

Since $X(t,\omega)$ is continuous with probability one, by Fernique [2] we obtain

$$a = E\big(\sup_{\bar t\in[0,1]^d}|X(\bar t)|\big) < \infty.$$

We have the following lemma.

LEMMA 2.1.

$$E\big(\sup_{\bar t\in[0,1]^d}|Y_r^*(\bar t)|\big) \leqslant E\big(\sup_{\bar t\in[0,1]^d}|Y_r(\bar t)|\big) \leqslant a.$$

Proof. Let us put $Z_r(t_1 \dots t_d) = Y_r(t_1 \dots t_d) - Y_r^*(t_1, t_2, \dots, t_d)$. Then Z_r has continuous paths and

$$Z_r(t_1, t_2, \dots, t_d) = \int_{(t_1, t_2, \dots, t_d) \in \square_{T_r} - \square_{T_{r-1}}} \dots \int e^{i(t_1\lambda_1 + \dots + t_d\lambda_d)} \theta_r(t_1, \dots, t_d) \times$$
$$\times d\Phi(\lambda_1, \lambda_2, \dots, \lambda_d),$$

where $\square_T = \{(\lambda_1, \dots, \lambda_d);\ |\lambda_i| \leqslant T\ (i = 1, 2, \dots, d)\}$. Therefore, we can easily see that $Z_r + Y_r^* = Y_r$; moreover Z_r and Y_r^* are mutually independent as continuous function-valued random variables, and using Lemma 3.2.4.A in Delporte ([1], p. 143), we have

$$E(\|Y_r^*\|) \leqslant E(\|Y_r\|),$$

where $\|\cdot\|$ denotes the uniform norm of continuous functions. Since $\left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} l_r(s_1, s_2, \dots, s_d)\, ds_1 ds_2 \dots ds_d = 1$, we have

$$E(\|Y_r\|) \leqslant a.$$

Define stationary Gaussian processes V_r and V_r^* by

$$V_r(t_1, t_2, \dots, t_d) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{\exists i, |s_i| \geqslant \frac{1}{\sqrt{T_r}}} \dots \int Y_{r+1}(t_1 - s_1, t_2 - s_2, \dots, t_d - s_d, \omega) \times$$
$$\times K_r(s_1, s_2, \dots, s_d)\, ds_1 ds_2 \dots ds_d,$$

$$V_r^*(t_1, t_2, \dots, t_d) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{\exists i, |s_i| \geqslant \frac{1}{\sqrt{T_r}}} \dots \int Y_{r+1}^*(t_1 - s_1, t_2 - s_2, \dots, t_d - s_d, \omega) \times$$
$$\times K_r(s_1, \dots, s_d)\, ds_1 ds_2 \dots ds_d.$$

The following estimates are crucial for the proof

$$K_r(t) = \frac{\sqrt{2}}{t^2 T_r \sqrt{\pi}} (1 - \cos T_r t) \geqslant 0,$$

$$\int_{\frac{1}{\sqrt{T_r}}}^{\infty} K_r(t)\, dt \leqslant \frac{2\sqrt{2}}{\sqrt{\pi}\sqrt{T_r}}, \qquad \int_{-\infty}^{\infty} K_r(t)\, dt = 1,$$

$$\int_{\frac{1}{\sqrt{T_r}}}^{\infty} l_r(t)\, dt \leqslant \frac{2\sqrt{2}}{\sqrt{\pi}\sqrt{T_r}} \quad \text{and} \quad \int_{-\infty}^{\infty} l_r(t)\, dt = 1.$$

By making use of the above estimates, we have

$$\int_{\exists i, |s_i| \geqslant \frac{1}{\sqrt{T_r}}} K_r(s_1, s_2, \dots, s_d)\, ds_1 ds_2 \dots ds_d \leqslant \frac{C}{\sqrt{T_r}}$$

and

$$\int_{\exists i, |s_i| \geqslant \frac{1}{\sqrt{T_r}}} l_r(s_1, s_2, \dots, s_d)\, ds_1 ds_2 \dots ds_d \leqslant \frac{C}{\sqrt{T_r}}$$

where C is an absolute constant.

Therefore, we have

$$E(\|V_r\|) \leqslant = \frac{C}{\sqrt{T_r}} \tag{2.1}$$

and

$$E(\|V_r^*\|) \leqslant \frac{C}{\sqrt{T_r}}, \tag{2.2}$$

where C is an absolute constant.

