STUDIA MATHEMATICA, T. XLIX. (1973)

A theorem of Cesari on multiple Fourier series

by

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Abstract. We extend an important result of L. Cesari on almost everywhere convergence of multiple Fourier series from 3 dimensions to n dimensions and extend the class of functions from generalized bounded variation, $GBV(T^n)$, with variation functions in $L_p, p > 1$, to variation functions in $L(\mathrm{Log}^+ L)^{n-2}$. The work of Cesari is rather inaccessible, difficult to read, and the extension to n variables is not obvious.

1. The primary purpose of this paper is to reacquaint the mathematical public with a little known but very interesting and important result of L. Cesari [3] on the almost everywhere convergence of multiple Fourier series. A secondary purpose is to extend Cesari's theorem from 3 dimensions to n dimensions and the class of functions from GBV with variation functions in L_p, $p > 1$, to variation functions in $L(\log^+ L)^{n-2}$, where the meaning of these classes is given in the sequel. The basic ideas are those of Cesari. In several places certain adaptations are necessary in order to obtain the extensions. The result of Cesari for $n = 3$ is remarkable and his proof shows an appreciation of the various subtle possible pitfalls. Nevertheless, the work is rather inaccessible, difficult to read, and the extension to n variables is not obvious. The result is sufficiently important to warrant a separate treatment for n variables.

2. Consider the n torus $T^n = [0, 2\pi) \times \ldots \times [0, 2\pi)$ of points $x = (x_1, \ldots, x_n)$. The class V_1 consists of those real functions on T^n whose partial derivatives, in the distribution sense, are totally finite measures. This class was introduced by Tonelli [7] for continuous functions and by Cesari [1] for the general case. It was used by Tonelli in area theory for nonparametric continuous surfaces (two variables) [7], and in double Fourier series [8]. It was used by Cesari in area theory for nonparametric arbitrary surfaces (two variables) [1], in double Fourier series [2], and triple Fourier series [3]. Tonelli in [9], [10] re-elaborated part of Cesari's work. G. Torrigiani [11] and A. M. Romano [6] later treated particular facets of the problem under discussion.

* The author is grateful to Professor Casper Goffman for his encouragement and invaluable help in making this paper possible.

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It is convenient to have the following notation: For each $i = 1, 2, \ldots, n$, we designate points in $(n-1)$ space with coordinates $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ as $\bar{x}_i$, and, for $i < j$, points in $(n-2)$ space with coordinates $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ as $\bar{x}_{ij}$. Thus a point in n space may be designated as $(x_i, \bar{x}_i)$, or $(x_i, x_j, \bar{x}_{ij})$, for $i, j = 1, \ldots, n$. We also write $dx = dx_i d\bar{x}_i = dx_i dx_j d\bar{x}_{ij}$. For any interval $I \subset \boldsymbol{R}^n$ we may write $I = I_i \times \bar{I}_i = I_i \times I_j \times \bar{I}_{ij}$, where $\bar{I}_i$ is an $(n-1)$ interval and $\bar{I}_{ij}$ an $(n-2)$ interval.

Suppose that f is a measurable function on $\boldsymbol{R}^n$ of period 2π in x_i, $i = 1, \ldots, n$. The class V_1 of functions $f(x)$, $x \in T^n$, under consideration (functions of *generalized bounded variation*) was introduced by Cesari in [1], [2], [3] by using suitably defined total variations $V_{x_i}(x_i, \bar{x}_i)$ of $f(x_i, \bar{x}_i)$ as a function of x_i alone. For the purpose of the present paper the form of the definition proposed by C. Goffman in [4] is preferable. Thus, we shall say that $f(x)$, $x \in T^n$, is of generalized bounded variation $\mathrm{GBV}(T^n)$, if for each $i = 1, \ldots, n$, there is an f_i equivalent to f, which is of bounded variation in x_i for almost all $\bar{x}_i$, and the variation function $V_{x_i}([0, 2\pi], x_i) = V_{x_i}(\bar{x}_i)$ is summable in $\bar{x}_i$ on T^{n-1}. For $(x_i, \bar{x}_i) \in T^n$ we designate by $V_{x_i}(x_i, \bar{x}_i)$ the total variation in $[0, x_i]$ of f_i considered as a function of x_i, and by E_f the set of which $f \neq f_i$ for some i. It is well known that $f \in \mathrm{GBV}(T^n)$ implies $f \in L_1(T^n)$.

The notion pseudo uniform convergence, introduced by Cesari [2], [3] refers to a sequence

$$k(x) \geqslant k_1(x) \geqslant k_2(x) \geqslant \ldots \geqslant k_r(x) \geqslant \ldots \geqslant 0 \tag{1}$$

of measurable functions on an interval $J \subset \boldsymbol{R}^n$. Let $g_{r,\varepsilon}$ be the characteristic function of the set for which $k_r(x) \geqslant \varepsilon$, $\varepsilon > 0$. Let

$$\chi(x) = \lim_{\varepsilon\to 0}\lim_{r\to\infty}\overline{\lim_{\substack{\delta_i\to 0\\ i=1,\ldots,n}}}\frac{1}{2^n\delta_1\delta_2\ldots\delta_n}\int_{J'_x} g_{r,\varepsilon}(a)k(a)\,da, \tag{2}$$

$$\chi'(x) = \lim_{\varepsilon\to 0}\lim_{r\to\infty}\overline{\lim_{\substack{\delta_i\to 0\\ i=1,\ldots,n}}}\frac{1}{2^n\delta_1\delta_2\ldots\delta_n}\int_{J'_x} g_{r,\varepsilon}(a)\,da, \tag{3}$$

where $x \in J$ and $J'_x = [x_1-\delta_1, x_1+\delta_1] \times \ldots \times [x_n-\delta_n, x_n+\delta_n] \subset J$. We say that $k_r(x)$ *converges pseudo uniformly* to 0 at x_0 with respect to $k(x)$ if $\chi(x_0) = 0$ and $\chi'(x_0) = 0$.

