On peut alors reproduire la fin de la démonstration du théorème III,1, pour conclure

$$\|\Phi \circ f\|_{A(E)} \geqslant \frac{1}{4\eta},$$

ce qui achève la démonstration puisque η est arbitraire.

Bibliographie

[1] J.-P. Kahane, *Séries de Fourier absolument convergentes*, 1970.
[2] Y. Meyer, *Algèbres de restriction non isomorphes*, Ann. Inst. Fourier 19 (1969), p. 117–124.
[3] A. Beurling et H. Helson, *Fourier–Stieltjes transforms with bounded powers*, Math. Scand. 1 (1953), p. 120–126.
[4] N. Leblanc, *Sur les isomorphismes dérivables des algèbres de restriction*, C. R. Acad. Sci. Paris 270 (1970), A520–A522.
[5] — *Les endomorphismes d'algèbre à poids*, Bull. Soc. Math. France 99 (1971), p. 387–396.
[6] Y. Meyer, *Isomorphismes entre certaines algèbres de restriction*, C. R. Acad. Sci. Paris 265 (1967), A18–A20.

CENTRE SCIENTIFIQUE ET POLYTECHNIQUE
SAINT-DENIS, FRANCE

Received July 5, 1972 (556)

STUDIA MATHEMATICA, T. XLIX. (1973)

On weak compactness*

by

MANUEL VALDIVIA (Valencia)

Abstract. In [1] E. E. Floyd and V. L. Klee have proved that a bounded closed convex subset C of a normed linear space fails to be weakly compact if and only if there is a decreasing sequence of closed linear manifolds whose intersection is empty and each of which intersects C. We obtain here some properties of certain locally convex spaces, analogous to that explained above, related with other results of V. Pták and R. C. James (see [5] and [2]).

We use here vector spaces over the field K of real or complex numbers. When we use the word "space" it means "locally convex separated vector space". For every space E we denote, as usual, by E' and E^* its topological and algebraical dual, respectively.

LEMMA. *Given a space E let M be a dense subset of $E'[\sigma(E', E)]$ with cardinal number a. Let z be a point of $(E')^*$. If G is the linear hull of $E \cup \{z\}$ there exists in $E'[\sigma(E', G)]$ a dense subset with cardinal number non-larger than a.*

Proof. Suppose that z is not in E. Let L be the subspace of $E'[\sigma(E', E)]$ generated by M. If the restriction of z to L is not continuous we put $L = P$. If the restriction of z to L is continuous let y be a point of E such that $\langle y, x'\rangle = \langle z, x'\rangle$, for all x' of L. Then there exists a $z' \in E'$ such that $\langle y, z'\rangle \neq \langle z, z'\rangle$ and therefore, if P is the subspace of $E'[\sigma(E', E)]$ generated by $L \cup \{z'\}$, we have that the restriction of z to P is not continuous. The subspace $P \cap z^{-1}(0)$ is dense in P and also in $E'[\sigma(E', E)]$. We take $x_0' \in P$, $x_0' \notin z^{-1}(0)$. Given any x' of E' we can write

$$x' = y' + \lambda x_0', \quad y' \in z^{-1}(0), \ \lambda \in K.$$

Let $\{x_d': d \in D\}$ be a net in $P \cap z^{-1}(0)$ which converges to y' for the topology $\sigma(E', E)$. We shall show that the net $\{x_d' + \lambda x_0': d \in D\}$ converges to x' for the topology $\sigma(E', G)$. Indeed, if $x \in E$ we have that

* Supported in part by the "Patronato para el Fomento de la Investigación en la Universidad".

$$\lim_{d\in D}\langle x'_d+\lambda x'_0, x\rangle = \lim_{d\in D}\langle x'_d, x\rangle+\lambda\langle x'_0, x\rangle = \langle y', x\rangle+\lambda\langle x'_0, x\rangle = \langle x', x\rangle$$

and

$$\lim_{d\in D}\langle x'_d+\lambda x'_0, z\rangle = \lim_{d\in D}\langle x'_d, z\rangle+\lambda\langle x'_0, z\rangle = \lambda\langle x'_0, z\rangle = \langle x', z\rangle.$$

