is complete, there exists in E a decreasing sequence \( \{L_n\}_n \) of closed linear manifolds, so that \( L_n \cap A \neq \emptyset \), \( n = 1, 2, \ldots \), and \( \bigcap_{n=1}^{\infty} L_n = \emptyset \).

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References


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PROJECTIONS IN DUAL WEAKLY COMPACTLY GENERATED BANACH SPACES

by

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Abstract. Modifying the ideas of D. Amir and J. Lindenstrauss in [1] and using some methods of [6] we prove an existence theorem for projections in weakly compactly generated (WCG) Banach spaces, which gives a construction of a Banach space generating the identity in such a way that given (not necessarily WCG) subspace is invariant under the projections and in the dual space the projections are \( w^* \) continuous. This is applied to the existence of shrinking Markushevich bases in such spaces, to \( w^* \) closed quasicomplements and to a certain rotundity renorming theorem for these spaces, extending [6].

1. Introduction. In papers [8], [9], [1] J. Lindenstrauss and D. Amir and J. Lindenstrauss construct, in spaces which are weakly compactly generated, a transfinite sequence of projections which decompose the space very suitably, and they show how it can be used in studying the structure of such spaces. In this note we show that in some results of J. Lindenstrauss ([8], [9]) the assumption that the subspaces are weakly compactly generated can be omitted. This requires a slightly different approach; unlike the Lindenstrauss, we construct the projections starting from finite dimensions, on the whole space, and they are constructed so that a given arbitrary subspace (not explicitly supposed to be weakly compactly generated) is invariant under them. In this connection, let us mention that it is not known whether any closed subspace of a WCG space must be WCG ([9], Problem 1, [10]). Moreover, combining this with an idea from [6], we work with three norms on \( X \) instead of two as in [1], to ensure the \( w^* \) continuity of projections in the dual case. In the next part of the paper we prove as an application the existence of some stronger type of the Markushevich basis in certain weakly compactly generated spaces (Propositions 5, 6), some results about \( w^* \) closed quasicomplements (Proposition 7) and a renorming result (Proposition 9), which extends the results of [6].

We would like to thank the referee for making the former versions of Proposition 7 stronger.
2. Notations and definitions. We shall deal with real Banach spaces (in short: B-spaces). A B-space X is weakly compactly generated (WCG) if there is a weakly compact set K ⊂ X such that X = bsp K — the closed linear hull of K. For A ⊂ X, X (or w*cl A if X = Y) denotes the norm closure (resp. the w* closure) of A. WCG spaces, forming a uniformization of the notions of separable and reflexive spaces, include for example c0(Γ) (Γ an arbitrary set); C(K) (K the Eberlein compact); (∑ pX) — the direct sum of WCG spaces Xp in the Ip(Γ) sense, Γ an arbitrary set, p ∈ (1, ∞) (cf. [9]).

If Y ⊂ X, then a set (fγ)γ∈Γ is an X-Marzkowski basis for Y if bsp fγ = Y and there is a set (aγ)γ∈Γ ⊂ X such that (aγ)γ∈Γ is total on Y and fγ(aγ) = δγ, (the Kronecker delta). Furthermore, if Y ⊂ X is a closed subspace of X, then a closed subspace Z of X will be called a quotient complement of Y in X if Y ∩ Z = (0) and Y + Z = X. For a B-space X, den X denotes the smallest cardinal number of a norm dense set in X. By a subspace of a linear space we mean a linear subspace.


Lemma 1. Let E, F, B be finite-dimensional subspaces of a normal linear space X such that B ⊂ E or E ⊂ F. Then there is a bounded linear projection P of X onto B such that PE ⊂ B and PF ⊂ E.

Proof. If B ⊂ E, let (a1) be a basis of B ∩ F. Let us complete (a1) by system of vectors (b2) and (c1) so as to obtain the bases of B and E ∩ F respectively. Then the set (a1) ∪ (b2) ∪ (c1) is linearly independent and we can complete it with a set (d1) to a basis of E ∩ F and we can complete it by some (e1) to basis of F. Then the set (a1) ∪ (b2) ∪ (c1) ∪ (d1) ∪ (e1) is again linearly independent and forms a basis of sp(ENF).

