*$\mathscr{B}$, is complete, there exists in $E$ a decreasing sequence $\{L_n\}_{n=1}^{\infty}$ of closed linear manifolds, so that $L_n \cap A \neq \emptyset$, $n = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} L_n = \emptyset$.*

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# Projections in dual weakly compactly generated Banach spaces

by

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**Abstract.** Modifying the ideas of D. Amir and J. Lindenstrauss in [1] and using some methods of [6] we prove an existence theorem for projections in weakly compactly generated (WCG) Banach spaces, which gives a construction of ordinal resolution of identity in such a way that a given (not necessarily WCG) subspace is invariant under the projections and in the dual case the projections are w* continuous. This is applied to the existence of shrinking Markuševič bases in such spaces, to w* closed quasicomplements and to a certain rotundity renorming theorem for these spaces, extending [6].

**1. Introduction.** In papers [8], [9], [1] J. Lindenstrauss and D. Amir and J. Lindenstrauss construct, in spaces which are weakly compactly generated, a transfinite sequence of projections which decompose the space very suitably, and they show how it can be used in studying the structure of such spaces. In this note we show that in some results of J. Lindenstrauss ([8], [9]) the assumption that the subspaces are weakly compactly generated can be omitted. This requires a slightly different approach; unlike Lindenstrauss, we construct the projections starting from finite dimensions, on the whole space, and they are constructed so that a given arbitrary subspace (not explicitly supposed to be weakly compactly generated) is invariant under them. In this connection, let us mention that it is not known whether any closed subspace of a WCG space must be WCG ([9], Problem 1, [10]). Moreover, combining this with an idea from [6], we work with three norms on $X$ instead of two as in [1], to ensure the w* continuity of projections in the dual case. In the next part of the paper we prove as an application the existence of some stronger type of the Markuševič basis in certain weakly compactly generated spaces (Propositions 5, 6), some results about w* closed quasicomplements (Proposition 7) and a renorming theorem (Proposition 9), which extends the results of [6].

We would like to thank the referee for making the former versions of Proposition 7 stronger.

**2. Notations and definitions.** We shall deal with real Banach spaces (in short: $B$-spaces). A $B$-space $X$ is *weakly complactly generated* (WCG) if there is a weakly compact set $K \subset X$ such that $X = \overline{\mathrm{sp}}\, K$ — the closed linear hull of $K$. For $A \subset X$, $\overline{A}$ (or $\mathrm{w^*cl}\, A$ if $X = Y^*$) denotes the norm closure (resp. the w* closure) of $A$. WCG spaces, forming a unification of the notions of separable and reflexive spaces, include for example $c_0(\Gamma)$ ($\Gamma$ — an arbitrary set); $C(K)$ ($K$ — the Eberlein compact), $\big(\sum_{\gamma\in\Gamma} \oplus X_\gamma\big)$ — the direct sum of WCG spaces $X_\mu$ in the $1_p(\Gamma)$ sense, $\Gamma$ an arbitrary set, $p \in (1, \infty)$ (cf. [9]).

If $Y \subset X^*$, then a set $\{f_a\}_{a\in\Gamma} \subset Y$ is called an *$X$-Markuševič basis for $Y$* if $\overline{\mathrm{sp}}\{f_a\} = Y$ and there is a set $\{x_a\}_{a\in\Gamma} \subset X$ such that $\{x_a\}_{a\in\Gamma}$ is total on $Y$ and $f_a(x_\beta) = \delta_{a,\beta}$ (the Kronecker delta). Furthermore, if $Y \subset X$ is a closed subspace of $X$, then a closed subspace $Z$ of $X$ will be called a *quasicomplement of* $Y$ in $X$ if $Y \cap Z = \{0\}$ and $\overline{Y+Z} = X$. For a $B$-space $X$, $\mathrm{dens}\, X$ denotes the smallest cardinal number of a norm dense set in $X$. By a subspace of a linear space we mean a linear subspace.

**3. Projections in dual WCG spaces.**

LEMMA 1. *Let $E, F, B$ be finite-dimensional subspaces of a normed linear space $X$ such that $B \subset E$ or $E \subset F$. Then there is a bounded linear projection $P$ of $X$ onto $B$ such that $PE \subset E$ and $PF \subset F$.*

Proof. If $B \subset E$, let $\{a_i\}$ be a basis of $B \cap F$. Let us complete $\{a_i\}$ by system of vectors $\{b_i\}$ and $\{c_i\}$ so as to obtain the bases of $B$ and $E \cap F$ respectively. Then the set $\{a_i\} \cup \{b_i\} \cup \{c_i\}$ is linearly independent and we can complete it with a set $\{d_i\}$ to a basis of $E$. $\{a_i\} \cup \{c_i\}$ forms a basis of $E \cap F$ and we can complete it by some $\{e_i\}$ to basis of $F$. Then the set $\{a_i\} \cup \{b_i\} \cup \{c_i\} \cup \{d_i\} \cup \{e_i\}$ is again linearly independent and forms a basis of $\mathrm{sp}(E \cup F)$.