Lemma 2.2 *Let $\{T_r\}$ be an increasing sequence of positive numbers such that $\sum_{r=1}^{\infty} \frac{1}{\sqrt{T_r}} < \infty$ and $T_r \geqslant 1$. Then*

$$\sum_{j=1}^{\infty} \left(\int_{(\lambda_1, \lambda_2, \dots, \lambda_d) \in \square_{T_{j+1}} - \square_{T_j}} \cdots\cdots \int \prod_{k=j+1}^{\infty} \prod_{i=1}^{d} \left(1 - \frac{|\lambda_i|}{T_k}\right)^2 dF(\lambda_1, \dots, \lambda_d)\right)^{\frac{1}{2}} < \infty.$$

Proof. We define successively the random variables S_j, S_j' and H_j, $j = 1, 2, \dots$, as follows:

$$S_1(\omega) \equiv 0,$$
$$H_1(\omega) \equiv Y_1(S_1(\omega), \omega),$$

$$S_1(\omega) \equiv \begin{cases} \text{some } \bar{t} \in \{\bar{t} = (t_1 \dots t_d);\ |t_i| \leqslant \tau_1, \\ \quad Y_2^*(t_1, \dots, t_d, \omega) = \min\limits_{(|s_i| \leqslant \tau_1 (i = 1, 2, \dots, d))} Y_2^*(s_1, s_2, \dots, s_d, \omega)\} \\ \quad \text{if } H_1(\omega) < \min\limits_{(|s_i| \leqslant \tau_1 (i = 1, 2, \dots, d))} Y_2^*(s_1, s_2, \dots, s_d, \omega), \\ \text{some } \bar{t} \in \{\bar{t};\ |t_i| \leqslant \tau_1\ (i = 1, 2, \dots, d), \\ \quad Y_2^*(t_1, t_2, \dots, t_d) = \max\limits_{(|t_i| < \tau_1 (i = 1, 2, \dots, d))} Y_2^*(s_1, s_2, \dots, s_d, \omega)\} \\ \quad \text{if } H_1(\omega) > \max\limits_{(|s_i| < \tau_1 (i = 1, 2, \dots, d))} Y_2^*(s_1, s_2, \dots, s_d, \omega), \\ \text{some } \bar{t} \in \{\bar{t};\ |t_i| \leqslant \tau_1\ (i = 1, 2, \dots, d), \\ \quad Y_2^*(t_1, \dots, t_d, \omega) = H_1(\omega)\}, \text{ otherwise,} \end{cases}$$

where $\tau_1 = 1 + \frac{1}{\sqrt{T_1}}$.

Obviously, $\tilde{S}_1(\omega)$ is measurable with respect to the Borel field $\mathscr{B}_1$ generated by $\{d\Phi(\lambda_1, \lambda_2, \ldots, \lambda_d);\ |\lambda_i| \leqslant T_1\ (i = 1, 2, \ldots, d)\}$.

$$S_{j+1}(\omega) \equiv \begin{cases} \tilde{S}_j(\omega) \text{ if } Y_{j+1}(\tilde{S}_j, \omega) \geqslant H_j(\omega), \\ \text{some } \bar{t} \in \{\bar{t} = (t_1, t_2, \ldots, t_d);\ |t_i| \leqslant \tau_j, \\ \quad Y_{j+1}(t_1, t_2, \ldots, t_d, \omega) = \max\limits_{|s_i| \leqslant \tau_j} Y_{j+1}(s_1, s_2, \ldots, s_d;\ \omega)\} \\ \quad \text{if } H_j(\omega) > \max\limits_{\{|s_i| \leqslant \tau_j\}} Y_{j+1}(s_1, s_2, \ldots, s_d, \omega), \\ \text{some } \bar{t} \in \{\bar{t} = (t_1, t_2, \ldots, t_d);\ |t_i| \leqslant \tau_j, \\ \quad Y_{j+1}(t_1, t_2, \ldots, t_d;\ \omega) = H_j(\omega)\}, \text{ otherwise}. \end{cases}$$

$H_{j+1}(\omega) \equiv Y_{j+1}(S_{j+1}(\omega), \omega)$.