LEMMA 1. *If the sequence $k_r(x)$ in (1) converges to 0 a. e. in J and $k(x) \in L(\log^+ L)^{n-1}(J)$, then $k_r(x)$ converges pseudo uniformly to 0 a. e. with respect to $k(x)$.*

Proof. The proof is essentially the same as that of Cesari [2], [3]. We need only note that since $k(x) \in L(\log^+ L)^{n-1}(J)$, the indefinite integral of $k(x)$ is strongly differentiable and the strong derivative is equal to $k(x)$ a.e. ([12], p. 306).

From the above lemma, we have the following corollary:

COROLLARY 1. *If $f(x) \in \mathrm{GBV}(T^n)$ and $V_{x_i}(\bar{x}_i) \in L(\log^+ L)^{n-2}(\bar{T}^n_i)$ for each $i = 1, 2, \ldots, n$, then for a.e. $x_0 = (x_1^0, \ldots, x_i^0, \ldots, x_n^0) \in T^n$*

$$\lim_{\theta\to 0+} V_{x_i}(x_i^0+\theta, \bar{x}_i) - V_{x_i}(x_i^0-\theta, \bar{x}_i) = 0, \tag{4}$$

where the limit exists with respect to a. e. $\bar{x}_i$. For a. e. $\bar{x}_i^0$ (4) is pseudo uniformly convergent to 0 with respect to $V_{x_i}(\bar{x}_i)$ at $\bar{x}_i^0$.

LEMMA 2. *If $f(x) \in \mathrm{GBV}(T^2)$ then for a. e. $x \in T^2$, the iterated limit $\lim_{\sigma_1\to 0+}\lim_{\sigma_2\to 0+} f(x_1+\sigma_1, x_2+\sigma_2)$ exists at x and is equal to $f(x)$, where a set of measure 0 may be ignored in taking the limits. Similarly, if $f(x) \in \mathrm{GBV}(T^n)$, then for a. e. $x \in T^n$ the $2^n n!$ iterated limits exist and are equal to $f(x)$.*

Proof. We need only consider $f(x) \in \mathrm{GBV}(T^2)$. Let

$$\psi(x_2) = \int_0^{2\pi} V_{x_2}(x_1, x_2)\,dx_1.$$

Since ψ is increasing, it is continuous a.e. in x_2, so that, for a.e. x_1, we have $V_{x_2}(x_1, x_2) = V_{x_2}(x_1, x_2+)$. So for a. e. x_1, we have $f(x_1, x_2) = f(x_1, x_2+)$ for a. e. x_2. Similarly, for a. e. x_2, we have $f(x_1, x_2) = f(x_1+, x_2)$ for a. e. x_1. By noting that these iterated limits are measurable functions defined for a. e. $x \in T^2$, ([5], Lemma 2, p. 363), the conclusion holds for a. e. $x \in T^2$.

3. In the remaining sections, we shall prove the following theorem:

THEOREM 1. *For $n \geqslant 2$, let $f(x) \in \mathrm{GBV}(T^n)$ and $V_{x_i}(\bar{x}_i) \in L(\mathrm{Log}^+ L)^{n-2}(\bar{T}^n_i)$ for each $i = 1, 2, \ldots, n$. The n-tuple Fourier series of $f(x)$ converges to $f(x)$ a.e., using rectangular summation.*

Remark. For $n = 2$, see [2] or [7]. We need only prove the theorem for $n \geqslant 3$.

4. Let $f(x) \in \mathrm{GBV}(T^n)$ and $x_0 = (x_1^0, \ldots, x_i^0, \ldots, x_n^0) \in T^n$ be chosen so that the following conditions hold:

(5) *All $2^n n!$ iterated limits of $f(x)$, defined as in Lemma 2, exist at x_0 and are equal to $f(x_0) = l$.*

(6) *For each $i = 1, 2, \ldots, n$, $\lim_{\theta\to 0+} V_{x_i}(x_i+\theta, \bar{x}_i) - V_{x_i}(x_i-\theta, \bar{x}_i) = 0$ is pseudo uniformly convergent to 0 with respect to $V_{x_i}(\bar{x}_i)$ at $\bar{x}_i^0$, where the limit is defined as in Corollary 1.*

PROPOSITION 1. *If (5) and (6) hold for $f(x)$ at a point $x_0 \in T^n$, then for each $\sigma > 0$, there exists $\delta = \delta(\sigma) > 0$ such that for any positive integers $h_1, \ldots, h_n$,*

$$|L| = \left| \int_0^\delta \ldots \int_0^\delta [f(x_0+2u)-l] \prod_{i=1}^n \frac{\sin h_i u_i}{\sin u_i} du \right| < \sigma.$$

In order to prove the proposition, we introduce some notation, definitions, and elementary lemmas.

