and similarly,

$$\|a_2 x\| \geqslant 2|\lambda_0| + 10|\lambda_1| + |\lambda_2| + 90\sum_{3}^{6} |\lambda_i|. \tag{6}$$

The required inequality clearly follows from inequalities (4)–(6).

Suppose now that there exist $B \supset A$ and $b_1, b_2 \in B$ such that $a_1 b_1 + a_2 b_2 = 1$, $\|b_1\|$, $\|b_2\| \leqslant 1$. Then

$$\begin{aligned} 10 = \|q_0\| &= \|q_0 + (q_0 + q_1 b_1 + q_2 b_2)(a_1 b_1 + a_2 b_2 - 1)\| \\ &\leqslant \|b_1(q_1 - q_0 a_1)\| + \|b_2(q_2 - q_0 a_2)\| + \\ &\qquad + \|b_1 b_2 (q_1 a_2 + q_2 a_1)\| + \|b_1^2 q_1 a_1\| + \|b_2^2 q_2 a_2\| \\ &\leqslant 5+1+1+1+1 = 9, \end{aligned}$$

and this contradiction proves that $\{A, a_1, a_2\}$ has property (ii). Thus the proof of the theorem is complete.

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H^p-spaces of conjugate systems on local fields

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Abstract. Properties of regular functions and subregular functions, analogous to harmonic functions and subharmonic functions, are studied. The local field variant of the Fatou–Calderón–Stein theorem on harmonic function and its Lusin area function is proved. Conjugate systems of regular functions are defined. The theory of H^p-spaces of conjugate systems in the sense of Stein–Weiss is presented. The F. and M. Riesz theorem is also treated.

INTRODUCTION

Stein and Weiss [10] have developed a theory of H^p-spaces for M. Riesz systems $F(x, y) = (f_0(x, y), f_1(x, y), \ldots, f_n(x, y))$ of conjugate harmonic functions on euclidean half-spaces $\boldsymbol{R}_+^{n+1}$ satisfying

$$\int_{\boldsymbol{R}^n} |F(x, y)|^p dx \leqslant A < \infty \quad \text{for all } y > 0.$$

Coifman and Weiss [2] extended the theory to Generalized Cauchy–Riemann systems. The basic result needed, common to all these systems, is the existence of a positive $p_0 < 1$ such that $|F|^{p_0}$ is subharmonic. It is our main objective in this paper to construct conjugate systems on local fields such that the analogue of the above basic result is valid which enable us to develop a theory of H^p-spaces on local fields.

Let K be a local field. That is, K is a locally compact, non-discrete, complete, totally disconnected field. Such a field is a p-adic field, a finite algebraic extension of a p-adic field, or a field of formal Laurent series over a finite field. See [8] for details. Various aspects of harmonic analysis on K and K^n, the n-dimensional vector spaces over K, have been studied in [4], [8], [12], [13], [14], [6], [7], and [5]. In particular, from [14], [6], and [7] we have the notion of singular integral operators and multipliers; from [13] we have the notion of regular functions on $K^n \times \boldsymbol{Z}$ which play the role of harmonic functions on $\boldsymbol{R}_+^{n+1}$.

In Part A, we study the theory of regular functions, including subregular functions and the Lusin area function. Conjugate systems of regular functions are introduced in Part B, so that we have the theory of H^p-spaces. The F. and M. Riesz theorem is also treated.

Most of the corresponding results in euclidean spaces we are generalizing can be found in [11] and [9]. The local field variants of the work of Fefferman and Stein [3] will be studied in a sequel to this paper.

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Notation and preliminaries. In general we follow the notation of [12]. The materials in [8]; § 2 and [12]; § 1 serve well as our preliminaries. However, we shall repeat them briefly as follows.

Let K be a fixed local field and let dx be a Haar measure on K^+ (the additive group of K). There is a natural non-archimedian norm on K such that $d(ax) = |a|\,dx$, $|x+y| \leqslant |x| \vee |y|$ $(= \max[|x|, |y|])$ and $|x+y| = |x| \vee |y|$ if $|x| \neq |y|$. The set $\mathcal{O} = \{x \in K\colon |x| \leqslant 1\}$ is the ring of integers in K. Haar measure is normalized such that the measure of $\mathcal{O}$ is 1, i.e., $|\mathcal{O}| = \int_{\mathcal{O}} dx = 1$. The set $\mathscr{P} = \{x \in K\colon |x| < 1\}$ is the (unique) maximal principal ideal in $\mathcal{O}$. $\mathcal{O}/\mathscr{P} \cong GF(q)$ where q is some prime power. Let $\mathfrak{p}$ be a generator of $\mathscr{P}$. Then $|\mathfrak{p}| = q^{-1}$ and for all $x \in K$, either $|x| = 0$ (when $x = 0$) or $|x| = q^k$ for some $k \in \mathbf{Z}$. The set $\mathscr{P}^k = \{x \in K\colon |x| \leqslant q^{-k}\}$ has measure q^{-k}. The collection $\{\mathscr{P}^k\}_{k=0}^{\infty}$ is a neighborhood basis for the identity in K^+. Cosets of $\mathscr{P}^k$ are called *spheres*. $\mathscr{P}_x^k = x + \mathscr{P}^k$ is the sphere with center x and radius q^{-k}. Every point in a sphere is its center. For any two spheres either they are disjoint or one contains the other. We note the existence of a nontrivial additive character χ such that χ is trivial on $\mathcal{O}$, but is not trivial on $\mathscr{P}^{-1}$. The Fourier transform for $f \in L^1(K)$ is defined as $\hat{f}(u) = \int_K f(x)\,\overline{\chi}(ux)\,dx$.