Let us define on $\mathrm{sp}(E \cup F)$ a projection $P$ by

$$P\Big(\sum a_i a_i + \sum \beta_i b_i + \sum \gamma_i c_i + \sum \delta_i d_i + \sum \varepsilon_i e_i\Big) = \sum a_i a_i + \sum \beta_i b_i .$$

Now we extend $P$ to the whole of $X$ by the Hahn–Banach theorem.

We proceed similarly in the case $E \subset F$.

LEMMA 2. *Let $X$ be a linear space with two norms $|\cdot|_1, |\cdot|_2$ such that $|x|_1 \leqslant |x|_2$ (for every $x \in X$) and let $|\cdot|_3$ be another norm defined on a subspace $N \subset X$ such that $|x|_1 \leqslant |x|_3$ for every $x \in N$. Further, suppose that we are given a finite-dimensional subspace $B \subset N$, $m$ elements $f_1, \ldots, f_m$ of $(X, |\cdot|_2)^*$, an integer $n > 0$ and a subspace $Y$ of $X$. Then there exists an $\aleph_0$-dimensional subspace $C \subset X$ containing $B$ such that, for every $\varepsilon > 0$, every subspace $Z$ of $X$ with $Z \supset B$, $\dim Z/B = n$, there is a linear operator $T\colon Z \to C$ with the properties:*

$$T(Z \cap Y) \subset Y, \quad T(Z \cap N) \subset N, \quad |T|_1 \leqslant 1+\varepsilon, \quad |T|_2 \leqslant 1+\varepsilon,$$

$$|T/(Z \cap N)|_3 \leqslant 1+\varepsilon, \quad Tb = b \quad \text{for every } b \in B$$

*and*

$$|f_k(z) - f_k(Tz)| \leqslant \varepsilon |z|_2 \quad \text{for every } z \in Z \text{ and } k = 1, 2, \ldots, m.$$



Proof. Let $r$ be a positive integer. Choose $b_1, \ldots, b_p \in B$ such that for every $b \in B$ we have:

(i) if $|b|_a \leqslant r$ then there is an $h$ ($1 \leqslant h \leqslant p$) such that $|b - b_h|_a < r^{-1}$ (for $a = 1, 2, 3$).

Consider the Euclidean space $\boldsymbol{R}^n$ with the norm $|\lambda| = \sum |\lambda_i|$. Choose elements $\lambda^1, \ldots, \lambda^q$ of the unit sphere $S^n = \{\lambda \in R^n;\ |\lambda| = 1\}$ in $\boldsymbol{R}^n$ such that for every $\lambda \in S^n$ there is a $j$, $1 \leqslant j \leqslant q$, with $|\lambda - \lambda^j| < r^{-1}$.

For any natural numbers $a, b, c \in \langle 0, n\rangle$ and any positive integer $r$ we define $Q = 3n + 3pq + mn$ real-valued functions of $(x_1, \ldots, x_a, x_{a+1}, \ldots, x_{a+b}, x_{a+b+1}, \ldots, x_{a+b+c}, \ldots, x_n) \in (N \cap Y)^a \times N^b \times Y^c \times X^{n-a-b-c} = H^{abc}$ as follows: $|x_i|_a$, $|Jx_i|_3$, $\big|b_h + \sum_{i=1}^n \lambda_i^j x_i\big|_a$, $\big|b_h + \sum_{i=1}^n \lambda_i^j J x_i\big|_3$, $f_k(x_i)$, ($1 \leqslant i \leqslant n$, $1 \leqslant a \leqslant 2$, $1 \leqslant h \leqslant p$, $1 \leqslant j \leqslant q$, $1 \leqslant k \leqslant m$). Here the function $J$ on $X$ is defined by $Jx = x$ for all $x \in N$ and $Jx = 0$ for $x \notin N$. These functions may be regarded as a function $\varphi$ from $H^{abc}$ into $R^Q$. Taking in $R^Q$ the metric $\varrho$ of maximal coordinate distance, we choose a sequence $\{\varphi(x^t)\}_t$, $x^t = (x_1^t, \ldots, x_n^t) \in H^{abc}$, which is $\varrho$-dense in $\varphi(H^{abc})$. This sequence is constructed for fixed $r, a, b, c$. Thus we have a sequence $\{x^t\} = \{x^{trabc}\}$ for each $r, a, b, c$. Let $C$ be the subspace spanned by $B$ and $\{x_i^{trabc}\}$, $i = 1, \ldots, n$, $t, r = 1, 2, \ldots$ and $a, b, c = 0, 1, \ldots, n$.

Now let $\varepsilon > 0$ and $Z \subset X$, with $Z \supset B$, $\dim Z/B = n$ be given.