Let ^+Z be the set of all non-negative integers and $^+Z^m$ the Cartesian product of m copies of ^+Z. If ν is any strictly increasing function from $\{1, \ldots, k\}$ into $\{1, \ldots, m\}$ and $a = (a_i) = (a_1, \ldots, a_{\nu_1}, \ldots, a_{\nu_j}, \ldots, a_{\nu_k}, \ldots, a_m) \in {}^+Z^m$, write $(a_{\nu_1}, \ldots, a_{\nu_j})$ as a_ν^j $[a_\nu$ if $j = k]$, $(a_1, \ldots, a_{\nu_j-1}, a_{\nu_j+1}, \ldots, a_m)$ as $\bar{a}_{\nu_j}$, and $(a_1, \ldots, a_{\nu_1-1}, a_{\nu_1+1}, \ldots, a_{\nu_j-1}, a_{\nu_j+1}, \ldots, a_{\nu_k}, \ldots, a_m)$ as $\bar{a}_\nu^j$ $[\bar{a}_\nu$ if $j = k]$. Also, $0 = (0, \ldots, 0)$ and $1 = (1, \ldots, 1)$. Thus, we may write $(a_\nu^j, \bar{a}_\nu^j)$ or $(a_\nu, \bar{a}_\nu)$ as a, $(a_1, \ldots, a_{\nu_j-1}, b_{\nu_j}, a_{\nu_j+1}, \ldots, a_m)$ as $(b_{\nu_j}, \bar{a}_{\nu_j})$, and $(a_1, \ldots \ldots, a_{\nu_1-1}, b_{\nu_1}, a_{\nu_1+1}, \ldots, a_{\nu_j-1}, b_{\nu_j}, a_{\nu_j+1}, \ldots, a_{\nu_k}, \ldots, a_m)$ as (b_ν^j, a_ν^j) $[(b_\nu, \bar{a}_\nu)$ if $j = k]$, where $b = (b_i) \in {}^+Z^m$.

We need the trivial modifications for the above notions if $j = k$ or $\nu_1 = 1$ or $\nu_k = m$.

Note that $a_\nu^j \in {}^+Z^j$ $(a_\nu \in {}^+Z^k)$, $\bar{a}_{\nu_j} \in {}^+Z^{m-1}$, $\bar{a}_\nu^j \in {}^+Z^{m-j}$ $(\bar{a}_\nu \in {}^+Z^{m-k})$.

Definition 1. If β is a complex function defined on $^+Z^m$, we write $\beta(d)$ as β_d for each $d \in {}^+Z^m$. Let a, b, ν be defined as the above. We define Δ inductively as follows: Let

$$\Delta_{(b_{\nu_j})} \beta_{(b_{\nu_j}, \bar{a}_{\nu_j})} = \beta_{(b_{\nu_j}, \bar{a}_{\nu_j})} - \beta_{(b_{\nu_j}+1, \bar{a}_{\nu_j})},$$

assuming that $\Delta_{b_\nu^j} \beta_{(b_\nu^j, \bar{a}_\nu^j)}$ has been defined for each $j = 1, \ldots, k-1$, let

$$\Delta_{b_\nu^{j+1}} \beta_{(b_\nu^{j+1}, \bar{a}_\nu^{j+1})} = \Delta_{b_\nu^j} [\Delta_{(b_{\nu_{j+1}})} \beta_{(b_{\nu_{j+1}}, \bar{c}_{\nu_{j+1}})}],$$

where $c = (c_i) \in {}^+Z^m$, satisfying $(b_{\nu_{j+1}}, \bar{c}_{\nu_{j+1}}) = (b_\nu^{j+1}, \bar{a}_\nu^{j+1})$.

Definition 2. Let $a = (a_i)$, $b = (b_i) \in {}^+Z^m$; we say that $a \leqslant (=) b$ if and only if $a_i \leqslant (=) b_i$ for each $i = 1, \ldots, m$. Obviously, $0 \leqslant a$ for any $a \in {}^+Z^m$.

Lemma 3. *Let α, β be complex functions defined on $^+Z^m$, and $c = (c_i)$, $d = (d_i)$, $p = (p_i) \in {}^+Z^m$. Let*

$$A_d = \sum_{c=0}^{d} \alpha_c \quad \text{and} \quad S_p = \sum_{d=0}^{p} \alpha_d \beta_d;$$

then

$$S_p = \sum_{d=0}^{p-1} A_d \Delta_d \beta_d + \ldots + \sum_\nu \Big[\sum_{\bar{d}_\nu=0}^{(\overline{p-1})_\nu} A_{(p_\nu, \bar{d}_\nu)} \Delta_{\bar{d}_\nu} \beta_{(p_\nu, \bar{d}_\nu)} \Big] + \ldots + A_p \beta_p,$$

where the summation $\sum_\nu$ is taken over those ν satisfying $1 \leqslant \nu_1 < \ldots < \nu_j \leqslant m$.

Proof. Simply apply the Abel-transformation for $m = 1$, and then use induction for the general case.

Definition 3. Let $d = (d_i) \in {}^+Z^m$; set

$$\eta_d = \begin{cases} 1 & \text{if } d = 0, \\ \dfrac{1}{\max\limits_i \{d_i\} \prod\limits_i d_i} & \text{otherwise,} \end{cases}$$

where the maximum $\max\limits_i$ and the product $\prod\limits_i$ are taken over those i for which $d_i \neq 0$.