Let K^n be the n-dimensional vector space over K.

$$K^n = \{x = (x_1, x_2, \ldots, x_n)\colon x_i \in K,\ i = 1, 2, \ldots, n\}.$$

The norm on K^n defined by $|x| = \max_{1 \leqslant i \leqslant n} |x_i|$, $x \in K^n$, is such that $|x+y| \leqslant |x| \vee |y|$ and $|x+y| = |x| \vee |y|$ when $|x| \neq |y|$ for $x, y \in K^n$. A Haar measure is given by $dx = dx_1 dx_2 \ldots dx_n$ where dx_i is the (additive) Haar measure on K as the ith coordinate space of K^n. $d(ax) = |a|^n dx$ for $a \in K$. We also denote

$$\mathscr{P}^k = \{x \in K^n\colon |x| \leqslant q^{-k}\},\ k \in \mathbf{Z}. \qquad |\mathcal{O}| = |\mathscr{P}^0| = 1 \quad \text{and} \quad |\mathscr{P}^k| = q^{-nk}.$$

For $x, y \in K^n$, let $x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_ny_n$. The Fourier transform for $f \in L^1(K^n)$ is defined by $\hat{f}(u) = \int_{K^n} f(x)\,\overline{\chi}(u \cdot x)\,dx$.

Let $\mathscr{S}$ be the space of test functions on K^n, i.e., those are constant on the cosets of some $\mathscr{P}^k$ and supported on some $\mathscr{P}^l$. $\mathscr{S}'$, the topological dual of $\mathscr{S}$, is called the *space of distributions*. For every $f \in \mathscr{S}'$, the Fourier transform of f is in $\mathscr{S}'$ and is defined by $\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$, for all $\varphi \in \mathscr{S}$.

Let Φ_k be the characteristic function of $\mathscr{P}^k$ in K. Then for $x = (x_1, x_2, \ldots, x_n) \in K^n$, $\Phi_k(x_1)\Phi_k(x_2) \ldots \Phi_k(x_n)$ is the characteristic function of $\mathscr{P}^k$ in K^n. We also denote it by Φ_k.

The following notation are used. For a set A, A' denotes the complement of A. $A \setminus B = A \cap B'$. We write $B \subsetapprox A$ if $|A \setminus B| = 0$; $A \cong B$ if $A \subsetapprox B$ and $B \subsetapprox A$. In the latter case we say that the two sets A and B are equivalent. We denote $\mathbf{Z}^+$ for the non-negative integers and $\mathbf{Z}^-$, the non-positive ones. For a sequence of real numbers $\{a_k\}_{k=1}^{\infty}$, we write $a_k \nearrow a$ as $k \to \infty$ if $a_k \leqslant a_l$ for $k \leqslant l$ and $a_k \to a$ as $k \to \infty$.

A. THE THEORY OF REGULAR FUNCTIONS

In § 1, we define regular functions and subregular functions on a domain in $K^n \times \mathbf{Z}$ and show that they behave very much like harmonic and subharmonic functions on euclidean spaces. We also prove the theorem on regular majorants of subregular functions. In § 2, we show that, for a regular function, the nontangential convergence is equivalent to the radial convergence and also, locally, equivalent to the radial boundedness and to the existence of the Lusin area function.

§ 1. We write $(\mathscr{P}_x^{-k}, l) = \{(y, l) \in K^n \times \mathbf{Z}\colon y \in \mathscr{P}_x^{-k}\}$ where $\mathscr{P}_x^{-k} = x + \mathscr{P}^{-k}$. A set $\mathscr{D} \subset K^n \times \mathbf{Z}$ is called a *domain* in $K^n \times \mathbf{Z}$ if

(i) $(x, k) \in \mathscr{D}$ implies $(\mathscr{P}_x^{-k}, k) \subset \mathscr{D}$;

(ii) $(x, k) \in \mathscr{D}$ and $(x, k-1) \in \mathscr{D}$ imply $(\mathscr{P}_x^{-k}, k-1) \subset \mathscr{D}$.