Let us define on sp(ENF) a projection P by

\[ P \left( \sum a_1a_1 + \sum b_1b_1 + \sum c_1c_1 + \sum d_1d_1 + \sum e_1e_1 \right) = \sum a_1a_1 + \sum b_1b_1. \]

Now we extend P to the whole of X by using the Hahn-Banach theorem.

We proceed similarly in the case B ⊂ F.

Lemma 2. Let X be a linear space with two norms \( \| \cdot \|_1, \| \cdot \|_2 \) such that \( \| \cdot \|_1 \leq \| \cdot \|_2 \) for every \( \alpha x + \beta y \) and let \( \| \cdot \|_3 \) be another norm defined on a subspace N ⊂ X such that \( \| \cdot \|_1 \leq \| \cdot \|_3 \) for every \( \alpha x + \beta y \). Further, suppose that we are given a finite-dimensional subspace B ⊂ N, m elements \( f_1, \ldots, f_m \) of \((X, \| \cdot \|_3)\), an integer \( n > 0 \) and a subspace Y of X. Then there exists an \( n \)-dimensional subspace G ⊂ X containing B such that, for every \( \varepsilon > 0 \),
The estimation of the $\| \cdot \|$ and $\| \cdot \|_1$ norms proceeds similarly.
If $s = b + \sum \beta_q x_q \in Z$, then
$$|f_s(x) - f_s(T_s)| = \sum l_q |(f_s(x) - f_s(x_q))| \leq M^{-1} |\lambda|,$$
while
$$|s - P_s| \leq \sum l_q x_q \leq (1 + K)^{-1} |\lambda|,$$

hence
$$|f_s(x) - f_s(T_s)|/|s| \leq M^{-1} (1 + K) < s.$$
COROLLARY 1. Every non-separable closed subspace of a WCG B-space is decomposable.

Proof. Let \( X \) be generated by a weakly compact set \( K_1 \). Let \( S \subseteq Y \) be a separable infinite-dimensional subspace of \( X \). Then \( S \) is generated by a (weakly) compact subset \( K_2 \subseteq S \). Put \( K = K_1 + K_2 \) and \( B = \text{sp} K \). We may now apply Proposition 1 to obtain a projection \( P \) in \( X \) with \( PX \cong B \) and \( PY \cong S \). Thus the restriction \( P|Y \) decomposes the space \( Y \).

Remark 1. Using the second part of Lemma 1, we have:

A) Lemma 2 holds if we change the assumption \( B \subseteq X \) to the assumption \( Y \subseteq X \).

B) Similarly Proposition 1 changes to the following

Proposition 3. Let \((X,\|\cdot\|,\cdot,\cdot,m,F)\) be as in Proposition 1. Let \( Y \subseteq sp K \) be a subspace of a weakly compact \( B \)-space of \( Y \) with \( \text{diam} B \leq m \). Then there is a linear projection \( F: X \to X \) with \( \|F\| = 1 \), \( P \subseteq X \), \( PB = \text{for every } b \in B, P^* f = f \) for every \( f \in F \), \( \text{diam} PX \leq m \) and \( PY \subseteq Y \).

In the proof we work on the whole \( X \), otherwise in the proof of Proposition 1, where we worked on \( X \subseteq sp K \). This proposition may be applied to arbitrary \( Y \subseteq X \) weakly compactly generated by \( K_1 \), because then \( Y_1 = sp K_1 = K_1 + K \).

The following proposition permits the application of Propositions 1 and 2 to dual spaces to get even \( w^* \)-continuous projections (see [6]).

Proposition 4. Let \((X,\|\cdot\|,\cdot,\cdot,m,F)\) be a weakly compact \( B \)-space of \( X \). Put \( f = \sup \{ \langle f(x) \rangle : x \in X \} \) for \( f \in X^* \). Then a linear operator \( T: X^* \to X^* \) is \( w^*-w^* \) continuous if it is continuous in both \( \|\cdot\| \) and \( \|\langle \cdot \rangle\| \) norm.