$$\tilde{S}_{j+1}(\omega) \equiv \begin{cases} \text{some } \bar{t} \in \{\bar{t} = (t_1, t_2, \ldots, t_d);\ |t_i| \leqslant \tau_{j+1}, \\ \quad Y^*_{j+2}(t_1, t_2, \ldots, t_d, \omega) = \min\limits_{|s_i| \leqslant \tau_{j+1}} Y^*_{j+2}(s_1, \ldots, s_d, \omega)\} \\ \quad \text{if } H_{j+1}(\omega) < \min\limits_{|s_i| \leqslant \tau_{j+1}} Y^*_{j+2}(s_1, s_2, \ldots, s_d, \omega), \\ \text{some } \bar{t} \in \{\bar{t} = (t_1, t_2, \ldots, t_d);\ |t_i| \leqslant \tau_{j+1}, \\ \quad Y^*_{j+1}(t_1, t_2, \ldots, t_d, \omega) = \max\limits_{|s_i| \leqslant \tau_{j+1}} Y^*_{j+2}(s_1, s_2, \ldots, s_d, \omega)\} \\ \quad \text{if } H_{j+1}(\omega) > \max\limits_{|s_i| \leqslant \tau_{j+1}} Y^*_{j+2}(s_1, s_2, \ldots, s_d, \omega), \\ \text{some } \bar{t} \in \{\bar{t} = (t_1, t_2, \ldots, t_d);\ |t_i| \leqslant \tau_{j+1}, \\ \quad Y^*_{j+2}(t_1, t_2, \ldots, t_d, \omega) = H_{j+1}(\omega)\}, \text{ otherwise}, \end{cases}$$

where $\tau_j = 1 + \dfrac{1}{\sqrt{T_1}} + \ldots + \dfrac{1}{\sqrt{T_j}}$. By definition, S_j and $\tilde{S}_j$ are measurable with respect to the Borel field $\mathscr{B}_j$, generated by $\{d\Phi(\lambda_1, \lambda_2, \ldots, \lambda_d);\ |\lambda_i| \leqslant T_j\}$. Now we shall prove that

$$P\left(\sup_{j=1,2,\ldots} |H_j(\omega)| < \infty\right) = 1. \tag{2.3}$$

Let us put

$$\tau = \lim_{j \to \infty} \tau_j.$$

$$Y_r(t_1, \ldots, t_d) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} X(t_1 - s_1, t_2 - s_2, \ldots, t_d - s_d) \times \times l_r(s_1, s_2, \ldots, s_d)\, ds_1 ds_2 \ldots ds_d$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{|s_i| \leqslant \frac{1}{\sqrt{T_r}}\ (i=1,\ldots,d)} \cdots \int X(t_1 - s_1, \ldots, t_d - s_d) \times l_r(s_1, \ldots, s_d)\, ds_1 \ldots ds_d +$$

$$+ \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{\exists i,\ |s_i| \geqslant \frac{1}{\sqrt{T_r}}} \cdots \int X_{r+1}(t_1 - s_1, \ldots, t_d - s_d) \times l_r(s_1, s_2, \ldots, s_d)\, ds_1 ds_2 \ldots ds_d.$$

Here we can see that

$$\sup_{|t_i| \leqslant \tau} \left| \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{|s_i| \leqslant \frac{1}{\sqrt{T_r}}\ (i=1,\ldots,d)} \cdots \int X(t_1 - s_1, \ldots, t_d - s_d) \times l_r(s_1, s_2, \ldots, s_d)\, ds_1, \ldots, ds_d \right| \leqslant \sup_{|t_i| \leqslant 2\tau} |X(t_1, \ldots, t_d)|$$

with probability one; also, since $\int\limits_{\exists i,\ |s_i| \leqslant \frac{1}{\sqrt{T_r}}} \cdots \int l_r(s_1, s_2, \ldots, s_d)\, ds_1 ds_2 \ldots ds_d \leqslant \mathrm{const}/\sqrt{T_r}$ we have

$$\sum_{r=1}^{\infty} E\left(\sup_{|t_i| \leqslant \tau} \left| \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{\exists i,\ |s_i| \geqslant \frac{1}{\sqrt{T_r}}} \cdots \int X(t_1 - s_1, \ldots, t_d - s_d) \times l_r(s_1, s_2, \ldots, s_d)\, ds_1 ds_2 \ldots ds_d \right| \right) < \infty.$$