Choose, according to Lemma 1, a $|\cdot|_1$-bounded projection $P$ of $X$ onto $B$ such that $P(Z \cap N) \subset N$ and $P(Z \cap Y) \subset Y$. Then $P$ is also $|\cdot|_2$-bounded and $P/E$ is $|\cdot|_3$-bounded. Let $K$ be such that $|P|_1 \leqslant K$, $|P|_2 \leqslant K$ and $|P/E|_3 \leqslant K$. Choose $M \geqslant 1$ such that $M > 6(1+K)\varepsilon^{-1}$.

In $(I-P)Z$ choose the basis $z_1, \ldots, z_n$ such that

(j) $\{z_1, \ldots, z_a\}$ is a basis in $(I-P)Z \cap N \cap Y$,

(jj) $\{z_1, \ldots, z_{a+b}\}$ is a basis in $(I-P)Z \cap N$,

(jjj) $\{z_1, \ldots, z_a, z_{a+b+1}, \ldots, z_{a+b+c}\}$ is a basis in $(I-P)Z \cap Y$.

We have $(z_1, \ldots, z_n) \in (N \cap Y)^a \times N^b \times Y^c \times X^{n-a-b-c} \subset H^{abc}$. Further we may suppose that $\big|\sum_{i=1}^n \lambda_i z_i\big|_a \geqslant |\lambda|$ and $\big|\sum_{i=1}^{a+b} \lambda_i z_i\big|_3 \geqslant |(\lambda_1, \ldots, \lambda_{a+b})|$

for $a=1,2$ and every $(\lambda_1,\ldots,\lambda_n)\in \mathbf{R}^n$. (It is sufficient to multiply all $z_i$ by a sufficiently great number.)

Let $s\geqslant 1$ be such that $|z_i|_a\leqslant s$, $a=1,2$ and $|Jz_i|_3\leqslant s$ for all $i=1,\ldots,n$. Now let us choose a positive integer $r$ such that $2s+1<\varepsilon(r-s)$ and $r^{-1}s<M^{-1}$. Thus we have the natural numbers $a$, $b$, $c$ and the positive integer $r$. These $r$, $a$, $b$, $c$ remain fixed in the rest of the proof.

Let $x=x^{trabc}$ be an element of the sequence defining $C$ such that $\varrho\big(\varphi(x),\varphi(z_1,\ldots,z_n)\big)<M^{-1}$. Define on $Z$

$$T\Big(b+\sum_{i=1}^{n}\lambda_iz_i\Big)=b+\sum_{i=1}^{n}\lambda_ix_i\qquad(b\in B).$$

If $z=b+\sum\lambda_iz_i\in(N\cap Z)$, then $b=Pz\in N$ and $\sum\lambda_iz_i\in N\cap(I-P)Z$. Thus $\lambda_{a+b+1}=\ldots=\lambda_n=0$ and $Tz=b+\sum_{i=1}^{a+b}\lambda_ix_i\in N$. This shows that $T(Z\cap N)\subset N$. Similarly it can be proved that $T(Z\cap Y)\subset Y$.

Now we prove that $|T/(Z\cap N)|_3\leqslant 1+\varepsilon$. It is sufficient to show that $|b+\sum\lambda_ix_i|_3\leqslant(1+\varepsilon)\,|b+\sum\lambda_iz_i|_3$ if

$$b+\sum\lambda_iz_i=b+\sum_{i=1}^{a+b}\lambda_iz_i\in N\cap Z\quad\text{ and }\quad|\lambda|=\sum_{i=1}^{n}|\lambda_i|=1.$$

If $|b|_3\geqslant r$ then $\big|b+\sum\lambda_iz_i\big|_3\geqslant r-s$, while

$$\Big|b+\sum\lambda_ix_i\Big|_3\leqslant\Big|b+\sum\lambda_iz_i\Big|_3+\Big|\sum\lambda_iz_i\Big|_3+\Big|\sum\lambda_ix_i\Big|_3\leqslant\Big|b+\sum\lambda_iz_i\Big|_3+s+(s+1)$$
$$\leqslant\Big|b+\sum\lambda_iz_i\Big|_3+\varepsilon(r-s)\leqslant(1+\varepsilon)\Big|b+\sum\lambda_iz_i\Big|_3.$$

Here the summation is taken over $i=1,\ldots,n$, or, which is the same, over $i=1,\ldots,a+b$. We have also used the fact that $\big||x_i|_3-|z_i|_3\big|\leqslant M^{-1}\leqslant 1$, hence $|x_i|_3\leqslant|z_i|_3+1\leqslant s+1$ for $i=1,\ldots,a+b$.