Proof. Using the fact that \( X \) is absolutely convex weakly compact we conclude, exactly as in the proof of Proposition 2 of [1], that the identity mapping of \( X^* \) is \( w^*-w^* \) continuous, where \( \overline{w} \) means the weak topology of the norm \( \|\cdot\| \). The \( \|\cdot\| \)-unit ball of \( X^* \) is \( w^* \)-compact, we see that the \( w^* \) and \( \overline{w} \) topologies coincide on \( X^* \). Since \( T \) is \( w^*-w^* \) continuous and \( TB \subseteq \text{sp} B \) for certain \( r \), it follows that \( T \) is \( w^*-w^* \) continuous on \( B \), and in virtue of the Banach–Dieudonné theorem \( T \) is \( w^*-w^* \) continuous on \( X^* \).

4. Dual Markul'čević bases.

Now we shall study certain types of Markul'čević bases in dual (WCG) spaces.

Proposition 4. Assume that \( X \) is an arbitrary separable \( B \)-space and \( Y \subseteq X^* \) a closed separable subspace of \( X^* \). Then there is an \( X \)-Markul'čević basis for \( Y \).

Proof. Let us denote by \( T \) the natural linear isometry of \( X(X)X^* \) onto \( X \). Then \( TX \subseteq X(X)X^* \) and \( TX \) is total to each other. Now we may use Mackey’s technique [11] (see also [2], Theorem 4, p. 8).

Proposition 5. Assume that \( X, X^* \) are both WCG \( B \)-spaces and \( Y \subseteq X^* \) is a norm closed subspace of \( X^* \). Then there is an \( X \)-Markul'čević basis for \( Y \).

Proof. We use transfinite induction on norm density \( \text{dens} X^* \).

If \( \text{dens} X^* = \aleph_0 \), then we have Proposition 4. Assume that \( X > \aleph_0 \) and suppose that Proposition 5 holds for any spaces \( X \) with \( \text{dens} X^* < \aleph_0 \). Let \( \nu \) be the first ordinal of cardinality \( \aleph_0 \). By Propositions 2 and 3 there is a system \( \{ P_{\alpha} \}_{\alpha < \nu} \) of projections in \( X \) such that \( \|P_{\alpha}\| = \nu \), \( P_{\alpha}P_{\beta} = \text{for every } \alpha, \beta < \nu \), \( P_{\alpha}X^* \subseteq X \) and \( P_{\alpha}^* Y \subseteq X \) for any \( \alpha < \nu \) and \( \beta < \nu \). Let us put \( P_0 = 0 \).

Let us consider the projection \( P_{\alpha+1} - P_{\alpha} : X \to (P_{\alpha+1} - P_{\alpha})X \) and its dual \( P_{\alpha+1}^* - P_{\alpha}^* : (P_{\alpha+1} - P_{\alpha})X^* \to (P_{\alpha+1}^* - P_{\alpha}^*)X^* \) for \( \alpha < \nu \). Then its inverse is the restriction of \( f \in (P_{\alpha+1}^* - P_{\alpha}^*)X^* \) to \( (P_{\alpha+1} - P_{\alpha})X \), and both are isomorphisms of \( (P_{\alpha+1}^* - P_{\alpha}^*)X^* \) and \( (P_{\alpha+1} - P_{\alpha})X^* \). By this we can easily see that there is an \( X \)-Markul'čević basis \( \{ x_\alpha \}_{\alpha < \nu} \) for \( (P_{\alpha+1} - P_{\alpha})X \) and \( (P_{\alpha+1}^* - P_{\alpha}^*)X^* \) with respect to the system \( \{ x_\alpha \}_{\alpha < \nu} \) and \( \delta \subseteq \alpha \). Then it is easy to see that there is an \( X \)-Markul'čević basis \( \{ x_\alpha \}_{\alpha < \delta} \) for \( Y \) with respect to \( \{ x_\alpha \}_{\alpha < \delta} \) and \( \delta \subseteq \alpha \). Indeed, by a simple inductive proof, \( P_{\alpha}x \in \text{sp} \cup (P_{\alpha+1}^* - P_{\alpha}^*)X^* \), for any \( \gamma < \mu \) and any \( f \). Thus \( \text{sp} \cup (P_{\alpha+1}^* - P_{\alpha}^*)X^* = Y \) and therefore \( f(x) < \mu \). Furthermore, if \( f \notin Y \), then \( (P_{\alpha+1} - P_{\alpha})f \notin \) for some \( a < \nu \) and then, since \( \{ x_\alpha \}_{\alpha < \nu} \) is total on \( (P_{\alpha+1} - P_{\alpha})X \), there is a \( \delta \) such that \( \langle (P_{\alpha+1} - P_{\alpha})f(x) \rangle < 0 \). Furthermore, if \( a \neq \alpha \), then \( \langle (P_{\alpha+1}^* - P_{\alpha}^*)f(x) \rangle = \langle (P_{\alpha+1}^* - P_{\alpha}^*)f(x) \rangle = (P_{\alpha+1} - P_{\alpha}^*)(P_{\alpha+1} - P_{\alpha}^*)(x) \rangle = (P_{\alpha+1}^* - P_{\alpha}^*)(P_{\alpha+1} - P_{\alpha}^*)x \rangle = 0 \).