Using the definitions Δ and η, we immediately have the following lemma:

Lemma 4. *Let $a, b \in {}^+Z^m$, and μ any strictly increasing function from $\{1, \ldots, k\}$ into $\{1, \ldots, m\}$. Then*

$$\Delta_{b_\mu} \eta_{(b_\mu, \bar{a}_\mu)} = 0 \quad \text{if any component of } b_\mu \text{ is } 0,$$

and

$$|\Delta_{\bar{b}_\mu} \eta_{(a_\mu, \bar{b}_\mu)}| < \frac{2^{k+1}}{\max\limits_i \{b_i\} \prod\limits_i b_i} \frac{1}{\prod\limits_i [b_i+1] \prod\limits_{j=1}^{k} [a_{\mu_j}+1]},$$

where all the components of $\bar{b}_\mu$ are positive, and the maximum $\max\limits_i$ and the product $\prod\limits_i$ are taken over all the components of $\bar{b}_\mu$.

We now prove Proposition 1.

Given $\sigma > 0$, there exists $0 < \varepsilon < 1$ such that

$$\pi^{2n} \left[\frac{3}{2^{n-2}} (n-1) + 1 \right] n! \left\{ 1 + \sum_{r=1}^{\infty} \frac{(1+\log r)^{n-2}}{r^2} \right\} \varepsilon < \sigma. \tag{7}$$

From (5) we can pick $\theta_0 > 0$ such that if $x_i^0 \leqslant x_i \leqslant x_i^0 + \theta_0$ for each $i = 1, \ldots \ldots, n$, then

$$|f(x_i^0+, \bar{x}_i) - l| < \varepsilon \quad \text{for a.e. } \bar{x}_i.$$

Let $g_{0,\varepsilon}(\bar{x}_i)$ be the characteristic function of the set for which

$$V_{x_i}(x_i^0 + 0, \bar{x}_i) - V_{x_i}(x_i^0+, \bar{x}_i) \geqslant \varepsilon.$$

From (6) there exists $\bar{\theta}(\varepsilon)$ and $\bar{\delta}(\bar{\theta}, \varepsilon)$ such that if $0 < \theta \leqslant \bar{\theta}(\varepsilon)$ and $0 < \delta_i \leqslant \bar{\delta}(\bar{\theta}, \varepsilon)$, then

$$\frac{1}{\delta_1 \ldots \delta_{i-1} \delta_{i+1} \ldots \delta_n} \int_{\dot{I}_i} g_{0,\varepsilon}(\bar{u}_i) [V_{x_i}(\bar{u}_i) + |l| + 1] d\bar{u}_i < \varepsilon$$

for each $i = 1, \ldots, n$, where $I = \prod_{i=1}^n [x_i^0, x_i^0 + \delta_i]$, and $u = (u_i) \in I$.

Choose $\delta > 0$ so that $3\delta < \min\{\theta_0, \overline{\theta}(\varepsilon), \overline{\delta}(\overline{\theta}, \varepsilon)\}$ and the hyperplanes $x_i = x_i^0 + 2\delta$, $i = 1, \ldots, n$, meet E_f in sets of measure 0. For simplicity we may assume $x_0 = 0$.

Let q_i be the largest integers for which $q_i\pi/h_i < \delta$.

Let $I_\delta^n = \underbrace{[0, \delta] \times \ldots \times [0, \delta]}_{n \text{ copies}}$, and $K_d = \prod_{i=1}^{n} \left[\frac{d_i\pi}{h_i}, \frac{(d_i+1)\pi}{h_i}\right]$ if $d = (d_i) \in {}^+Z^n$.

Set

$$D_d = K_d \cap I_\delta^n.$$

Designate

$$\lambda_d = \int_{D_d} [f(2u) - l] \prod_{i=1}^{n} \frac{\sin h_i u_i}{\sin u_i} du.$$

Note that our integral may be written as

$$L = \sum_{d=0}^{q} \lambda_d,$$

where $q = (q_i) \in {}^+Z^n$.

For $n = 3$, Cesari [3] splits L into 3 different groups. $L_1^{(1)}, L_1^{(2)}$; $L_2^{(1)}, L_2^{(2)}$; $L_3^{(1)}, L_3^{(2)}$, and analyses each group

$$L = L_1^{(1)} + L_1^{(2)} + \ldots + L_3^{(2)},$$

where $L_i^{(k)}$ is indicated as follows:

$L_1^{(1)}$ is the sum of those $\lambda_{(d_i)}$ with components satisfying $d_1 \leqslant d_2 \leqslant d_3$, $L_1^{(2)}$ is the sum of those $\lambda_{(d_i)}$ satisfying $d_2 < d_1 \leqslant d_3$. Similarly, $L_2^{(1)}, L_2^{(2)}$, $L_3^{(1)}, L_3^{(2)}$ are taken over those components satisfying $d_1 \leqslant d_3 < d_2$, $d_3 < d_1 \leqslant d_2$, $d_3 \leqslant d_2 < d_1$, $d_2 < d_3 < d_1$, respectively. See Figure 1.