A domain $\mathscr{D}$ in $K^n \times \mathbf{Z}$ is bounded if there exists a $k_0 \in \mathbf{Z}$ such that $k \geqslant k_0$ for all $(x, k) \in \mathscr{D}$. For a domain $\mathscr{D}$ in $K^n \times \mathbf{Z}$, let $\partial\mathscr{D} = \{(x, k) \in \mathscr{D}\colon (x, k-1) \notin \mathscr{D}\}$ and let $m(x) = \sup\{k;\ (x, k) \in \mathscr{D}\}$, $(m(x) = \infty$ if $(x, k) \notin \mathscr{D}$ for all $k \in \mathbf{Z})$ and $m(\mathscr{D}) = \inf_x m(x)$. A domain in $K^n \times \mathbf{Z}$ is said to be *simple* provided that $(x, k) \in \partial\mathscr{D}$ implies $(x, l) \notin \mathscr{D}$ for all $l < k$ and that $m(\mathscr{D}) > -\infty$.

DEFINITION. A function $f(x, k)$ defined on a domain $\mathscr{D} \subset K^n \times \mathbf{Z}$ is said to be *regular* on $\mathscr{D}$ if, for all $(x, k) \in \mathscr{D} \setminus \partial\mathscr{D}$, $f(x, k)$ is constant on $(\mathscr{P}_x^{-k}, k)$ and

$$f(x, k) = \frac{1}{|\mathscr{P}_x^{-k}|} \int_{\mathscr{P}_x^{-k}} f(y, k-1)\,dy. \tag{1.1}$$

A function $f(x, k)$ is *subregular* (*superregular*, respectively) on $\mathscr{D}$ if it is real-valued and "$=$" in (1.1) is replaced by "$\leqslant$" ("$\geqslant$", respectively).

Note that the defining property (1.1) of a regular function is the analogue of the mean-value property of a harmonic function. For $\mathscr{D} = K^n \times \boldsymbol{Z}$, this is the same as the definition of regularity in [13].

PROPOSITION 1.2 (MAXIMUM AND MINIMUM PRINCIPLE). (a) *If $f(x, k)$ is subregular on a bounded domain $\mathscr{D} \subset K^n \times \boldsymbol{Z}$, then*

$$\sup_{(x,k)\in\mathscr{D}} f(x, k) = \sup_{(x,k)\in\partial\mathscr{D}} f(x, k).$$

(b) *If $f(x, k)$ is superregular on a bounded domain $\mathscr{D} \subset K^n \times \boldsymbol{Z}$, then*

$$\inf_{(x,k)\in\mathscr{D}} f(x, k) = \inf_{(x,k)\in\partial\mathscr{D}} f(x, k).$$

Proof. Since domain $\mathscr{D}$ is a union of the sets $(\mathscr{P}_x^{-k}, k)$ and $\mathscr{D}$ is bounded, it suffices to consider the special case that $\mathscr{D} = (\mathscr{P}_x^{-k}, k) \cup (\mathscr{P}_x^{-k}, k-1)$ with $\partial\mathscr{D} = (\mathscr{P}_x^{-k}, k-1)$. But in this case, the conclusions follow immediately from the subregularity or the superregularity of f.

COROLLARY 1.3 (Uniqueness of boundary values of regular functions). *If f and g are regular functions on a bounded domain $\mathscr{D} \subset K^n \times \boldsymbol{Z}$ and agree on $\partial\mathscr{D}$, then $f(x, k) = g(x, k)$ for all $(x, k) \in \mathscr{D}$.*

Proof. Apply Proposition 1.2 to the real and the imaginary parts of the regular function $f-g$.

The following are also obvious extensions of results about subharmonic functions.

PROPOSITION 1.4. (a) *The linear combination of regular functions on $\mathscr{D}$ is regular on $\mathscr{D}$. If f and g are subregular on $\mathscr{D}$ and $a, b \geqslant 0$, then $af+bg$ and $f \vee g$ are subregular on $\mathscr{D}$.*

(b) *If f is subregular on $\mathscr{D}$ and φ is a non-decreasing convex function defined on an interval containing the range of f. Then the composition $\varphi \circ f$ is subregular on $\mathscr{D}$.*

Proof. (a) Immediate.

(b) For $(x, k) \in \mathscr{D} \setminus \partial\mathscr{D}$,

$$\varphi \circ f(x, k) \leqslant \varphi\left(\frac{1}{|\mathscr{P}_x^{-k}|} \int_{\mathscr{P}_x^{-k}} f(y, k-1)\, dy\right) \leqslant \frac{1}{|\mathscr{P}_x^{-k}|} \int_{\mathscr{P}_x^{-k}} \varphi \circ f(y, k-1)\, dy$$

as follows from Jensen's inequality. Therefore $\varphi \circ f$ is subregular on $\mathscr{D}$.