If $|b|_3\leqslant r$, let $b_h\in B\cap N$ be the $r^{-1}$-approximation to $b$ (according to (i)) and let $\lambda^j\in S^n$ be also the $r^{-1}$-approximation to $\lambda\in S^n$. We have

$$\Big|b+\sum\lambda_ix_i\Big|_3-\Big|b+\sum\lambda_iz_i\Big|_3$$
$$\leqslant 2\,|b-b_h|_3+\Big|b_h+\sum_{i=1}^{a+b}\lambda_i^jx_i\Big|_3-\Big|b_h+\sum_{i=1}^{a+b}\lambda_i^jz_i\Big|_3+\Big|\sum_{i=1}^{a+b}(\lambda_i^j-\lambda_i)x_i\Big|_3+\Big|\sum_{i=1}^{a+b}(\lambda_i^j-\lambda_i)z_i\Big|_3$$
$$\leqslant 2r^{-1}+M^{-1}+(s+1)r^{-1}+sr^{-1}\leqslant 6M^{-1},$$

while

$$\varepsilon\Big|b+\sum\lambda_iz_i\Big|_3\geqslant\varepsilon\,|I-P|_3^{-1}\cdot\Big|\sum_{i=1}^{a+b}\lambda_iz_i\Big|_3\geqslant\varepsilon(1+K)^{-1}>6M^{-1}.$$



The estimation of the $|\cdot|_1$ and $|\cdot|_2$ norms proceeds similarly.

If $z=b+\sum\lambda_iz_i\in Z$, then

$$|f_k(z)-f_k(Tz)|=\Big|\sum\lambda_i\big(f_k(z_i)-f_k(x_i)\big)\Big|\leqslant M^{-1}|\lambda|,$$

while

$$|z|_2\geqslant|I-P|_2^{-1}\Big|\sum\lambda_iz_i\Big|_2\geqslant(1+K)^{-1}|\lambda|;$$

hence

$$|f_k(z)-f_k(Tz)|/|z|_2\leqslant M^{-1}(1+K)<\varepsilon.$$

In all that follows all topological terms (weak topology, density character, closure) will refer to the $\|\cdot\|$-norm.

PROPOSITION 1. *Let $(X,\|\cdot\|)$ be a B-space which is generated by a weakly compact absolutely convex subset $K$, let $|\cdot|$ be another norm on $X$ such that $|x|\leqslant\|x\|$ for every $x\in X$ and let $\mathfrak{m}$ be an infinite cardinal number. Let $Y$ be a closed subspace of $X$, $B$ a subspace of $\operatorname{sp}K$ with $\operatorname{dens}B\leqslant\mathfrak{m}$ and $F$ a subspace of $(X,\|\cdot\|)^*$ with $w^*$-$\operatorname{dens}F\leqslant\mathfrak{m}$. Then there is a linear projection $P\colon X\to X$ with $|P|=\|P\|=1$, $PK\subset K$, $Pb=b$ for every $b\in B$, $P^*f=f$ for every $f\in F$, $\operatorname{dens}PX\leqslant\mathfrak{m}$ and $PY\subset Y$.*

Proof. We proceed exactly as in the proof of Lemma 4 in [1] with the difference that the construction is performed on $N=\operatorname{sp}K\subset X$ instead of the whole $X$. Also the notation of the norms is changed — our $|\cdot|$-norm is new and the $|||\cdot|||$-norm in [1] is the norm on $N$ generated by $K$. The operators $\{T_n\}$ in this proof are defined on the whole $X$ and $T_nY\subset Y$, but for the construction of $P$ exactly as in [1], we consider only their restrictions on $N$. Now as $\{T_n\}$ are equicontinuous on $X$ and $P$ is a weak cluster point of $\{T_n/N\}$, we infer that the continuous extension of $P$ to $X$ is a weak cluster point of $\{T_n\}$. Thus $PY\subset Y$. $P$ is a projection because its restriction to $N$ is a projection. Similarly for $P=\lim P_a$ in the next part of the proof of Lemma 4 in [1].

PROPOSITION 2. *Let $X$, $|\cdot|$, $\|\cdot\|$, $K\subset X$, $Y\subset X$ be as in Proposition 1, let $\mu$ be the first ordinal of cardinality $\operatorname{dens}X$, and let $\{x_a;\ a<\mu\}$ be a dense subset of $\operatorname{sp}K$. Then there is a "long sequence" of linear projections $\{P_a;\ \omega\leqslant a\leqslant\mu\}$ satisfying $|P_a|=\|P_a\|=1$, $P_aK\subset K$, $x_a\in P_{a+1}X$, $P_aY\subset Y$, $\operatorname{dens}P_aX\leqslant\overline{\overline{a}}$ for every $a$, $P_aP_\beta=P_\beta P_a=P_\beta$ whenever $\beta<a$, and $\bigcup_{\beta<a}P_{\beta+1}X$ is dense in $P_aX$ for every $a>\omega$.*

Proof. We proceed exactly as in the proof of Lemma 6 of [1], restricting ourselves again to $N=\operatorname{sp}K\subset X$ as in the preceding proof.

A $B$-space $X$ is called *decomposable* (see [10]) if there is a bounded linear projection $P$ in $X$ such that $\dim PX$ and $\dim(I-P)X$ are both infinite. It was shown in [10] that every non-separable WCG $B$-space is decomposable.