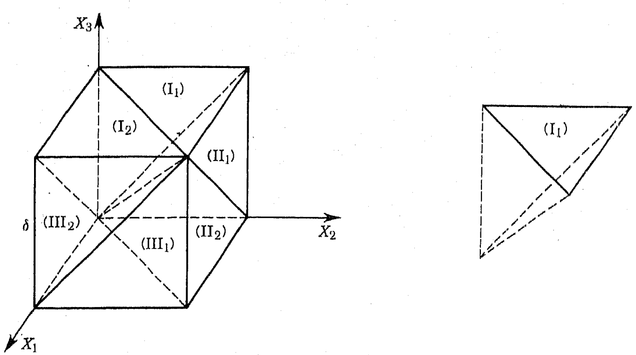

Figure 1

Geometrically, $L_1^{(1)}$ is the sum of all integrals over those rectangular parallelepiped cuboids with vertices $\left(\frac{d_1\pi}{h_1}, \frac{d_2\pi}{h_2}, \frac{d_3\pi}{h_3}\right)$ and $\left(\frac{(d_1+1)\pi}{h_1}, \frac{(d_2+1)\pi}{h_2}, \frac{(d_3+1)\pi}{h_3}\right)$ which meet the tetrahedron (I_1) with non-zero volume; $L_1^{(2)}$ is over those which are not in $L_1^{(1)}$ and meet the tetrahedron (I_2) with non-zero volume. We can interpret $L_2^{(1)}, L_2^{(2)}, L_3^{(1)}$ and $L_3^{(2)}$, similarly. For $n > 3$, we extend Cesari's method to split L into n different groups as follows:

$$L_j^{(1)}, \ldots, L_j^{(k)}, \ldots, L_j^{(n-1)!}, \quad j = 1, 2, \ldots, n,$$

where $L_j^{(1)}$ is the sum of those λ_d with components satisfying

(8) $$d_1 \leqslant \ldots \leqslant d_{n-j} \leqslant d_n \leqslant d_{n-j+2} \leqslant \ldots \leqslant d_{n-2} < d_{n-j+1},$$

and each $L_j^{(k)}$ is formed by a suitable permutation of the components of $\bar{d}_{n-j+1}$ in (8) with proper sign "$\leqslant$" or "$<$" between them. We need the obvious modifications in (8) for $j = 1, 2$ or n.

We consider $L_1^{(1)}$ in detail. Let

$$f^0(x) = \begin{cases} f(x) & \text{if } g_{2\delta,s}(\bar{x}_n) = 0, \\ l & \text{if } g_{2\delta,s}(\bar{x}_n) = 1, \end{cases}$$

where $x = (x_1, \ldots, x_n) \in I_{2\delta}^n = [0, 2\delta] \times \ldots \times [0, 2\delta]$. Write

$$f(x) = f^0(x) + \psi(x) \quad \text{if } x \in I_{2\delta}^n,$$

$$\lambda_d^0 = \int_{D_d} [f^0(2u) - l] \prod_{i=1}^{n} \frac{\sin h_i u_i}{\sin u_i} du,$$

$$\gamma_d = \int_{D_d} \psi(2u) \prod_{i=1}^{n} \frac{\sin h_i u_i}{\sin u_i} du.$$

Thus, $\lambda_d = \lambda_d^0 + \gamma_d$.

The decomposition $f = f^0 + \psi$ gives the decomposition

$L_i^{(k)} = L_i^{0(k)} + l_i^{(k)}$ for each $i = 1, \ldots, n$ and each $k = 1, \ldots, (n-1)!$.

Let

$$\Phi(\bar{\varepsilon}_n') = \int_0^{\varepsilon_1'} \ldots \int_0^{\varepsilon_{n-1}'} g_{2\delta,s}(2\bar{u}_n) V_{x_n}(2\bar{u}_n) + |l| + 1 \, d\bar{u}_n,$$

$$\varphi_{\bar{d}_n} = \Phi\left(\frac{d_1\pi}{h_1}, \ldots, \frac{d_{n-1}\pi}{h_{n-1}}\right),$$

$$\tau_{\bar{d}_n} = \begin{cases} 0 & \text{if } \bar{d}_n = 0 \\ \dfrac{h_1 \ldots h_{n-1}}{\Pi^{n-1} d_1 \ldots d_{n-1}} \varphi_{\bar{d}_n} & \text{otherwise,} \end{cases}$$

where $\varepsilon' = (\varepsilon_i')$, $u = (u_i)$ and $d = (d_i)$.

From arguments similar to those for the 2-dimensional case [2], we obtain the following:

$$\Big|\sum_{d_n=0}^{q_n}\lambda_a^0\Big| < \frac{\pi^{2n}}{2^{n-2}}\,\varepsilon,\ \Big|\sum_{d_n=d_{n-1}}^{q_n}\lambda_b^0\Big| < \frac{\pi^{2n-j-1}}{2^{n-2}}\,\frac{\varepsilon}{d_{n-1}^2 d_{n-2}\dots d_{n-j}},$$

$$\Big|\sum_{d_n=0}^{q_n}\gamma_a\Big| \leqslant \frac{3\pi^{2n}}{2^n}\,\frac{h_1\dots h_{n-1}}{\pi^{n-1}}\,\eta_{(0,\dots,0)}(-1)^{n-1}\Delta_{(0,\dots,0)}\varphi_{(0,\dots,0)},$$

$$\Big|\sum_{d_n=d_{n-1}}^{q_n}\gamma_b\Big| \leqslant \frac{3\pi^{2n-j-1}}{2^n}\,\frac{h_1\dots h_{n-1}}{\pi^{n-1}}\,\frac{1}{d_{n-1}^2 d_{n-2}\dots d_{n-j}}\,C_1(\Delta,\varphi),$$

where $a=(0,\dots,0,d_n)$, $b=(b_i)=(0,\dots,0,d_{n-j},\dots,d_n)$ with $d_{n-j},\dots,d_n>0$, and $C_1(\Delta,\varphi)=(-1)^{n-1}\Delta_{\bar b_n}\varphi_{\bar b_n}$. Note that

$$L_1^{(1)} = \sum_{d_{n-1}=0}^{q_{n-1}}\dots\sum_{d_{n-j}=0}^{d_{n-j+1}}\dots\sum_{d_n=d_{n-1}}^{q_n}\lambda_{(d_1,\dots,d_{n-j},\dots,d_n)}.$$