Remark. A useful consequence of Proposition 1.4 (b) is that if $f(x, k)$ is regular on $\mathscr{D}$ and $p \geqslant 1$, then $|f(x, k)|^p$ is subregular on $\mathscr{D}$. This is no longer true if $p < 1$. In fact, if $f(x, k) \geqslant 0$ is regular on $\mathscr{D}$ and $p \leqslant 1$, then $|f(x, k)|^p$ is superregular on $\mathscr{D}$ as follows from the inequality

$$\left[\frac{1}{m}\sum_{i=1}^{m} a_i\right]^p \geqslant \frac{1}{m}\sum_{i=1}^{m} a_i^p \quad \text{for } 0 < p \leqslant 1,\ a_i \geqslant 0,\ i = 1, 2, \ldots, m.$$

The following being defined in [13] generalizes the notion of Poission kernel and Poission integral:

DEFINITION. $R(x, k) = R_k(x) = q^{-nk}\Phi_{-k}(x)$ is called the *regularization kernel*. For $f \in \mathscr{S}'$, the space of distributions, let $f(x, k) = R_k * f(x)$ and is said to be the *regularization* (*integral*) of f.

Note that $R(x, k) \in \mathscr{S}$ for all k and $R_k * f(x)$ is well-defined for $f \in \mathscr{S}'$. Also, $R(x, k)$ and $R_k * f(x)$, with $f \in \mathscr{S}'$, are regular on $K^n \times \boldsymbol{Z}$. Moreover, regular functions on $K^n \times \boldsymbol{Z}$ stand in one-to-one correspondence with distributions on K^n ([13], Lemma 1).

PROPOSITION 1.5. (a) *If $f(x, k)$ is regular on $K^n \times \boldsymbol{Z}$, then*

$$R_k * f(\cdot, l)(x) = f(x, k \vee l).$$

(b) *If $f(x, k)$ is subregular on $K^n \times \boldsymbol{Z}$, then*

$$R_k * f(\cdot, l)(x) \geqslant f(x, k \vee l).$$

Proof.

$$R_k * f(\cdot, l)(x) = \frac{1}{|\mathscr{P}_x^{-k}|} \int_{\mathscr{P}_x^{-k}} f(y, l)\, dy.$$

If $f(x, k)$ is regular, this is just $f(x, k \vee l)$. If $f(x, k)$ is subregular, then

$$\frac{1}{|\mathscr{P}_x^{-k}|} \int_{\mathscr{P}_x^{-k}} f(y, l)\, dy \begin{cases} = f(x, l) & \text{if } k \leqslant l; \\ \geqslant f(x, k) & \text{if } k \geqslant l. \end{cases}$$

Thus, $R_k * f(\cdot, l)(x) \geqslant f(x, k \vee l)$.

The following two results can be found in [13]:

PROPOSITION 1.6. I. *Let $f \in L^p(K^n)$ with $1 \leqslant p \leqslant \infty$ and $f(x, k) = R_k * f(x)$.*

(a) *$f(x, k) \to f(x)$ a.e. as $k \to -\infty$.*

(b) *$\|f(\cdot, k)\|_p \nearrow \|f\|_p$ as $k \to -\infty$, $1 \leqslant p \leqslant \infty$.*

(c) *$\|f(\cdot, k) - f(\cdot)\|_p \to 0$ as $k \to -\infty$, $1 \leqslant p < \infty$ and $f(x, k) \to f(x)$ in the ω^*-topology as $k \to -\infty$ if $p = \infty$.*

II. *If μ is a finite Borel measure with total variation $\|\mu\|$, then $\|\mu(\cdot, k)\|_1 \nearrow \|\mu\|$ as $k \to -\infty$, and $\mu(x, k) \to \mu(x)$ in the ω^*-topology as $k \to -\infty$ where $\mu(x, k) = \int_{K^n} R_k(x-y)\, d\mu(y)$.*

PROPOSITION 1.7. *Suppose $f(x, k)$ is regular on $K^n \times \boldsymbol{Z}$ and $\sup_{k\in\boldsymbol{Z}} \|f(\cdot, k)\|^p \leqslant A < \infty$ where $1 \leqslant p \leqslant \infty$.*

(a) *If $1 < p \leqslant \infty$, $f(x, k)$ is the regularization of a function in $L^p(K^n)$.*

(b) *If $p = 1$, $f(x, k)$ is the regularization of a finite Borel measure on K^n.*

The following is an immediate consequence (compare with Lemma 11 in [13]).