If \( Y \) is total on \( X \), we have a stronger result.

Proposition 6. If \( X, X^* \) are WCG, and \( Y \subseteq X^* \) is a norm-closed subspace of \( X^* \) which is total on \( X \), then there is an \( X \)-Markul'čević basis \( \{ x_\alpha \}_{\alpha < \delta} \), \( \delta \subseteq \alpha \), for \( Y \) such that \( \text{sp} \{ x_\alpha \} = Y \).

Proof. We again use transfinite induction on \( \text{dens} X^* \). In the separable case see Proposition 5. Suppose that \( \aleph_0 > \aleph_0 \) is a cardinal number and assume that the assertion is true for all \( X \) with \( \text{dens} X^* < \aleph_0 \). As in the proof of Proposition 5, take the system of projections \( \{ P_{\alpha} \}_{\alpha < \delta} \). Then using the observations that \( (P_{\alpha+1} - P_{\alpha})X \) is total on \( (P_{\alpha+1} - P_{\alpha})X \) and the method of proof of Proposition 5, we easily obtain our statement.

By W. Johnson a Markul'čević basis \( \{ x_\alpha \}_{\alpha < \delta} \) is a \( B \)-space \( X \) is called shrinking if \( \text{sp} \{ x_\alpha \} = X \).
COROLLARY. If \( X, X^* \) are WCG B-spaces, then there is a shrinking Markov-Lojewski basis of \( X \).

5. \( w^* \)-closed quasi-complements. J. Lindenstrauss proved in [9], Theorem 2.5, that if \( X \) is WCG and \( Y \subset X \) a closed WCG subspace of \( X \), then \( Y \) has a quasi-complement. Using this proof and Proposition 2, we may prove the same result without the assumption that the subspace \( Y \) is WCG (see also the proof of Proposition 7).

Here we shall construct the \( w^* \)-closed quasi-complement of every norm-closed \( Y \subset X^* \) if \( X, X^* \) are WCG.

**Proposition 7.** Assume that \( X, X^* \) are WCG B-spaces. Then for any norm-closed subspace \( Y \subset X^* \) there is a \( w^* \)-closed quasi-complement of \( Y^* \) in \( X^* \).

**Proof.** We use transfinite induction on \( \text{dens} X^* \). If \( \text{dens} X^* = \aleph_0 \), then the assertion follows by [7], Theorem 3. Let \( \aleph > \aleph_0 \) be a cardinal number and suppose that Proposition 7 is proved for all \( X \) with \( \text{dens} X^* < \aleph \).