Therefore,

$$\sup_r \sup_{|t_i| \leqslant \tau} |Y_r(t_1, \ldots, t_d)| < \infty \quad \text{with probability one},$$

which combined with the definition of H_j yields (2.3). Now we observe the following relations:

$$H_{j+1}(\omega) - H_j(\omega) = \{(H_{j+1}(\omega) - H_j(\omega)) \vee 0\} - \{(Y_j(S_j(\omega), \omega) - \sup_{|s_i| \leqslant \tau_j} Y_{j+1}(s_1, s_2, \ldots, s_d, \omega)) \vee 0\}.$$

For $(t_1, t_2, \ldots, t_d)$ satisfying $|t_i| \leqslant \tau_{j-1}$ $(i = 1, \ldots, d)$, we have

$$Y_j(t_1, t_2, \ldots, t_d) \leqslant \sup_{|t_i| \leqslant \tau_j\ (i=1,2,\ldots,d)} Y_{j+1}(t_1, t_2, \ldots, t_d) + \sup_{|t_i| \leqslant \tau_{j-1}} V_j(t_1, t_2, \ldots, t_d).$$

Therefore, for $\bar{t}$ such that $|t_i| \leqslant \tau_{j-1}$ $(i = 1, 2, \ldots, d)$, we obtain

$$Y_{j+1}(t_1, t_2, \ldots, t_d) - \sup_{|s_i| \leqslant \tau_j} Y_{j+1}(s_1, s_2, \ldots, s_d) \leqslant \sup_{|s_i| \leqslant \tau_{j-1}\ (i=1,2,\ldots,d)} V_j(s_1, s_2, \ldots, s_d),$$

$$\left(Y_j(S_j(\omega)) - \sup_{|s_i| \leqslant \tau_j} Y_{j+1}(s_1, s_2, \ldots, s_d)\right) \vee 0 \leqslant \sup_{|s_i| \leqslant \tau_{j-1}\ (j=1,2,\ldots,d)} |V_j(s_1, s_2, \ldots, s_d)|$$

with probability 1.

Again, by (2.1), we have

$$\sum_{j=1}^{\infty} E\Big\{\Big(Y_j(S_j) - \sup_{|s_i|\leqslant\tau_j\,(i=1,2,\ldots,d)} Y_{j+1}(s_1, s_2, \ldots, s_d)\Big)\vee 0\Big\} < \infty .$$

By the use of (2.3) and the relation

$$\sum_{j=1}^{n}(H_{j+1}-H_j)\vee 0$$

$$= H_{n+1}-H_1+\sum_{j=1}^{n}\Big(Y_j(S_j) - \sup_{|s_i|\leqslant\tau_j\,(i=1,2,\ldots,d)} Y_{j+1}(s_1, s_2, \ldots, s_d)\Big)\vee 0,$$

we have

$$\sum_{j=1}^{\infty}(H_{j+1}-H_j)\vee 0 < \infty, \quad \text{with probability } 1.$$

On the other hand,

(2.4) $$(H_{j+1}-H_j)\vee 0 = \big(Y_{j+1}(\tilde S_j)-H_j\big)\vee 0$$
$$\geqslant \big\{\big(Y_{j+1}(\tilde S_j)-Y^*_{j+1}(\tilde S_j)\big)\vee 0\big\} -$$
$$-\big\{\big(H_j-\sup_{|t_i|\leqslant\tau_j} Y^*_{j+1}(t_1, t_2, \ldots, t_d)\big)\vee 0\big\}.$$

Also, observing that for $(t_1, t_2, \ldots, t_d)$ satisfying $|t_i|\leqslant\tau_{i-1}$ $(i=1,2,\ldots,d)$

$$Y_j(t_1, t_2, \ldots, t_d) \leqslant \sup_{|t_i|\leqslant\tau_j\,(i=1,\ldots,d)} Y^*_{j+1}(t_1, t_2, \ldots, t_d) +$$
$$+\sup_{|t_i|\leqslant\tau_{j-1}\,(i=1,\ldots,d)} V^*_j(t_1, t_2, \ldots, t_d),$$

we have

$$\Big(Y_j(t_1, t_2, \ldots, t_d) - \sup_{|s_i|\leqslant\tau_j\,(i=1,2,\ldots,d)} Y^*_{j+1}(s_1, s_2, \ldots, s_d)\Big)\vee 0$$
$$\leqslant \sup_{|t_i|\leqslant\tau_{j-1}} |V^*_j(t_1, t_2, \ldots, t_d)|,$$

and consequently

$$\sum_{j=1}^{\infty} E\Big(\Big(Y_j(t_1, t_2, \ldots, t_d) - \sup_{|s_i|\leqslant\tau_j\,(i=1,2,\ldots,d)} Y^*_{j+1}(s_1, s_2, \ldots, s_d)\vee 0\Big)\Big) < \infty .$$