Corollary 1. *Every non-separable closed subspace of a WCG B-space is decomposable.*

Proof. Let $X$ be generated by a weakly compact set $K_1$. Let $S \subset Y$ be a separable infinite-dimensional subspace of $Y$. Then $S$ is generated by a (weakly) compact subset $K_2 \subset S$. Put $K = K_1 + K_2$ and $B = \mathrm{sp}\, K_2$. We may now apply Proposition 1 to obtain a projection $P$ in $X$ with $PX$ separable, $PY \supset B$ and $PY \subset Y$. Thus the restriction $P/Y$ decomposes the space $Y$.

Remark 1. Using the second part of Lemma 1, we have:

A) Lemma 2 holds if we change the assumption $B \subset N$ to the assumption $Y \subset N$.

B) Similarly Proposition 1 changes to the following

Proposition. *Let* $(X, \|\cdot\|), |\cdot|, \mathfrak{m}, F$ *be as in Proposition* 1. *Let* $Y \subset \mathrm{sp}\, K$ *be a subspace and* $B$ *an arbitrary subspace of* $X$ *with* $\mathrm{dens}\, B \leqslant \mathfrak{m}$. *Then there is a linear projection* $P\colon X \to X$ *with* $|P| = \|P\| = 1$, $PK \subset K$, $Pb = b$ *for every* $b \in B$, $P^* f = f$ *for every* $f \in F$, $\mathrm{dens}\, PX \leqslant \mathfrak{m}$ *and* $PY \subset Y$.

In the proof we work on the whole $X$, otherwise than in the proof of Proposition 1, where we worked on $N = \mathrm{sp}\, K$. This proposition may be applied to arbitrary $Y \subset X$ weakly compactly generated by $K_1$, because then $Y_1 = \mathrm{sp}\, K_1 \subset K_1 + K$.

The following proposition permits the application of Propositions 1 and 2 to dual spaces to get even w* continuous projections (see [6]).

Proposition 3. *Let* $(X, \|\cdot\|)$ *be a B-space generated by a weakly compact absolutely convex subset* $K$. *Put* $|f| = \sup\{|f(x)|;\, x \in K\}$ *for* $f \in X^*$. *Then a linear operator* $T\colon X^* \to X^*$ *is* w*-w* *continuous if it is continuous in both* $|\cdot|$ *and* $\|\cdot\|$ *norms.*

Proof. Using the fact that $K$ is absolutely convex weakly compact we conclude, exactly as in the proof of Proposition 2 of [1], that the identity mapping of $X^*$ is w*-$\tilde{\mathrm{w}}$ continuous, where $\tilde{\mathrm{w}}$ means the weak topology of the norm $|\cdot|$. The $\|\cdot\|$-unit ball $B$ of $X^*$ being w* compact, we see that the w* and $\tilde{\mathrm{w}}$ topologies coincide on each $rB$ ($r$ real positive). Since $T$ is $\tilde{\mathrm{w}}$-$\tilde{\mathrm{w}}$ continuous and $TB \subset rB$ for certain $r$, it follows that $T$ is w*-w* continuous on $B$, and in virtue of the Banach–Dieudonné theorem $T$ is w*-w* continuous on $X^*$.

**4. Dual Markuševič bases.** Now we shall study certain types of Markuševič bases in dual (WCG) spaces.

Proposition 4. *Assume that* $X$ *is an arbitrary separable B-space and* $Y \subset X^*$ *a closed separable subspace of* $X^*$. *Then there is an* $X$*-Markuševič basis for* $Y$.



Proof. Let us denote by $T$ the natural linear isometry of $\mathrm{w^*cl}\, Y$ onto $(X/Y_\perp)^*$. Then $TY \subset (X/Y_\perp)^*$ and $X/Y_\perp$ and $TY$ are total to each other. Now we may use Mackey's technique [11] (see also [4], Theorem 4, p. 8).

Proposition 5. *Assume that* $X, X^*$ *are both WCG B-spaces and* $Y \subset X^*$ *is a norm closed subspace of* $X^*$. *Then there is an* $X$*-Markuševič basis for* $Y$.

Proof. We use transfinite induction on norm density $\mathrm{dens}\, X^*$. If $\mathrm{dens}\, X^* = \aleph_0$, then we have Proposition 4. Assume that $\aleph < \aleph_0$ is a cardinal number and suppose that Proposition 5 holds for any spaces $X$ with $\mathrm{dens}\, X^* < \aleph$. Let $\mu$ be the first ordinal of cardinality $\aleph$. By Propositions 2 and 3 there is a system $\{P_\alpha\}_{\alpha \leqslant \mu}$ of projections in $X$ such that $\|P_\alpha\| = 1$, $P_\alpha P_\beta = P_\beta P_\alpha = P_{\min(\alpha,\beta)}$, $P_\mu =$ identity, $\mathrm{dens}\, P_\alpha^* X^* < \aleph$, $P_\beta^* x \in \overline{\mathrm{sp}}\, (P_{\alpha+1}^* x)_{\alpha<\beta}$ and $P_\alpha^* Y \subset Y$ for any $\alpha < \mu$ and $\beta \leqslant \mu$. Let us put $P_0 = 0$.