Since $z\notin E$ the dimension of P is infinite and its cardinal number is not larger than α. Therefore, it is obvious that there is in P a dense subset for the topology $\sigma(E', G)$, with cardinal number non-larger than α. ■

We shall apply next the following result that has been used by V. Pták in [4]: a) *Let A be a bounded convex set in a space E. If A is not weakly relatively compact, there exists a point z of the weak closure of A in $(E')^*$, $z\notin E$, so that given any finite set $\{x'_1, x'_2, \ldots, x'_n\}$ of E', there exists an $x_0\in A$ such that $x'_j(x_0) = x'_j(z)$, $j = 1, 2, \ldots, n$.*

THEOREM 1. *Let A be a bounded convex set in a space E. If A is not weakly relatively compact and if in $E'[\sigma(E', E)]$ there exists a dense subset with cardinal number α, then there is a filter $\mathscr{F}$ in E which satisfies the following conditions:*

1) *$\mathscr{F}$ has not any weakly adherent point in E.*

2) *$\mathscr{F}$ induces a filter in A.*

3) *$\mathscr{F}$ has a sub-basis $\{H_i: i\in I\}$ of closed hyperplanes, so that the cardinal number of I is not larger than α.*

Proof. Since A is not weakly relatively compact, we take a point z of the weak closure of A in $(E')^*$, which satisfies the condition of result a). Let G be the linear hull of $E\cup\{z\}$. According to Lemma, there exists a dense set in $E'[\sigma(E', G)]$, $\{x'_i: i\in I\}$, so that the cardinal number of I is not larger than α. If $V_i = \{x: x\in G, |\langle x'_i, x\rangle|\leqslant 1\}$ then the family $\{z+V_i: i\in I\}$ is a sub-basis of neighbourhoods for a separated topology and, therefore, $\bigcap_{i\in I}(z+V_i) = \{z\}$. In G let $L_i = z+x_i'^{-1}(0)$, $i\in I$. The filter $\mathscr{F}$ generated by $\{H_i = L_i\cap E: i\in I\}$ induces, according to the result a), a filter in A. Furthermore $L_i\subset z+V_i$ and, therefore,

$$\bigcap_{i\in I} H_i = \Big(\bigcap_{i\in I} L_i\Big)\cap E\subset\Big[\bigcap_{i\in I}(z+V_i)\Big]\cap E = \{z\}\cap E = \emptyset,$$

i.e. $\mathscr{F}$ has not any weakly adherent point in E. ■

COROLLARY 1.1. *Let A be a bounded convex set in a space E. If A is not weakly relatively compact and $E'[\sigma(E', E)]$ is separable, then there exists in E a sequence $\{H_n\}_{n=1}^{\infty}$ of closed hyperplanes, so that $\Big[\bigcap_{n=1}^{p} H_n\Big]\cap A\neq\emptyset$, $p = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} H_n = \emptyset$.*

THEOREM 2. *Let α be a cardinal number. Let A be a bounded convex set of a space E. If A is not weakly relatively compact and E', equipped with the topology of Mackey $\mu(E', E)$, is the inductive limit of a family $\mathscr{M}$ of closed subspaces of $E'[\mu(E'E)]$, so that if $M\in\mathscr{M}$ there is in M a dense subset, with cardinal number non-larger than α, then there exists in E a filter which satisfies the conditions 1, 2 and 3 of Theorem 1.*

Proof. In $(E')^*$ let z be a point of the weak closure of A, which is not in E. Since $E'[\mu(E', E)]$ is the inductive limit of $\mathscr{M}$ and $z\notin E$, there exists an $M\in\mathscr{M}$ such that the restriction of z to M is not continuous. If $M^{\perp}$ is the orthogonal subspace of M in E and φ is the canonical mapping of E onto $E/M^{\perp}$, we have that $\varphi(A)$ is not weakly relatively compact, since the restriction of z to M is a point of the weak closure of $\varphi(A)$ in $(M')^*$, which is not in $E/M^{\perp}$. We apply Theorem 1 and we find a family $\{K_i: i\in I\}$ of closed hyperplanes in $E/M^{\perp}$, so that the finite intersections intersect $\varphi(A)$ and $\bigcap_{i\in I} K_i = \emptyset$, being the cardinal number of I non-larger than α. If $H_i = \varphi^{-1}(K_i)$, $i\notin I$, the filter $\mathscr{F}$ generated by $\{H_i: i\in I\}$ satisfies the conditions of the theorem. ■