Now, using (2.4) and the relation

$$\big(H_j(\omega)-\sup_{|t_i|\leqslant\tau_j} Y^*_{j+1}(t_1, t_2, \ldots, t_d)\big)\vee 0$$
$$= \big(Y_j(S_j(\omega), \omega)-\sup_{|t_i|\leqslant\tau_j} Y^*_{j+1}(t_1, t_2, \ldots, t_d)\big)\vee 0,$$

we get

$$\sum_{j=1}^{\infty}\big(Y_{j+1}(\tilde S_j)-Y^*_{j+1}(\tilde S_j)\big)\vee 0 < \infty \quad \text{with probability } 1.$$

Put

$$\gamma_j = Y_{j+1}(\tilde S) - Y^*_{j+1}(\tilde S_j)$$
$$= \int\limits_{(\lambda_1,\ldots,\lambda_d)\in\square_{T_{j+1}}-\square_{T_j}}\cdots\int \exp\Big(i\sum_{i=1}^{d}\tilde S^i_j\lambda_i\Big)\,\theta_{j+1}(\lambda_1, \ldots, \lambda_d)\,d\Phi(\lambda_1, \lambda_2, \ldots, \lambda_d)$$

and

$$v_j = \int\limits_{(\lambda_1,\lambda_2,\ldots,\lambda_d)\in\square_{T_{j+1}}-\square_{T_j}}\cdots\int \prod_{k=j+1}^{\infty}\prod_{i=1}^{d}\Big(1-\frac{|\lambda_i|}{T_k}\Big)^2 dF(\lambda_1, \lambda_2, \ldots, \lambda_d),$$

where $\tilde S_j = (\tilde S^1_j, \tilde S^2_j, \ldots, \tilde S^d_j)$. Then, in the same way as in Nisio [7], we obtain

$$\sum_{j=1}^{\infty}\sqrt{v_j} < \infty,$$

which completes the proof of Lemma 2.2.

We shall now pass to the proof of Theorem 1. Take 2^{-k} as T_k and $a = \prod_{k=0}^{\infty}(1-3\cdot 2^{-k-2})^{2d}$.

Evidently, we get

$$a\int\limits_{(\lambda_1,\ldots,\lambda_d)\in\square_{(3/2)2^j}-\square_{2^j}}\cdots\int dF(\lambda_1, \lambda_2, \ldots, \lambda_d)$$
$$\leqslant \int\limits_{(\lambda_1,\ldots,\lambda_d)\in\square_{(3/2)2^j}-\square_{2^j}}\cdots\int \prod_{k=j+1}^{\infty}\prod_{i=1}^{d}\Big(1-\frac{|\lambda_i|}{2^k}\Big)^2 dF(\lambda_1, \lambda_2, \ldots, \lambda_d).$$

Therefore

(2.5) $$\sum_{j=0}^{\infty}\Big(\int\limits_{(\lambda_1,\lambda_2,\ldots,\lambda_d)\in\square_{(3/2)2^j}-\square_{2^j}}\cdots\int dF(\lambda_1, \lambda_2, \ldots, \lambda_d)\Big)^{1/2} < \infty .$$

Also, let us take $T_k = 2\cdot 2^{k-1}$ and $a = \prod_{k=0}^{\infty}(1-\frac{1}{3}2^{-k+1})^{2d}$.

Since

$$2a \int_{(\lambda_1,\lambda_2,\ldots,\lambda_d)\in\square_{2^{j+1}}-\square_{(3/2)2^j}} \!\!\cdots\!\! \int dF(\lambda_1, \lambda_2, \ldots, \lambda_d)$$
$$\leqslant \int_{(\lambda_1,\lambda_2,\ldots,\lambda_d)\in\square_{2^{j+1}}-\square_{2^j}} \!\!\cdots\!\! \int \prod_{k=j+1}^{\infty} \prod_{i=1}^{d} \left(1-\frac{|\lambda_i|}{3\cdot 2^{k-1}}\right)^2 dF(\lambda_1, \lambda_2, \ldots, \lambda_d),$$

we have

$$\sum_{j=1}^{\infty}\Big(\int_{(\lambda_1,\ldots,\lambda_d)\in\square_{2^{j+1}}-\square_{(3/2)2^j}} \!\!\cdots\!\! \int dF(\lambda_1, \lambda_2, \ldots, \lambda_d)\Big)^{1/2} < \infty. \tag{2.6}$$