Therefore, we have

$$|L_1^{0(1)}| < \frac{\pi^{2n}\varepsilon}{2^{n-2}}\Big\{1+\sum_{d_{n-1}=1}^{\infty}\frac{1}{d_{n-1}^2}[\dots]\Big[1+\sum_{d_{n-j}=1}^{d_{n-j+1}}\frac{1}{d_{n-j}}\Big][\dots]\Big[1+\sum_{d_1=1}^{d_2}\frac{1}{d_1}\Big]\Big\}$$

$$< \pi^{2n}\Big\{1+\sum_{r=1}^{\infty}\frac{(1+\log r)^{n-2}}{r^2}\Big\}\varepsilon,$$

and

$$|l_1^{(1)}| \leqslant C_2\Big|\sum_{d_{n-1}=0}^{q_{n-1}}\dots\sum_{d_{n-j}=0}^{d_{n-j+1}}\dots\sum_{d_1=0}^{d_2}\eta_{\bar d_n}\Delta_{\bar d_n}\varphi_{\bar d_n}\Big|,$$

where $C_2 = C_2(h_1,\dots,h_{n-1}) = \dfrac{3\pi^{2n}}{2^n}\,\dfrac{h_1\dots h_{n-1}}{\pi^{n-1}}$, and $d=(d_i)\in{}^+Z^n$. Similarly, we have

$$|L_j^{0(k)}| < \pi^{2n}\Big\{1+\sum_{r=1}^{\infty}\frac{(1+\log r)^{n-2}}{r^2}\Big\}\varepsilon$$

for each $j=1,\dots,n$ and each $k=1,\dots,(n-1)!$.

Analogous inequalities hold for $l_1^{(2)},\dots,l_1^{(k)},\dots,l^{(n-1)!}$ and we obtain

$$\Big|\sum_{k=1}^{(n-1)!} l_1^{(k)}\Big| \leqslant C_2\Big|\sum_{\bar d_n=0}^{\bar q_n}\eta_{\bar d_n}\Delta_{\bar d_n}\varphi_{\bar d_n}\Big| = C_2\Big|\sum_{c=0}^{p}\eta_c\Delta_c\varphi_c\Big|,$$

where $c=\bar d_n$, $p=\bar q_n\in{}^+Z^{n-1}$.

Note that if $c'=(c_i')\in{}^+Z^{n-1}$, then

$$\Big|\sum_{c'=0}^{c-1}\Delta_{c'}\varphi_{c'}\Big| = \varphi_c.$$

From Lemma 3 and the first part of Lemma 4, we now have

$$\Big|\sum_{k=1}^{(n-1)!} l_1^{(k)}\Big| \leqslant C_2\Big\{\Big|\sum_{c=1}^{p-1}\varphi_{c+1}\Delta_c\eta_c\Big| + \dots +$$

$$+\sum_{\nu}\Big|\sum_{\bar c_\nu=1}^{(\overline{p-1})_\nu}\varphi_{((p+1)_\nu,(\overline{c+1})_\nu)}\Delta_{\bar c_\nu}\eta_{(p_\nu,\bar c_\nu)}\Big| + \dots + \varphi_{p+1}\eta_p\Big\},$$

where the summation $\sum_\nu$ is taken over those ν satisfying $1\leqslant\nu_1<\dots<\nu_j\leqslant n-1$.

Using the second part of Lemma 4 and noting that $|\tau_c|<\varepsilon$ for $1\leqslant c\leqslant p+1$, we have

$$\frac{h_1\dots h_{n-1}}{\pi^{n-1}}\Big|\sum_{\bar c_\nu=1}^{(\overline{p-1})_\nu}\varphi_{((p+1)_\nu,(\overline{c+1})_\nu)}\Delta_{\bar c_\nu}\eta_{(p_\nu,\bar c_\nu)}\Big|$$

$$< 2^{j+1}[n-(j+1)]!\Big\{1+\sum_{r=1}^{\infty}\frac{(1+\log r)^{n-j-2}}{r^2}\Big\}\varepsilon.$$

Hence, by direct computation, we obtain

$$\Big|\sum_{k=1}^{(n-1)!} l_1^{(k)}\Big| < \frac{3\pi^{2n}}{2^{n-2}}(n-1)!\,(n-1)\Big\{1+\sum_{r=1}^{\infty}\frac{(1+\log r)^{n-2}}{r^2}\Big\}\varepsilon.$$

Similarly, the above inequality holds for $\sum_{k=1}^{(n-1)!} l_i^{(k)}$. Hence, $\Big|\sum_{i,k} l_i^{(k)}\Big| < \dfrac{3\pi^{2n}}{2^{n-2}}n!\,(n-1)\Big\{1+\sum_{r=1}^{\infty}\dfrac{(1+\log r)^{n-2}}{r^2}\Big\}\varepsilon$. Thus, $|L| < \pi^{2n}\Big\{\dfrac{3}{2^{n-2}}(n-1)+1\Big\}n!\Big\{1+\sum_{r=1}^{\infty}\dfrac{(1+\log r)^{n-2}}{r^2}\Big\}\varepsilon < \sigma$.

This completes the proof of Proposition 1.