Let us consider now the projection $(P_{\alpha+1} - P_\alpha)\colon X \to (P_{\alpha+1} - P_\alpha) X$ and its dual $P_{\alpha+1}^* - P_\alpha^*\colon \big((P_{\alpha+1} - P_\alpha) X\big)^* \to (P_{\alpha+1}^* - P_\alpha^*) X^* \subset X^*$, $(0 \leqslant \alpha \leqslant \mu)$. Then its inverse is the restriction of $f \in (P_{\alpha+1}^* - P_\alpha^*) X^*$ to $(P_{\alpha+1} - P_\alpha) X$ and both are isomorphisms of $(P_{\alpha+1}^* - P_\alpha^*) X^*$ and $\big((P_{\alpha+1} - P_\alpha) X\big)^*$. By this we can easily see that there is an $X$-Markuševič basis $\{f_\alpha^\delta\}$ for $(P_{\alpha+1}^* - P_\alpha^*) Y \subset Y \subset X^*$ with respect to the system $\{x_\alpha^\delta\} \subset (P_{\alpha+1} - P_\alpha) X \subset X$, $\delta \in \Lambda_\alpha$. Then it is easy to see that the system $\{f_\alpha^\delta;\, 0 \leqslant \alpha < \mu,\, \delta \in \Lambda_\alpha\}$ is an $X$-Markuševič basis for $Y$ with respect to $\{x_\alpha^\delta,\, \alpha < \mu,\, \delta \in \Lambda_\alpha\}$. Indeed, by a simple inductive proof, $P_\gamma f \in \overline{\mathrm{sp}} \bigcup_{0 \leqslant \alpha < \mu} (P_{\alpha+1}^* - P_\alpha^*) Y$, for any $\gamma \leqslant \mu$ and any $f \in Y$. Thus $\overline{\mathrm{sp}} \bigcup_{0 \leqslant \alpha < \mu} (P_{\alpha+1}^* - P_\alpha^*) Y = Y$ and therefore $\overline{\mathrm{sp}}\{f_\alpha^\delta;\, 0 \leqslant \alpha < \mu,\, \delta \in \Lambda_\alpha\} = Y$. Furthermore, if $f \in Y$, $f \neq 0$, then $(P_{\alpha+1}^* - P_\alpha^*) f \neq 0$ for some $\alpha < \mu$ and then, since $\{x_\alpha^\delta\}$ is total on $(P_{\alpha+1}^* - P_\alpha^*) Y$, there is a $\delta$ such that $\big((P_{\alpha+1}^* - P_\alpha^*) f\big)(x_\alpha^\delta) = f(x_\alpha^\delta) \neq 0$. Furthermore, if $\alpha_1 \neq \alpha_2$, then $(f_{\alpha_1}^{\delta_1})(x_{\alpha_2}^{\delta_2}) = \big((P_{\alpha_1+1}^* - P_{\alpha_1}^*) f\big)(x_{\alpha_2}^{\delta_2}) = \big((P_{\alpha_1+1} - P_{\alpha_1}) f_{\alpha_1}^{\delta_1}\big)(P_{\alpha_2+1} - P_{\alpha_2})(x_{\alpha_2}^{\delta_2}) = f_{\alpha_1}^{\delta_1}\big((P_{\alpha_1+1} - P_{\alpha_1})(P_{\alpha_2+1} - P_{\alpha_2}) x_{\alpha_2}^{\delta_2}\big) = f_{\alpha_1}^{\delta_1}(0) = 0$.

If $Y$ is total on $X$, we have a stronger result.

Proposition 6. *If* $X, X^*$ *are WCG, and* $Y \subset X^*$ *is a norm-closed subspace of* $X^*$ *which is total on* $X$, *then there is an* $X$*-Markuševič basis* $\{f_i\}, \{x_i\}$, $i \in I$, *for* $Y$ *such that* $\overline{\mathrm{sp}}\{x_i\} = X$.

Proof. We again use transfinite induction on $\mathrm{dens}\, X^*$. In the separable case see Proposition 5. Suppose that $\aleph > \aleph_0$ is a cardinal number and assume that the assertion is true for all $X$ with $\mathrm{dens}\, X^* < \aleph$. As in the proof of Proposition 5, take the system of projections $\{P_\alpha\}_{\alpha \leqslant \mu}$. Then using the observations that $(P_{\alpha+1}^* - P_\alpha^*) Y$ is total on $(P_{\alpha+1} - P_\alpha) X$ and the method of proof of Proposition 5, we easily obtain our statement.

By W. Johnson a Markuševič basis $\{x_i\}$, $\{f_i\}$, $i \in I$, of a $B$-space $X$ is called shrinking if $\overline{\mathrm{sp}}\{f_i\} = X$.

COROLLARY. *If $X, X^*$ are WCG B-spaces, then there is a shrinking Markuševič basis of $X$.*

**5. w* closed quasicomplements.** J. Lindenstrauss proved in [9], Theorem 2.5, that if $X$ is WCG and $Y \subset X$ a closed WCG subspace of $X$, then $Y$ has a quasicomplement. Using this proof and Proposition 2, we may prove the same result without the assumption that the subspace $Y$ is WCG (see also the proof of Proposition 7).