Moreover (2.5) and (2.6) yield

$$\sum_{j=0}^{\infty}\Big(\int_{(\lambda_1,\ldots,\lambda_d)\in\square_{2^{j+1}}-\square_{2^j}} \!\!\cdots\!\! \int dF(\lambda_1, \ldots, \lambda_d)\Big)^{1/2} < \infty.$$

Finally, we note that

$$\Big(\int_{2^j\leqslant|\lambda|<2^{j+1}} \!\!\cdots\!\! \int dF(\lambda_1, \ldots, \lambda_d)\Big)^{1/2}$$
$$\leqslant \Big(\int_{(\lambda_1,\ldots,\lambda_d)\in\square_{2^{j+1}}-\square_{2^j}} \!\!\cdots\!\! \int dF(\lambda_1, \lambda_2, \ldots, \lambda_d)\Big)^{1/2} + \Big(\int_{(\lambda_1,\ldots,\lambda_d)\in\square_{2^j}-\square_{2^{j-1}}} \!\!\cdots\!\! \int dF(\lambda_1, \ldots, \lambda_d)\Big)^{1/2}$$
$$\ldots + \Big(\int_{(\lambda_1,\lambda_2,\ldots,\lambda_d)\in\square_{2^{j-N'+1}}-\square_{2^{j-N'}}} \!\!\cdots\!\! \int dF(\lambda_1, \lambda_2, \ldots, \lambda_d)\Big)^{1/2},$$

where N' is defined by the relation $2^{N'-1} \leqslant \sqrt{d} \leqslant 2^{N'}$.

Consequently

$$\sum_{j=0}^{\infty}\Big(\int_{2^j\leqslant|\bar\lambda|\leqslant 2^{j+1}} \!\!\cdots\!\! \int dF(\lambda_1, \ldots, \lambda_d)\Big)^{1/2}$$
$$\leqslant N' \sum_{j=1}^{\infty}\Big(\int_{(\lambda_1,\ldots,\lambda_d)\in\square_{2^{j+1}}-\square_{2^j}} \!\!\cdots\!\! \int dF(\lambda_1, \ldots, \lambda_d)\Big)^{1/2} < \infty,$$

which proves Theorem 1.

§ 3. Proof of Theorem 2. Proof of Theorem 2 can be carried out in the same way as in Nisio [7], but for the sake of completeness, we will give it with a slight modification. At first, suppose that $\{s_n\}$ itself is decreasing and $\sum_{n=1}^{\infty} \sqrt{s_n} < \infty$.

Let $c(j) = c^{2^j}$ and put, for $j = 1, 2, \ldots$:

$$X_j(\bar t) = \int_{c(j-1)\leqslant|\bar\lambda|\leqslant c(j)} \!\!\cdots\!\! \int e^{i(\bar t,\bar\lambda)}\, dF(\bar\lambda).$$

Since $E(|(\bar t+\bar h)-x_j(\bar t)|^2) \sim h^2$, $X_j(\bar t)$ has continuous paths. For $\bar t = (t_1, \ldots, t_d)$, we can write for $i = 1, 2, \ldots, d$, and $j = 1, 2, \ldots$, $t_i = \sum_{k=j+1}^{\infty} \frac{l(i,k)}{c(k)}$, where $l(i,k)$ are integers satisfying $0 \leqslant l(i, j+1) < c(j+1)$ and $0 \leqslant l(i,k) < \frac{c(k)}{c(k-1)}$ $(k = j+2, \ldots)$. Now, putting

$$\bar t_k = \Big(\sum_{r=j+1}^{\infty} \frac{l(1,r)}{c(r)}, \ldots, \sum_{r=j+1}^{\infty} \frac{l(d,k)}{c(r)}\Big),$$

we have

$$X_j(\bar t) = X_j(\bar t_{j+1}) + \sum_{k=j+1}^{\infty} \big(X(\bar t_{k+1}) - X(\bar t_k)\big).$$