Using the procedure of the preceding section, we obtain the following corollary:

Corollary 2. *If* (5) *and* (6) *hold for $f(x)$ at $x_0\in T^m$, then for each $\sigma>0$, there exists $\delta>0$ such that for any positive integers $h_1,\dots,h_n$,*

$$|L^*| = \Big|\int_{I_\delta^n}[F(x_0+2u)-2^n l]\prod_{i=1}^{n}\frac{\sin h_i u_i}{\sin u_i}\,du\Big| < \sigma,$$

where $F(x_0+2u) = f(x_0+2u)+\ldots+\sum_{\nu} f\big((x_0-2u)_\nu, \overline{(x_0+2u)_\nu}\big)+\ldots+$ $+f(x_0-2u)$ and the summation $\sum_{\nu}$ is taken over those ν satisfying $1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_j \leqslant n$. Note that here we adapt obvious extensions of the notions for integer coordinates.

5. We now finish the proof of the main theorem stated in § 3.

For fixed i and k such that $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n-2$, and for any strictly increasing function ν from $\{1, \ldots, k\}$ into $\{1, \ldots, n\}$ such that $\nu_j \neq i$ for any $j = 1, \ldots, k$, designate

$$W^k_{x_i}(\bar{x}_\nu) = \underbrace{\int_0^{2\pi} \ldots \int_0^{2\pi}}_{(k\ \text{times})} V_{x_i}(x_\nu, \bar{x}_\nu)\,dx_\nu = \int_0^{2\pi} \ldots \int_0^{2\pi} V_{x_i}\big(x_i, x_\nu, (\bar{x}_\nu)_i\big)\,dx_\nu,$$

where we recall that $x_\nu \in \boldsymbol{R}^k$, $\bar{x}_\nu \in \boldsymbol{R}^{n-k}$ and $(\bar{x}_\nu)_i \in \boldsymbol{R}^{n-k-1}$.

Remark 1. $W^k_{x_i}(\bar{x}_\nu) = W^k_{x_i}\big(x_i, (\bar{x}_\nu)_i\big)$ is non-decreasing in x_i, and

$$W^k_{x_i}(\bar{x}_\nu) \leqslant \int_0^{2\pi} \ldots \int_0^{2\pi} V_{x_i}\big(x_\nu, (\bar{x}_\nu)_i\big)\,dx_\nu = \int_0^{2\pi} \ldots \int_0^{2\pi} V_{x_i}(\bar{x}_i)\,dx_\nu.$$

Remark 2. Let $V_{x_i}(\bar{x}_i) \in L(\mathrm{Log}^+L)^{n-2}(T^{n-1})$ and consider $g(u) = u(\log^+ u)^{n-2}$ $(n \geqslant 3)$. Then $g(u)$ is convex and increasing on $[0, \infty)$ and for each $s \neq \nu_1, i$ we have

$$\int_0^{2\pi} g\left(\frac{W^1_{x_i}(\bar{x}_\nu)}{2\pi}\right) dx_s \leqslant \int_0^{2\pi} g\left(\frac{\int_0^{2\pi} V_{x_i}(\bar{x}_i)\,dx_\nu}{2\pi}\right) dx_s$$

$$\leqslant \frac{1}{2\pi} \int_0^{2\pi}\int_0^{2\pi} g\big(V_{x_i}(\bar{x}_i)\big)\,dx_\nu\,dx_s,$$

by Jensen's inequality.

This implies that $W^1_{x_i}(\bar{x}_\nu) \in L(\mathrm{Log}^+L)^{n-2}(T^{n-2})$. Hence $W^1_{x_i}\big(2\pi, (\bar{x}_\nu)_i\big) \in L(\mathrm{Log}^+L)^{n-3}(T^{n-2})$. Similarly, $W^k_{x_i}\big(2\pi, (\bar{x}_\nu)_i\big) \in L(\mathrm{Log}^+L)^{n-k-2}(T^{n-k-1})$ by induction.

Remark 3. For a.e. $x_0 \in T^n$, by using Fubini's theorem, we can conclude that $f(x_\mu, \bar{x}^0_\mu)$ considered as a function of x_μ on T^r is integrable for any μ and r, where μ is a strictly increasing function from $\{1, \ldots, r\}$ into $\{1, \ldots, n\}$, and $1 \leqslant r \leqslant n-1$.

LEMMA 5. *If $f(x) \in \mathrm{GBV}(T^n)$ and $V_{x_i}(\bar{x}_i) \in L(\mathrm{Log}^+L)^{n-2}(T^{n-1})$, then for a. e. $x_0 = (x^0_1, \ldots, x^0_i, \ldots, x^0_n) = \big(x^0_i, x^0_\nu, (\bar{x}^0_\nu)_i\big) \in T^n$, for any ν*

$$\lim_{\theta \to 0+} \big[W^k_{x_i}\big(x^0_i+\theta, (\bar{x}_\nu)_i\big) - W^k_{x_i}\big(x^0_i-\theta, (\bar{x}_\nu)_i\big)\big] = 0$$

for a. e. $(\bar{x}_\nu)_i \in T^{n-k-1}$ and the convergence is pseudo-uniform with respect to $W^k_{x_i}\big(2\pi, (\bar{x}_\nu)_i\big)$ at $(\bar{x}^0_\nu)_i$.

Proof. From Remark 2, it is an immediate consequence of Lemma 1.