COROLLARY 1.2. *Let A be a bounded convex set in a space E. If A is not weakly relatively compact and E', equipped with the topology of Mackey $\mu(E', E)$, is the inductive limit of the family of all the closed separable subspaces of $E'[\mu(E', E)]$, then there is in E a sequence $\{H_n\}_{n=1}^{\infty}$ of closed hyperplanes, so that $\Big[\bigcap_{n=1}^{p} H_n\Big]\cap A\neq\emptyset$, $p = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} H_n = \emptyset$.*

THEOREM 3. *Let α be an infinite cardinal number. Let A be a bounded convex set in a space E. Let $\mathscr{B}$ be a saturated family of absolutely convex, closed and bounded sets of $E'[\sigma(E', E)]$, such that $E = \bigcup\{B: B\in\mathscr{B}\}$ and if $B\in\mathscr{B}$ there exists in B a dense subset, with cardinal number non-larger than α. If A is not weakly relatively compact and E, equipped with the topology of the uniform convergence on the sets of $\mathscr{B}$, is complete, there exists a filter $\mathscr{F}$ in E which satisfies the conditions 1, 2 and 3 of Theorem 1.*

Proof. In $E'[\mu(E', E)]$ let $\mathscr{M}$ be the family of all the closed subspaces generated by the elements of $\mathscr{B}$. It is easy to prove that $E'[\mu(E', E)]$ is the inductive limit of $\mathscr{M}$ and, therefore, Theorem 2 can be applied. ■

COROLLARY 1.3. *Let A be a bounded convex set in a space E. Let $\mathscr{B}$ be the family of all the closed bounded spearable absolutely convex sets of $E'[\sigma(E', E)]$. If A is not weakly relatively compact and E, equipped with the topology of the uniform convergence on the sets of $\mathscr{B}$, is complete, then there exists in E a sequence $\{H_n\}_{n=1}^{\infty}$ of closed hyperplanes, so that $\Big(\bigcap_{n=1}^{p} H_n\Big)\cap\cap A\neq\emptyset$, $p = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} H_n = \emptyset$.*

COROLLARY 2.3. *Let A be a bounded convex set in a space E. If A is not weakly relatively compact and $E'[\mu(E', E)]$ is bornological, then there* *exists in E a sequence $\{H_n\}_{n=1}^{\infty}$ of closed hyperplanes, so that $(\bigcap_{n=1}^{p} H_n)\cap \cap A \neq \emptyset$, $p = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} H_n = \emptyset$.*

Proof. It is inmediate since E is complete for the topology of uniform convergence on every sequence of $E'[\sigma(E', E)]$ which converges to the origin for the convergence of Mackey, (see [3], p. 388). ■

We shall apply next the following result that we have obtained in [6]: b) *Let E be a locally convex space. Suppose that in $E'[\sigma(E', E)]$ there is a family $\mathscr{B} = \{B_i: i \in I\}$ of compact absolutely convex sets, so that $\bigcup_{i\in I} B_i$ is dense in $E'[\sigma(E', E)]$, being a the cardinal number of I. If A is any weakly (relatively) a-compact set of E, then A is weakly (relatively) compact.*

THEOREM 4. *Let A be a bounded convex set in a space E. Suppose that in $E'[\sigma(E', E)]$ there is a family $\mathscr{B} = \{B_i: i \in I\}$ of compact absolutely convex sets, so that $\bigcup_{i\in I} B_i$ is dense in $E'[\sigma(E', E)]$, a being the cardinal number of I. If A is not weakly relatively compact there exists in E a filter $\mathscr{F}$, which satisfies the following conditions:*