COROLLARY 1.8. *If $f(x, k)$ is regular on $K^n \times \mathbf{Z}$, $\lim_{k\to-\infty} f(x, k) = 0$ a.e., and $f(x, k)$ is bounded, then f must be identically zero.*

Note that Corollary 1.8 can be regarded as a result on the uniqueness of regular functions on the unbounded domain $K^n \times \mathbf{Z}$. The result is not true unless a restriction, such as boundedness, is imposed on f. $R(x, k)$ is such an example.

If a regular function $m(x, k)$ majorizes the function $f(x, k)$ on a domain in $K^n \times \mathbf{Z}$, we say that m is a *regular majorant* of f. If $m \leqslant h$ whenever h is another regular majorant of f, m is called the *least regular majorant* of f.

THEOREM 1.9. *If $f(x, k)$ is a non-negative subregular function on $K^n \times \mathbf{Z}$ and $\sup_{k\in\mathbf{Z}} \|f(\cdot, k)\|_p \leqslant A < \infty$ where $1 \leqslant p \leqslant \infty$. Then $f(x, k)$ has the least regular majorant $m(x, k)$. Moreover,*

(a) *if $1 < p \leqslant \infty$, $m(x, k)$ is the regularization of a function in $L^p(K^n)$;*

(b) *if $p = 1$, $m(x, k)$ is the regularization of a finite Borel measure on K^n.*

Proof. For fixed $k \in \mathbf{Z}$, let $m_l(x, k) = R_k * f(\cdot, l)(x)$. Since $f(x, l)$ is subregular, $m_l(x, k) \leqslant R_k * f(\cdot, l-1) = m_{l-1}(x, k)$ for $l \leqslant k$. Thus $m_l(x, k) \nearrow m(x, k)$, say (as $l \to -\infty$). By Proposition 1.5, for $l \leqslant k$, we have $f(x, k) \leqslant m_l(x, k) \leqslant m(x, k)$. Moreover, applying the monotone convergence theorem, we have

$$m(x, k) = \lim_{l\to-\infty} m_l(x, k) = \lim_{l\to-\infty} \frac{1}{|\mathcal{P}_x^{-k}|} \int_{\mathcal{P}_x^{-k}} m_l(x, k-1)\, dx$$

$$= \frac{1}{|\mathcal{P}_x^{-k}|} \int_{\mathcal{P}_x^{-k}} m(x, k-1)\, dx.$$

That is, $m(x, k)$ is regular on $K^n \times \mathbf{Z}$.

Now, from Proposition 1.6, we know that $\|m_l(\cdot, k)\|_p \leqslant \|f(\cdot, l)\|_p \leqslant A$. If $1 \leqslant p < \infty$, by the monotone convergence theorem,

$$\int_{K^n} m^p(x, k)\, dx = \lim_{l\to-\infty} \int_{K^n} m_l^p(x, k)\, dx \leqslant A^p;$$

that is, $\sup_{k\in\mathbf{Z}} \|m(\cdot, k)\|_p \leqslant A$. For the case $p = \infty$, $\|m(\cdot, k)\|_\infty \leqslant A$ for all $k \in \mathbf{Z}$, is obvious. Therefore, by Proposition 1.7,

(a) if $1 < p \leqslant \infty$, $m(x, k)$ is the regularization of a function in $L^p(K^n)$;

(b) if $p = 1$, $m(x, k)$ is the regularization of a finite Borel measure on K^n.

It remains to show that the majorant m is in fact the least one. Suppose $h(x, k)$ is a regular majorant of $f(x, k)$, then for $l \leqslant k$,

$$\begin{aligned} m_l(x, k) &= R_k * f(\cdot, l)(x) \\ &\leqslant R_k * h(\cdot, l)(x) \\ &= h(x, k \vee l) = h(x, k). \end{aligned}$$

Letting $l \to -\infty$, we thus have $m(x, k) \leqslant h(x, k)$. This completes the proof.

§ 2. We identify K^n with $K^n \times \{-\infty\}$. For $l \in \mathbf{Z}^+$ and $z \in K^n$, let $\Gamma_l(z) = \{(x, k) \in K^n \times \mathbf{Z}\colon |x - z| \leqslant q^{k+l}\}$. If $f(x, k)$ is defined on $K^n \times \mathbf{Z}$, we say that it has a *nontangential limit* L at $z \in K^n$ if, for each $l \in \mathbf{Z}^+$, $\lim f(x, k) = L$ as (x, k) tends to $(z, -\infty)$ within $\Gamma_l(z)$. We write, simply, $\text{n.t.}\lim_{(x,k)\to z} f(x, k) = L$.