THEOREM 2. *Suppose that $f(x) \in \mathrm{GBV}(T^n)$ and $V_{x_i}(\bar{x}_i) \in L(\mathrm{Log}^+L)^{n-2}(T^{n-1})$ for each $i = 1, \ldots, n$. If conditions 5, 6, Remark 3, and Lemma 5 for any $1 \leqslant k \leqslant n-2$, hold at x_0, then the n-tuple Fourier series of $f(x)$ converges to $f(x)$ at x_0 by using rectangular summation.*

Proof. It is enough to show that

$$(9) \qquad \int^{\delta} \ldots \int^{\delta} \underbrace{\int_{\delta}^{\pi/2} \ldots \int_{\delta}^{\pi/2}}_{\text{ad least two}} \to 0$$

and

$$(10) \qquad \int_0^{\delta} \ldots \int_0^{\delta} \int_{\delta}^{\pi/2} \to 0$$

as $h_1, \ldots, h_n \to \infty$, where the integrands in (9) and (10) are $[f(x_0+2x) - l]\dfrac{\sin h_1 x_i}{\sin x_i}$. Let

$$\Lambda_{h_i}(\bar{x}_i) = \int_{\delta}^{\pi/2} [f(x_0+2x)-l]\,\frac{\sin h_i x_i}{\sin x_i}\,dx_i.$$

Denote by $\overline{W}_{x_j}(x_j, \bar{x}_{ij})$ the total variation of $\Lambda_{h_i}(\bar{x}_i)$ considered as a function of x_j, $i \neq j$.

Observe that

$$\overline{W}_{x_j}(x''_j, \bar{x}_{ij}) - \overline{W}_{x_j}(x'_j, \bar{x}_{ij}) \leqslant \frac{1}{\sin\delta}\,[W^1_{x_j}(x''_j, \bar{x}_{ij}) - W^1_{x_j}(x'_j, \bar{x}_{ij})]$$

and $\overline{W}_{x_j}(\bar{x}_{ij}) = \overline{W}_{x_j}(2\pi, \bar{x}_{ij}) \leqslant \dfrac{1}{\sin\delta}\, W^1_{x_j}(2\pi, \bar{x}_{ij})$ for x''_j, x'_j $(x''_j \geqslant x'_j)$ in $[0, 2\pi]$.

Also, note that the iterated limits of $\Lambda_{h_i}(\bar{x}_i)$ are equal to $\int_{\delta}^{\pi/2} \Big[f\big((x_0+2x)_i, (\bar{x}_0)_i\big) - l\Big]\dfrac{\sin h_i x_i}{\sin x_i}\,dx_i$.

For (9): we use induction.

For (10): the conditions of Proposition 1 hold for $\Lambda_{h_i}(\bar{x}_i)$ and $n-1$ in place of f and n, respectively. The conclusion (10) follows immediately.

Proof of Theorem 1. Since for a.e. $x \in T^n$, Lemmas 1, 2, Corollary 1, Remark 3, and Lemma 5 hold, then by Theorem 2, for a.e. $x \in T^n$, the n-tuple Fourier series of $f(x)$ converges to $f(x)$. By periodicity of $f(x)$, we complete the proof of our main theorem.

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Received October 17, 1972 (599)

STUDIA MATHEMATICA, T. XLIX. (1973)

On the continuity property of Gaussian random fields

by

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Abstract. The conditions for sample paths to be continuous are considered for Gaussian random fields. Especially, the necessary conditions are described.

§ 1. Introduction. Let $X = \{X(\bar{t}), \bar{t} \in \boldsymbol{R}^d\}$ be a zero mean, real, stationary, separable, mean continuous, Gaussian random field with a d-dimensional Euclidean parameter space. Then, the covariance function $\varrho(\bar{t}) = E\big(X(\bar{t}+\bar{x})X(\bar{s})\big)$ is expressed by $\int_{\boldsymbol{R}^d} \cos(\bar{t}, \bar{\lambda})\, dF(\bar{\lambda})$, where $(\,,)$ denotes the inner product, $\bar{t}, \bar{s} \in \boldsymbol{R}^d$ and $F(\cdot)$ (1) is a bounded positive measure.

The purpose of this paper is to describe the continuity condtitions of path functions (which are known for the 1-dimensional parameter case) for random fields. Most sufficient conditions for sample functions to be continuous are already described for random fields. Thus, we shall be concerned mainly with sufficient conditions for sample functions to be discontinuous.

In the case of the 1-dimensional parameter space, the conditions in terms of the spectral measure $F(\cdot)$ were given by Kahane [4] and Nisio [7]. The corresponding results for random fields are the following. Let $s_n = F(\bar{\lambda} \in S_{2^{n+1}}) - F(\bar{\lambda} \in S_{2^n})$, where $S_{2^{n+1}} = \{\lambda;\ |\bar{\lambda}| \leqslant 2^{n+1}\}$, $n = 0, 1, 2, \ldots$

THEOREM 1. *If $X(\bar{t})$ is continuous, then* $\sum_{n=1}^{\infty} s_n^{\frac{1}{2}} < \infty$.

THEOREM 2. *If there exists a decreasing sequence $\{M_n\}$ such that $s_n \leqslant M_n$ and $\sum_{n=1}^{\infty} M_n^{\frac{1}{2}} < \infty$, then X has continuous paths.*

As is shown by Marcus [5] and Marcus and Shepp [6], these conditions are neither too strong, nor necessary and sufficient. However, they give a simple criterion for some cases. In § 2 and § 3, we shall give the proof of the above theorems.

A result corresponding to theorem of Marcus and Shepp ([6], p. 380) is as follows.

(1) F is occasionally used as a measure or as a point function.