If we denote

$$\eta_j = \max_{0\leqslant l_i<c(j+1)} \Big|X_j\Big(\frac{l_1}{c(j+1)}, \ldots, \frac{l_d}{c(j+1)}\Big)\Big|,$$

$$\zeta(\bar p, \bar q, k) = X_j\Big(\frac{p_1}{c(k)} + \frac{q_1}{c(k+1)}, \ldots, \frac{p_d}{c(k)} + \frac{q_d}{c(k+1)}\Big) - X_j\Big(\frac{p_1}{c(k)}, \ldots, \frac{p_d}{c(k)}\Big),$$

and

$$\theta_k = \max_{\substack{0\leqslant p_i<c(k)\\ 1\leqslant q_i<\frac{c(k+1)}{c(k)}}} |S(\bar p, \bar q, k)|,$$

we have

$$\sup_{t\in\square_1} |X_j(\bar t)| \leqslant \eta_j + \sum_{k=j+1}^{\infty} \theta_k.$$

Now, using the estimate of Delporte (Theorem 3.5.2.A, (p. 154) in [1]), we have, if we put $\sigma_j^2 = F(S_{c(j)}) - F(S_{c(j+1)})$,

$$E(\eta_j) \leqslant A\{\log c(j+1)\}^{1/2}\sigma_j$$

and

$$E(\theta_k) \leqslant A\{\log c(k+1)\}^{1/2} \sup_{\substack{0\leqslant p_i<c(k)\\ 0\leqslant q_i<\frac{c(k+1)}{c(k)}}} E\big(|S(\bar p, \bar q, k)|^2\big)^{1/2},$$

where A is an absolute constant.

On the other hand, we have

$$E(|S(\bar{p},\bar{q},k)|^2) = 2 \int\limits_{c(j-1)\leqslant|\bar{\lambda}|\leqslant c(j)} \left(1-\cos\left(\frac{\bar{q}}{c(k+1)},\bar{\lambda}\right)\right) dF(\bar{\lambda}) \leqslant \left(\frac{c(j)}{c(k)}\right)^2 \sigma_j^2.$$

Therefore,

$$E(\theta_k) \leqslant A\{\log c(k+1)\}^{1/2}\left(\frac{c(j)}{c(k)}\right)\sigma_j,$$

and consequently

$$E(\|X\|) \leqslant \sum_{j=1}^{\infty} E(\|X_j\|) + E\left(\sup_{t\in\square_1}\left|\int\limits_{0\leqslant|\lambda|\leqslant 2^2} e^{i(\bar{t},\bar{\lambda})} dF(\bar{\lambda})\right|\right).$$

The second term of the right-hand side is finite. On the other hand,

$$\sum_{j=1}^{\infty} E(\|X_j\|) \leqslant A\sum_{j=1}^{\infty}\{\log c(j+1)\}^{1/2}\sigma_j + A\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty}\{\log c(k+1)\}^{1/2}\left(\frac{c(j)}{c(k)}\right)\sigma_j \leqslant 2A\sum_{j=1}^{\infty} 2^{j/2}\sigma_j.$$

Since we can show that

$$2^{j/2}\sigma_j \leqslant 4\sum_{k=2^{j-2}}^{2^{j-1}}\sqrt{s_k},$$

we have

$$E(\|X\|) < \infty,$$

which implies the continuity of paths by Yu. K. Belyaev's theorem (see, for example, Theorem 5 in Jain and Kallianpur [3]).

Let us now pass to the general case. Put

$$G(A) = F(A) + \sum_{n=0}^{\infty}(M_n - s_n)\,\delta_{\partial S_{2n+1}}(A), \qquad A \in B(\boldsymbol{R}^n),$$

where $\delta_{\partial S_{2n+1}}(\cdot)$ is the uniform probability measure concentrated on $\partial S_{2n+1} = \{\bar{\lambda};\ |\lambda| = 2^{n+1}\}$ and $B(\boldsymbol{R}^n)$ are Borel sets in $\boldsymbol{R}^n$. We can construct independent stationary Gaussian processes X_1 and X_2 such that

$$\varrho_1(\bar{t}) = E\big(X_1(\bar{t}+\bar{s})X_1(\bar{s})\big) = \int\limits_{\boldsymbol{R}^n} e^{i(\bar{t},\bar{\lambda})} dF(\bar{\lambda}),$$