Here we shall construct the w* closed quasicomplement of every norm-closed $Y \subset X^*$ if $X, X^*$ are WCG.

PROPOSITION 7. *Assume that $X, X^*$ are WCG B-spaces. Then for any norm-closed subspace $Y \subset X^*$ there is a w* closed quasicomplement in $X^*$.*

Proof. We use transfinite induction on $\operatorname{dens} X^*$. If $\operatorname{dens} X^* = \aleph_0$, then the assertion follows by [7], Theorem 3. Let $\aleph > \aleph_0$ be a cardinal number and suppose that Proposition 7 is proved for all $X$ with $\operatorname{dens} X^* < \aleph$. Let $\operatorname{dens} X^* = \aleph$ and let $Y$ be a norm-closed subspace of $X^*$. Let $\{P_\alpha\}_{\alpha \leqslant \mu}$ ($\mu$ being the first ordinal of cardinality $\aleph$) be a system of projections as in the proof of Proposition 6. By the induction hypothesis, and the natural isomorphism of $(P^*_{\alpha+1}-P^*_\alpha)X^*$ onto $\big((P_{\alpha+1}-P_\alpha)X\big)^*$ mentioned in the proof of Proposition 5, we easily see that there is for any $\alpha < \mu$ a w* closed subspace $Z_\alpha \subset (P^*_{\alpha+1}-P^*_\alpha)X^*$ which is a quasicomplement of $(P^*_{\alpha+1}-P^*_\alpha)Y$ in $(P^*_{\alpha+1}-P^*_\alpha)X^*$. Let $Z = \mathrm{w}^*\operatorname{clsp}(\bigcup Z_\alpha)$ and $z \in Z \cap Y$. Then $(P^*_{\alpha+1}-P^*_\alpha)z \in Z_\alpha$ for any $\alpha < \mu$, since $P^*_\alpha$ are w*-w* continuous and also $(P^*_{\alpha+1}-P^*_\alpha)z \in (P^*_{\alpha+1}-P^*_\alpha)Y$. Thus $(P^*_{\alpha+1}-P^*_\alpha)z = 0$ for any $\alpha < \mu$. From this it easily follows by the behaviour of $\{P^*_\alpha\}$ that $z = 0$. Also, $\overline{\operatorname{sp}}(Z+Y) \supset \overline{\operatorname{sp}}\big(Z_\alpha + (P^*_{\alpha+1}-P^*_\alpha)Y\big) = (P^*_{\alpha+1}-P^*_\alpha)X^*$ for any $\alpha < \mu$. Thus, $\overline{\operatorname{sp}}(Z+Y) = X^*$ (cf. the proof of Proposition 6).

**6. A renorming theorem.** Here we prove a renorming result for certain (WCG) spaces. First we need two auxiliary lemmas.

LEMMA 3. *Suppose that $T$ and $T_1$ are continuous linear operators acting from $X$ and $X^{**}$ respectively into $c_0(\Gamma)$, such that $T_1$ is w*-w continuous and $T = T_1$ on $X$. Then $T_1 = T^{**}$.*

Proof. $T^{**}$, $T_1$: $X^{**} \to m(\Gamma)$ and both are w*-w* continuous. Furthermore, $T = T_1$ on a w* dense set $X$ in $X^{**}$.

LEMMA 4. *Let $X$, $Y$ be B-spaces and $T$ a continuous linear operator of $X$ into $Y$. Then*

$$\{x \in X;\ \|x\| + \|Tx\| \leqslant 1\}^{00} = \{x^{**} \in X^{**};\ \|x^{**}\| + \|T^{**}x^{**}\| \leqslant 1\}.$$

Proof. Let $K, K'$ denote the closed unit balls in $X^*$, $Y^*$, respectively. Then, clearly, it suffices to prove:

(i) $K + T^*(K') = \{x \in X;\ \|x\| + \|Tx\| \leqslant 1\}^0$, and

(ii) $\big(K + T^*(K')\big)^0 = \{x^{**} \in X^{**};\ \|x^{**}\| + \|T^{**}x^{**}\| \leqslant 1\}$.



($\|x^{**}\|$ and $\|T^{**}x^{**}\|$ denote the second dual norms on $X^{**}$ and $Y^{**}$, respectively.)

We prove (i). Since $T^*(K')$ is w* compact, then $K + T^*(K')$ is w* closed, and thus it suffices to prove that $\big(K + T^*(K')\big)_0 = \{x \in X;\ \|x\| + \|Tx\| \leqslant 1\}$. But the last fact is a matter of simple direct computation.

Similarly (ii) is proved.