Let \( \text{dens} X^* = \aleph \) and let \( Y \) be a norm-closed subspace of \( X^* \). Let \( \{ P_i \}_{i=0}^{\alpha} \) (\( \alpha \) being the first ordinal of cardinality \( \aleph \)) be a system of projections as in the proof of Proposition 6. By the induction hypothesis, and the natural isomorphism of \( (P_{i+1} - P_i) X^* \) onto \( (P_{i+1} - P_i) Y \) mentioned in the proof of Proposition 5, we easily see that there is for any \( \alpha < \mu \) a \( w^* \)-closed subspace \( Z_\alpha \subset (P_{\alpha+1} - P_\alpha) X^* \) which is a quasi-complement of \( (P_{\alpha+1} - P_\alpha) Y \). Let \( Z = \omega \cup \bigcup \{ Z_\alpha \} \) and \( z \in Z \cap X^* \).

Then \( (P_{\alpha+1} - P_\alpha) z \in Z_\alpha \) for any \( \alpha < \mu \), since \( P_\mu = 1 \) are \( w^* \)-continuous and also \( (P_{\alpha+1} - P_\alpha) (P_{\alpha+1} - P_\alpha) Y \). Thus \( (P_{\alpha+1} - P_\alpha) Y = 0 \) for any \( \alpha < \mu \). From this it easily follows by the behaviour of \( P_\mu \) that \( z = 0 \).

Also, \( \overline{P}(Z + Y) = \overline{P}(Z + (P_{\alpha+1} - P_\alpha) Y) = (P_{\alpha+1} - P_\alpha) X^* \) for any \( \alpha < \mu \). Thus, \( \overline{P}(Z + Y) = X^* \) (cf. the proof of Proposition 6).

6. A renorming theorem. Here we prove a renorming result for certain (WCG) spaces. First we need two auxiliary lemmata.

**Lemma 3.** Suppose that \( T \) and \( T_1 \) are continuous linear operators acting from \( X \) and \( X^* \) respectively into \( c_0(\Gamma) \), such that \( T \) is \( w^* \)-continuous and \( T_1 \) is \( w^* \)-continuous and \( T = T_1 \) on \( X \). Then \( T_1 = T \).

**Proof.** \( T^* \), \( T_1^* \) are \( w^* \)-continuous and \( T_1 = T \) on \( X \). Furthermore, \( T - T_1 \) is a \( w^* \)-continuous on \( X^* \).

**Lemma 4.** Let \( \overline{X} \) be B-spaces and \( T \) a continuous linear operator of \( X \) into \( X^* \). Then

\[
\{ \{ x \in X; \| x \| + \| T x \| \leq 1 \} \} = \{ \{ x \in X^*; \| x \| + \| T x \| \leq 1 \} \}.
\]

**Proof.** Let \( K, K' \) denote the closed unit balls in \( X^* \), \( X^* \), respectively. Then, clearly, it suffices to prove:

(i) \( K + T(K') = \{ x \in X; \| x \| + \| T x \| \leq 1 \} \), and

(ii) \( (K + T(K'))^0 = \{ x^* \in X^*; \| x^* \| + \| T x^* \| \leq 1 \} \).

We prove (i). Since \( T'(K') \) is \( w^* \)-compact, then \( K + T'(K') \) is \( w^* \)-closed, and thus it suffices to prove that \( \| T x \| + \| T x \| \leq 1 \). But the last fact is a matter of simple direct computation.

Similarly (ii) is proved.

**Proposition 8.** If \( X \) is WCG, then there is an equivalent norm on \( X \), the second dual norm of which on \( X^* \) is rotund (i.e. strictly convex).

**Proof.** Let \( X \) be a weakly compact absolutely convex set in \( X^* \) such that \( E \) is \( X^* \). Let \( T \) be a \( w^* \)-continuous linear one-to-one operator of \( X^* \) into \( c_0(\Gamma) \) for some \( \Gamma \) constructed by D. Amir and J. Lindenstrauss [1], Proposition 2, p. 37, and let \( T \) be its restriction to \( X \). Then by Lemma 5 we have \( T_1 = T_{\text{rot}} \). Now let us define a new equivalent norm \( ||| \cdot ||| \) on \( X \) by \( ||| x ||| = || x || + || T x || \), where the norm \( ||| \cdot ||| \) on \( c_0(\Gamma) \) is Day's rotund norm (§3, Theorem 10). Then by Lemma 6, the second dual norm of this on \( X^* \) is \( ||| T x ||| = || x || + || T x || \), which is rotund.