The nontangential convergence is obviously stronger than the "radial" convergence, i.e., $\lim_{k\to-\infty} f(z, k) = L$. We shall show that for regular functions they are equivalent. Let us first consider the case for regularizations:

PROPOSITION 2.1. (a) *If f is locally integrable on K^n, then $f(x, k) \to f(x)$ as $k \to -\infty$ for a.e. $x \in K^n$.*

(b) *If f is locally integrable on K^n, then $f(x, k) \to f(z)$ as $(x, k) \to z$ nontangentially for a.e. $z \in K^n$.*

Proof. (a) See [13] for a proof.

(b) It follows from (a) that

$$\frac{1}{|\mathcal{P}_x^{-k}|} \int_{\mathcal{P}_x^{-k}} [f(t) - f(x)]\, dt \to 0 \quad \text{as } k \to -\infty \text{ for a.e. } x \in K^n. \tag{2.2}$$

We claim, moreover, that

$$\frac{1}{|\mathcal{P}_x^{-k}|} \int_{\mathcal{P}_x^{-k}} |f(t) - f(x)|\, dt \to 0 \quad \text{as } k \to -\infty \text{ for a.e. } x \in K^n. \tag{2.3}$$

The desired result follows from the above claim. In fact, for $(x, k) \in \Gamma_l(z)$, we have $\mathcal{P}_x^{-k} \subset \mathcal{P}_x^{-(k+l)} = \mathcal{P}_z^{-(k+l)}$. Let $z \in K^n$ be a point such that (2.3) is valid. Then, for $(x, k) \in \Gamma_l(z)$,

$$|f(x,k)-f(z)| = \frac{1}{|\mathscr{P}_x^{-k}|}\Big|\int_{\mathscr{P}_x^{-k}} [f(t)-f(z)]\,dt\Big|$$

$$\leqslant \frac{1}{|\mathscr{P}_x^{-k}|}\int_{\mathscr{P}_x^{-k}} |f(t)-f(z)|\,dt$$

$$\leqslant \frac{q^{nl}}{|\mathscr{P}_z^{-(k+l)}|}\int_{\mathscr{P}_z^{-(k+l)}} |f(t)-f(z)|\,dt \to 0 \quad \text{as } k\to-\infty.$$

Thus $f(x,k)\to f(z)$ as $(x,k)\to z$ nontangentially.

It remains to show (2.3). Note that $|f(t)-\varrho|$ is locally integrable for any rational (complex) number ϱ. Hence, by (2.2),

$$\frac{1}{|\mathscr{P}_x^{-k}|}\int_{\mathscr{P}_x^{-k}} [|f(t)-\varrho|-|f(x)-\varrho|]\,dt\to 0 \quad \text{a.e. as } k\to-\infty. \tag{2.4}$$

Let F_ϱ be the exceptional set for (2.4). $F = \bigcup_\varrho F_\varrho$ has measure 0. If $x\notin F$, for any $\varepsilon>0$, let ϱ be such that $|f(x)-\varrho|<\varepsilon/2$. Then

$$\frac{1}{|\mathscr{P}_x^{-k}|}\int_{\mathscr{P}_x^{-k}} |f(t)-f(x)|\,dt \leqslant \frac{1}{|\mathscr{P}_x^{-k}|}\int_{\mathscr{P}_x^{-k}} [|f(t)-\varrho|+|f(x)-\varrho|]\,dt<\varepsilon$$

for $-k$ large enough as follows from (2.4). This completes the proof of (2.3) and (b).

Before proceeding the theorem on equivalence, we introduce the following:

Definition. Let $\mathscr{D}$ be a simple domain in $K^n\times \mathbf{Z}$ and $m = m(\mathscr{D})$ $(=\inf_x\sup\{k\colon (x,k)\in\mathscr{D}\}<-\infty$. See §1.). For $f(x,k)$ regular on $\mathscr{D}$, we define $\bar f(x,k)$ on $K^n\times\mathbf{Z}$ as follows:

$$\bar f(x,k) = \begin{cases} f(x,k) & \text{if } (x,k)\in\mathscr{D},\\ f(x,l) & \text{if } (x,l)\in\partial\mathscr{D} \text{ and } k\leqslant l,\end{cases} \tag{2.5}$$

for other values of (x,k), $\bar f(x,k) = 0$ if $k\leqslant m$ and, finally, $\bar f(x,k) = R_k * \bar f(\cdot,m)(x)$ if $k\geqslant m$. $\bar f(x,k)$ which is obviously well-defined and regular on $K^n\times\mathbf{Z}$ is called the *extension* of $f(x,k)$ on $K^n\times\mathbf{Z}$.