PROPOSITION 8. *If $X^*$ is WCG, then there is an equivalent norm on $X$, the second dual norm of which on $X^{**}$ is rotund (i.e. strictly convex).*

Proof. Let $K$ be a weakly compact absolutely convex set in $X^*$ such that $\overline{\operatorname{sp}}\, K = X^*$. Let $T_1$ be a w*-w continuous linear one-to-one operator of $X^{**}$ into $c_0(\Gamma)$ for some $\Gamma$ constructed by D. Amir and J. Lindenstrauss in [1], Proposition 2, p. 37, and let $T$ be its restriction to $X$. Then by Lemma 5 we have $T_1 = T^{**}$. Now let us define a new equivalent norm $|||\cdot|||$ on $X$ by $|||x||| = \|x\| + \|Tx\|$, where the norm $\|\cdot\|$ on $c_0(\Gamma)$ is Day's rotund norm ([3], Theorem 10). Then by Lemma 6, the second dual norm of this on $X^{**}$ is $|||x^{**}||| = \|x^{**}\| + \|T^{**}x^{**}\|$, which is rotund.

Before we proceed to the main result of this section, we recall the notion of local uniform rotundity of a $B$-space $X$, introduced by R. Lovaglia. $X$ is said to be locally uniformly rotund, LUR, if whenever $x_n$, $x \in X$, $\|x_n\| = \|x\| = 1$, $\|x_n + x\| \to 2$ then $\|x_n - x\| \to 0$.

Now we may state

PROPOSITION 9. *Assume $X, X^*$ are both WCG B-spaces. Then there is an equivalent norm $|||\cdot|||$ on $X$ with the following properties:*

(i) *$|||\cdot|||$ is LUR,*

(ii) *the dual norm of $|||\cdot|||$ on $X^*$ is LUR,*

(iii) *the second dual norm of $|||\cdot|||$ on $X^{**}$ is rotund.*

Proof. By [6], Corollary 1, there is an equivalent norm $|||\cdot|||_1$ on $X$ with properties (i) and (ii). Let $|||\cdot|||_2$ be the norm from Proposition 8. Then we may combine the norms $|||\cdot|||_1$ and $|||\cdot|||_2$ by Asplund's averaging procedure [2] to obtain a norm $|||\cdot|||$ with properties (i), (ii) and (iii) simultaneously.

Remark. In connection with this theorem let us mention the following example. The second dual of $c_0$ cannot be renormed either to be Gâteaux smooth ([3], Theorem 9) or to be LUR ([9], Theorem 5.8).

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# Sur le comportement asymptotique de systèmes aléatoires généralisés à liaisons complètes non homogènes

par

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**Résumé.** On étudie pour des systèmes aléatoires généralisés à liaisons complètes non homogènes des conditions nécessaires et suffisantes d'ergodicité faible, des conditions nécessaires et suffisantes d'ergodicité forte et des liens entre ces deux modes d'ergodicité.

**I. Introduction.** C'est dans l'article [4] de LeCalvé et Theodorescu que la notion de systèmes aléatoires généralisés à liaisons complètes a été introduite. Sa définition est la suivante:

On appelle système aléatoire généralisé à liaisons complètes, une suite $\big((W_t, \mathscr{W}_t), (\mathscr{X}_{t+1}, \mathscr{B}_{t+1}), {}^t\Pi, {}^tP\big)_{t\in\boldsymbol{T}}$, ($\boldsymbol{T}$ désignant soit l'ensemble $\boldsymbol{N}$ des entiers positifs ou nul, soit l'ensemble $\boldsymbol{Z}$ des entiers relatifs), telle que pour tout $t\in\boldsymbol{T}$,

a) $(W_t, \mathscr{W}_t)$ soit un espace mesurable,

b) $(\mathscr{X}_{t+1}, \mathscr{B}_{t+1})$ soit un espace mesurable appelé „espace des états" à l'instant $t+1$,

c) ${}^t\Pi$ soit une probabilité de transition de l'espace mesurable $(W_t\times\mathscr{X}_{t+1}, \mathscr{W}_t\otimes\mathscr{B}_{t+1})$ dans l'espace mesurable $(W_{t+1}, \mathscr{W}_{t+1})$,

d) ${}^tP$ soit une probabilité de transition de l'espace mesurable $(W_t, \mathscr{W}_t)$ dans l'espace des états $(\mathscr{X}_{t+1}, \mathscr{B}_{t+1})$.

Lorsque, pour tout $(w, x)\in W_t\times\mathscr{X}_{t+1}$, la probabilité de transition ${}^t\Pi[(w, x), \cdot]$ est la probabilité de Dirac $\delta_{u_t(w,x)}(\cdot)$ dont la masse est concentrée au point $u_t(w, x)$, (où $u_t$ est une application mesurable de $(W_t\times\mathscr{X}_{t+1}, \mathscr{W}_t\otimes\mathscr{B}_{t+1})$ dans $(W_{t+1}, \mathscr{W}_{t+1})$), on retrouve la notion de systèmes aléatoires à liaisons complètes étudiée depuis longtemps par divers auteurs roumains (voir par exemple [3]).

Comme déjà indiqué dans le résumé, notre contribution porte sur l'ergodicité faible, sur l'ergodicité forte et sur les liens entre ces notions.

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