Before we proceed to the main result of this section, we recall the notion of local uniform rotundity of a B-space \( X \), introduced by R. Lovačija. \( X \) is said to be locally uniformly rotund, LUR, if whenever \( \epsilon \), \( x \in X \), \( \| x \| - \| x - \epsilon y \| < 2 \) then \( \| x - \epsilon y \| \to 0 \).

Now we may state

**Proposition 9.** Assume \( X, X^* \) are both WCG B-spaces. Then there is an equivalent norm \( ||| \cdot ||| \) on \( X \) with the following properties:

(i) \( ||| \cdot ||| \) is LUR,

(ii) the dual norm of \( ||| \cdot ||| \) on \( X^* \) is LUR,

(iii) the second dual norm of \( ||| \cdot ||| \) on \( X^* \) is rotund.

**Proof.** By [6], Corollary 1, there is an equivalent norm \( ||| \cdot ||| \) on \( X \) with properties (i) and (ii). Let \( ||| \cdot ||| \) be the norm from Proposition 8.

Then we may combine the norms \( ||| \cdot ||| \), and \( ||| \cdot ||| \) by Asplund's averaging procedure [2] to obtain a norm \( ||| \cdot ||| \) with properties (i), (ii), and (iii) simultaneously.

**Remark.** In connection with this theorem let us mention the following example. The second dual of \( c_0 \) cannot be renormed either to be Gaussian smooth (§3) or to be LUR (§9, Theorem 5.5).

**References**


Sur le comportement asymptotique de
systèmes aléatoires généralisés à liaisons complètes non homogènes

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Résumé. On étudie pour des systèmes aléatoires généralisés à liaisons complètes non homogènes, les conditions nécessaires et suffisantes d'ergodicité faible, des conditions nécessaires et suffisantes d'ergodicité forte et des liens entre ces deux modes d'ergodicité.

I. Introduction. C'est dans l'article [4] de LeCalvé et Theodorescu que la notion de systèmes aléatoires généralisés à liaisons complètes a été introduite. Sa définition est la suivante:

On appelle système aléatoire généralisé à liaisons complètes, une suite \((W_1, \omega_1), (X_{t+1}, A_{t+1})\) telle que \(T (T\) désignant soit l'ensemble \(N\) des entiers positifs ou nul, soit l'ensemble \(Z\) des entiers relatifs), telle que pour tout \(t \in T\),

a) \((W_1, \omega_1)\) soit un espace mesurable,

b) \((X_{t+1}, A_{t+1})\) soit un espace mesurable appelé "espace des états" à l'instant \(t+1\),

c) \(\mathcal{T}\) soit une probabilité de transition de l'espace mesurable \((W_t \times \mathcal{X}_{t+1}, \mathcal{A}_t \otimes A_{t+1})\) dans l'espace mesurable \((W_{t+1}, \mathcal{A}_{t+1})\),

d) \(\mathcal{P}\) soit une probabilité de transition de l'espace mesurable \((W_t, \omega_t)\) dans l'espace des états \((\mathcal{X}_{t+1}, A_{t+1})\).

Lorsque, pour tout \((w, \omega) \in W_t \times X_{t+1}\), la probabilité de transition \(\mathcal{P}((w, \omega), (w', \omega'))\) est la probabilité de Dirac \(\delta_{W_t(w), w'}\), dont la masse est concentrée au point \(u_t(w, \omega)\) (où \(u_t\) est une application mesurable de \((W_t \times X_{t+1}, \omega_t \otimes A_{t+1})\) dans \((W_{t+1}, \omega_{t+1})\)), on retrouve la notion de systèmes aléatoires à liaisons complètes étudiées depuis longtemps par divers auteurs russes (voir par exemple [3]).

Comme déjà indiqué dans le résumé, notre contribution porte sur l'ergodicité faible, sur l'ergodicité forte et sur les liens entre ces notions.

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