For $l\in\mathbf{Z}^+$ and $h\in\mathbf{Z}$ let $\Gamma_l^h(z)$ denote the truncated cone $\Gamma_l^h(z) = \{(x,k)\in K^n\times\mathbf{Z}\colon\ |x-z|\leqslant q^{k+l},\ k\leqslant h\} = \bigcup_{k=-\infty}^{h} (\mathscr{P}_z^{-(k+l)},k)$. The height h of the truncation is not essential. For a regular function $f(x,k)$ on $K^n\times\mathbf{Z}$, let

$d_kf(x) = f(x,k)-f(x,k+1)$. The *Lusin area function* of f with respect to the fixed cone $\Gamma_l^h(z)$ is given by

$$S^{(l)}(f)(z) = \Big(\sum |d_kf(x)|^2\Big)^{\frac12}$$

where the summation runs over distinct $(\mathscr{P}_x^{-k},k)\subset\Gamma_l^h(z)$. We single out the cone $\Gamma_0^0(z)$ and $S(f)(z) = S^{(0)}(f)(z) = \big(\sum_{k=-\infty}^{0}|d_kf(z)|^2\big)^{\frac12}$ is just a truncated Littlewood–Paley function.

We are now going to prove the following version of the Fatou–Calderón–Stein theorem:

Theorem 2.6. *If $f(x,k)$ is regular on $K^n\times\mathbf{Z}$, then the following sets are equivalent*:

$$A = \{x\in K^n\colon\ \lim_{k\to-\infty} f(x,k)\ \textit{exists}\};$$

$$B = \{z\in K^n\colon\ \operatorname*{n.t.lim}_{(x,k)\to z} f(x,k)\ \textit{exists}\};$$

$$C = \{x\in K^n\colon\ \sup_{k\in\mathbf{Z}^-}|f(x,k)|<\infty\};$$

$$D = \{x\in K^n\colon\ S(f)(x)<\infty\};$$

$$L = \{z\in K^n\colon\ S^{(l)}(f)(z)<\infty\}.$$

We need the following lemmas:

Lemma 2.7. *Let $\{(x_j,k_j)\}_{j=1}^\infty\subset\Gamma_l(z)$, $l\in\mathbf{Z}^+$, be such that $(x_j,k_j)\to(z,-\infty)$ as $j\to\infty$. If z is a point of density of F, then $(x_j,k_j)\in\bigcup_{y\in F}\Gamma_0(y)$ for all j large enough.*

Proof. We first note that if ξ_F is the characteristic function of a (measurable) set $F\subset K^n$, then z is a point of density of F if $\frac{1}{|\mathscr{P}_z^m|}\int_{\mathscr{P}_z^m}\xi_F(x)\,dx\to 1$ as $m\to\infty$. Observe that Proposition 2.1 (a) implies that almost every point in F is a point of density of F.

Now, for $(x,k)\notin\bigcup_{y\in F}\Gamma_0(y)$, we have $\mathscr{P}_x^{-k}\cap F = \emptyset$. If, moreover, $(x,k)\in\Gamma_l(z)$, then $x\in\mathscr{P}_z^{-(k+l)}$ and $\mathscr{P}_x^{-k}\subset\mathscr{P}_z^{-(k+l)}$. Let $E = \mathscr{P}_z^{-(k+l)}\setminus\mathscr{P}_x^{-k}$. Then

$$\frac{1}{|\mathscr{P}_z^{-(k+l)}|}\int_{\mathscr{P}_z^{-(k+l)}}\xi_F(y)\,dy = \frac{|\mathscr{P}_z^{-(k+l)}\cap F|}{|\mathscr{P}_z^{-(k+l)}|} = \frac{|E\cap F|}{q^{n(k+l)}}$$

$$\leqslant\frac{|E|}{q^{n(k+l)}} = 1-q^{-nl}.$$

Suppose the conclusion is not true, then there exists a subsequence $\{(x_t,k_t)\}\subset\{(x_j,k_j)\}$ such that $(x_t,k_t)\to(z,-\infty)$ as $t\to\infty$ and

$(x_l, k_l) \notin \bigcup_{y \in F} \Gamma_0(y)$. But then

$$\lim_{l\to\infty} \frac{1}{|\mathscr{P}_z^{-(k_l+l)}|} \int_{\mathscr{P}_z^{-(k_l+l)}} \xi_F(y)\,dy \leqslant 1-q^{-nl} < 1.$$

This contradicts the fact that z is a point of density of F. The proof of Lemma 2.7 is therefore completed.