1. *$\mathscr{F}$ has not any weakly adherent point in E.*
2. *$\mathscr{F}$ induces a filter in A.*
3. *$\mathscr{F}$ has a basis $\{L_j: j \in J\}$ of closed linear manifolds, so that the cardinal number of J is not larger than a.*

Proof. According to result b), the set A is not weakly relatively a-compact and, therefore, there exists in A a subset $\{x_p: p \in P\}$ which is not weakly relatively compact, so that the cardinal number of P is not larger than a. Let B be the convex hull of $\{x_p: p \in P\}$. If F is the closed subspace generated by B, its dual F' can be identified to $E'/F^{\perp}$, being $F^{\perp}$ the subspace of E' orthogonal to F. If φ is the canonical mapping of E' onto $E'/F^{\perp}$ we have that in $F'[\sigma(F', F)]$, $\{\varphi(B_i): i \in I\}$ is a family of compact sets with dense union in $F'[\sigma(F', F)]$. Since in F there exists a dense set with cardinal number non-larger than a, it results that each $\varphi(B_i)$, $i \in I$, has a dense subset with cardinal number non-larger than a. Since F has dimension non-zero, then a is infinite and, therefore, $F'[\sigma(F', F)]$ has a dense subset with cardinal number β so that $\beta \leqslant a\cdot \mathrm{card}\ I = a\cdot a = a$.

The set B is not weakly relatively compact and, therefore, applying Theorem 1, there exists a filter $\mathscr{F}_1$ in F which has not any weakly adherent point and it induces a filter on B, so that $\mathscr{F}_1$ has a sub-basis $\{H_q: q \in Q\}$ of closed hyperplanes, being the cardinal number of Q non-larger than a. If $\{L_j: j \in J\}$ is the family of all the finite intersections of the elements of $\{H_q: q \in Q\}$, then the filter $\mathscr{F}$ of E, generated by $\{L_j: j \in J\}$, satisfies the theorem. ■

COROLLARY 1.4. *Let A be a bounded convex set in a space E. Suppose that in E there exists a topology $\mathscr{T}$, coarser than the initial one, so that $E[\mathscr{T}]$ is a metrizable locally convex space. If A is not weakly relatively compact there exists in E a decreasing sequence $\{L_n\}_{n=1}^{\infty}$ of closed linear manifolds, so that $L_n \cap A \neq \emptyset$, $n = 1, 2, \ldots$; and $\bigcap_{n=1}^{\infty} L_n = \emptyset$.*

THEOREM 5. *Let a be a cardinal number. Let A be a bounded convex set of a space E. Suppose that E', equipped with the topology of Mackey $\mu(E', E)$, is the inductive limit of the family $\mathscr{M}$ of closed subspaces of $E'[\mu(E', E)]$, such that $M \in \mathscr{M}$ if and only if there exists in M a family $\{M_i: i \in I\}$ of absolutely convex weakly compact subsets with dense union in M, so that the cardinal number of I is not larger than a. If A is not weakly relatively compact there exists a filter $\mathscr{F}$ in E, which satisfies the conditions 1, 2 and 3 of the Theorem 4.*

Proof. It is analogous to the proof of Theorem 2, but using Theorem 4 instead of Theorem 1. ■

COROLLARY 1.5. *Let A be a bounded convex set in a space E. If A is not weakly relatively compact and E', equipped with the topology of Mackey $\mu(E', E)$, is the inductive limit of the family $\mathscr{M}$ of closed subspaces of $E'[\mu(E', E)]$, such that $M \in \mathscr{M}$ if and only if there exists in M a sequence $\{M_n\}_{n=1}^{\infty}$ of absolutely convex weakly compact sets, whose union is dense in M, then there exists in E a decreasing sequence $\{L_n\}_{n=1}^{\infty}$ of closed linear manifolds so that $L_n \cap A \neq \emptyset$, $n = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} L_n = \emptyset$.*