$$\varrho_2(\bar{t}) = E\big(X_2(\bar{t}+\bar{s})X_2(\bar{s})\big) = \int\limits_{\boldsymbol{R}^n} e^{i(\bar{t},\bar{\lambda})} dH(\bar{\lambda}),$$

where $H(A) = \sum_{n=0}^{\infty}(M_n - s_n)\,\delta_{\partial S_{2n+1}}(A)$. Then, G is the spectral measure of the covariance function of the process X_1+X_2 and $G(S_{2^{n+1}}) - G(S_{2^n}) = M_n$. We have therefore

$$P(\|X_1+X_2\| < \infty) = 1.$$

Using again Delporte's lemma, we obtain

$$P(\|X_1\| < \infty) = 1,$$

which implies the path continuity of the process X_1.

§ 4. Examples.

EXAMPLE 1. Let X be a real, separable, stationary Gaussian process with the covariance function $\varrho(\bar{h}) = \varrho_1(h_1)\varrho_2(h_2)\ldots\varrho_d(h_d)$, where $\bar{h} = (h_1, h_2, \ldots, h_d)$ and ϱ_i is the covariance function of a stationary Gaussian process with a 1-dimensional parameter space. When some ϱ_i are covariance functions which statisfy the condition of Theorem 3, X has discontinuous paths in virtue of Theorem 3.

Let $X = \{X(p), p\in H\}$ be a stationary Gaussian process with zero mean, where H is a Hilbert space. If $\varrho(p) = E\big(X(p+q)X(q)\big)$, $p, q\in H$, is continuous with respect to the S-topology (see, for example, Parthasarathy [8]), $\varrho(p)$ take the form

$$\varrho(p) = \int\limits_{H} e^{i(\lambda,p)} dF(\lambda).$$

Then, we can propose similar problems to Theorems 1 and 2.

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Received November 10, 1972 (615)

STUDIA MATHEMATICA, T. XLIX. (1973)

Sections induced from weakly sequentially complete spaces*

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Abstract. It is shown that function algebras are never weakly sequentially complete (unless finite dimensional) and then sections induced from maps from weakly sequentially complete spaces onto function algebras are studied. As a result, it is shown that for an infinite Helson set E the restriction map ϱ of the Fourier algebra $A(G)$ (that is, $L^2(G)*L^2(G)$) of a locally compact (not necessarily abelian) group onto the space $C(E)$ of continuous functions on E never admits a section π, (that is, a continuous linear map $\pi\colon C(E)\to A(G)$ with $\varrho\circ\pi = \mathrm{id}$). A set $E\subset G$ is called a Helson set provided $A(G)|E = C(E)$. A similar application to Sidon sets in the dual of a compact group is also given.

THEOREM 1. *Let A be a weakly sequentially complete commutative Banach algebra. If A is isomorphic to a closed subalgebra $\tilde{A}$ of $C_0(S)$, the continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space, then A is finite-dimensional.*

Proof. If A is infinite-dimensional, then there exists an infinite-dimensional separable subalgebra which is weakly sequentially complete. Thus we may assume that A is separable.

If $\tilde{A}$ does not separate the points of S, we embed A instead into $C_0(S/\sim)$, where for $s,t\in S$, $s\sim t$ if and only if $\tilde{f}(s)=\tilde{f}(t)$ for all $f\in A$. Thus we may assume that $\tilde{A}$ separates the points in S and hence in the Shilov boundary ∂A (since $\partial A\subset S$). Thus $\partial A\subset S$ is a metrizable locally compact space.

Let $P\subset\partial A\subset S$ denote the set of peak points of A. The set P is dense in ∂A (Bishop's theorem ([6], p. 56) since A is metrizable. It will thus suffice to show that P is finite: for then ∂A will be finite (and equal to P), and A is isomorphic to $\tilde{A}|\partial A$.

By the Lebesgue dominated convergence theorem, given a sequence $\{f_n\}\subset A$ with $\|\tilde{f}_n\|_\infty\leqslant 1$ and $\tilde{f}_n\xrightarrow{n}\chi_p$ (the characteristic function of the set $\{p\}$, $p\in P$) pointwise on S, it follows that $\{\tilde{f}_n\}$ is weakly Cauchy in $\tilde{A}$ ($\cong A$). Hence, by the weak sequential completeness of A, $\chi_p\in\tilde{A}$. Thus P consists of isolated points.

* This research was supported in part by NSF contract GP-31483X.