THEOREM 6. *Let a be an infinite cardinal number. Let A be a bounded convex set in a space E. Let $\mathscr{B}$ be the family of closed bounded absolutely convex sets of $E'[\sigma(E', E)]$, so that $B \in \mathscr{B}$ if and only if there exists in B a family $\{B_i: i \in I\}$ of compact absolutely convex subsets, with dense union in B, so that the cardinal number of I is not larger than a. If A is not weakly relatively compact and E, equipped with the topology of the uniform convergence on every set of $\mathscr{B}$, is complete, there exists in E a filter $\mathscr{F}$ which satisfies the conditions 1, 2 and 3 of the Theorem 4.*

Proof. It is analogous to the proof of Theorem 3, but using Theorem 5 instead Theorem 2. ■

COROLLARY 1.6. *Let A be a bounded convex set in a space E. Let $\mathscr{B}$ be the family of closed bounded absolutely convex sets of $E'[\sigma(E', E)]$ such that $B \in \mathscr{B}$ if and only if there exists in B a sequence $\{B_n\}_{n=1}^{\infty}$ of compact absolutely convex sets with dense union in B. If A is not weakly relatively compact and E, equipped with the topology of the uniform convergence on every set of*

$\mathscr{B}$, is complete, there exists in E a decreasing sequence $\{L_n\}_{n=1}^{\infty}$ of closed linear manifolds, so that $L_n \cap A \neq \emptyset$, $n = 1, 2, \ldots$, and $\bigcap_{n=1}^{\infty} L_n = \emptyset$.

The author is grateful to the referee for his suggestions.

References

[1] E. E. Floyd and V. L. Klee, *A characterisation of reflexivity*, Proc. Amer. Math. Soc. 5 (1954), pp. 655–661.

[2] J. R. James, *Weak compactness and reflexivity*, Israel J. Math. 2 (1964), pp. 101–119.

[3] G. Köthe, *Topologische lineare Räume I*, Berlin-Heidelberg-New York 1966.

[4] V. Pták, *O polnych topologičeskich linejnych prostranstvach*, Czechosl. Mat. J. 3 (78) (1953), pp. 301–364.

[5] — *Two remarks on weak compactness*, Czechosl. Math. J. 5 (80) (1955), pp. 532–545.

[6] M. Valdivia, *Some criteria for weak compactness*, J. Reine Angew. Math. 255 (1972), pp. 165–169.

FACULTAD DE CIENCIAS
VALENCIA, ESPAÑA

Received July 5, 1972 (559)

STUDIA MATHEMATICA, T. XLIX. (1973)

Projections in dual weakly compactly generated Banach spaces

by

K. JOHN and V. ZIZLER (Prague)

Abstract. Modifying the ideas of D. Amir and J. Lindenstrauss in [1] and using some methods of [6] we prove an existence theorem for projections in weakly compactly generated (WCG) Banach spaces, which gives a construction of ordinal resolution of identity in such a way that a given (not necessarily WCG) subspace is invariant under the projections and in the dual case the projections are w* continuous. This is applied to the existence of shrinking Markuševič bases in such spaces, to w* closed quasicomplements and to a certain rotundity renorming theorem for these spaces, extending [6].

1. Introduction. In papers [8], [9], [1] J. Lindenstrauss and D. Amir and J. Lindenstrauss construct, in spaces which are weakly compactly generated, a transfinite sequence of projections which decompose the space very suitably, and they show how it can be used in studying the structure of such spaces. In this note we show that in some results of J. Lindenstrauss ([8], [9]) the assumption that the subspaces are weakly compactly generated can be omitted. This requires a slightly different approach; unlike Lindenstrauss, we construct the projections starting from finite dimensions, on the whole space, and they are constructed so that a given arbitrary subspace (not explicitly supposed to be weakly compactly generated) is invariant under them. In this connection, let us mention that it is not known whether any closed subspace of a WCG space must be WCG ([9], Problem 1, [10]). Moreover, combining this with an idea from [6], we work with three norms on X instead of two as in [1], to ensure the w* continuity of projections in the dual case. In the next part of the paper we prove as an application the existence of some stronger type of the Markuševič basis in certain weakly compactly generated spaces (Propositions 5, 6), some results about w* closed quasicomplements (Proposition 7) and a renorming theorem (Proposition 9), which extends the results of [6].

We would like to thank the referee for making the former versions of Proposition 7 stronger.