LEMMA 2.8. (a) *Suppose $f(x, k)$ is regular on $K^n \times \boldsymbol{Z}$ such that $f(x, k) \to 0$ as $k \to \infty$ for each x and $\big(\sum_{k=-\infty}^{\infty} |d_k f(x)|^2\big)^{\frac12} \in L^p(K^n)$ for some $1 < p < \infty$. Then $f(x, k)$ is the regularization of a function $F \in L^p(K^n)$.*

(b) *If $f \in L^p(K^n)$ with $1 < p < \infty$, then $\big(\sum_{k=-\infty}^{\infty} |d_k f(x)|^2\big)^{\frac12}$ exists for a.e. $x \in K^n$ and is in $L^p(K^n)$.*

(c) *If $f \in L^2(K^n)$ and $S^{(l)}(f)$ is the Lusin area function of f with respect to Γ_l^h, then $S^{(l)}(f)(z)$ exists for a.e. $z \in K^n$ and $\|S^{(l)}(f)\|_2^2 = q^{nl} \int_{K^n} \sum_{k=-\infty}^{h} |d_k f(x)|^2\,dx < \infty$.*

Proof. (a) and (b) are known. See [13] for a proof.

(c) For fixed $k \in \boldsymbol{Z}$,

$$\int_{K^n} \sum_{i=1}^{q^{nl}} |d_k f(x_i)|^2\,dz = q^{nl} \int_{K^n} |d_k f(z)|^2\,dz$$

where x_i are such that $\mathscr{P}_{x_i}^{-k}$ being distinct cosets in $\mathscr{P}_z^{-(k+l)}$. Thus (c) follows from (b) by taking the summation with respect to k.

Proof of Theorem 2.6. The fact that $B \subset A \subset C$ and $L \subset D$ is trivial. We start by showing that $C \subseteq B$ and $C \subseteq L$.

For $M \in \boldsymbol{Z}^+$, let $E_M = \{x \in K^n\colon \sup_{k \in \boldsymbol{Z}^-} |f(x, k)| \leqslant M\} \cap \mathcal{O}$. We observe that it suffices to consider E_M instead of C. For $m \in \boldsymbol{Z}^-$, let $E_M^m = \{z \in E_M\colon \Gamma_{l+1}^m(z) \subset \bigcup_{y \in E_M} \Gamma_0(y)\}$ where $\Gamma_{l+1}^m(z) = \{(x, k) \in \Gamma_{l+1}(z)\colon k \leqslant m\}$. From Lemma 2.7, we see that for almost all $z \in E_M$, there exists an $j(z) \in \boldsymbol{Z}$ such that $\Gamma_{l+1}^j(z) \subset \bigcup_{y \in E_M} \Gamma_0(y)$. Thus, $\lim_{m\to-\infty} E_M^m \cong E_M$. Hence it is sufficient for us to consider the set E_M^m.

Now notice that $\mathscr{D} = \bigcup \{\Gamma_{l+1}^m(z)\colon z \in E_M^m\}$ is a simple domain in $K^n \times \boldsymbol{Z}$ with $\partial\mathscr{D} = \mathscr{D} \setminus \bigcup \{\Gamma_l^m(z)\colon z \in E_M^m\}$. Also that $|f(x, k)| \leqslant M$ for all $(x, k) \in \mathscr{D}$ since $\mathscr{D} \subset \bigcup_{y \in E_M} \Gamma_0(y)$.

Consider $f(x, k)$ as a regular function defined on $\mathscr{D}$, let $\bar{f}(x, k)$ be its extension on $K^n \times \boldsymbol{Z}$. Thus $|\bar{f}(x, k)| \leqslant M$ on $K^n \times \boldsymbol{Z}$. Hence, by Proposition 1.7, $\bar{f}(x, k)$ is the regularization of a function $F \in L^\infty(K^n)$. And Proposition 2.1 (b) says that $\text{n.t.}\lim_{(x,k)\to z} \bar{f}(x, k) = F(z)$ for a.e. $z \in K^n$. Therefore, $\text{n.t.}\lim_{(x,k)\to z} f(x, k)$ exists for a.e. $z \in E_M^m$. That is, $C \subseteq B$.

Moreover, from the construction of $\bar{f}(x, k)$, we see that $F(x)$ is supported on $\mathcal{O}$. Hence $F \in L^2(K^n)$. By Lemma 2.8 (c), we have $S^{(l)}(f)(z) = S^{(l)}(\bar{f})(z) < \infty$ for a.e. $z \in E_M^m$. Therefore $C \subseteq L$.

To complete the proof of the theorem, it suffices to show $D \subseteq A$.

As before without loss of generality, we may consider the set $E_M^m = \{x \in E_M\colon \Gamma_1^m(x) \subset \bigcup_{y \in E_M} \Gamma_0(y)\}$ where $E_M = \{x \in \mathcal{O}\colon S(f)(x) \leqslant M\}$. Let $\mathscr{D} = \bigcup \{\Gamma_1^m(x)\colon x \in E_M^m\}$. Consider $f(x, k)$ as a regular function on the simple domain $\mathscr{D}$, let $\bar{f}(x, k)$ be its extension on $K^n \times \boldsymbol{Z}$. Hence $\bar{f}(x, k)$ is regular on $K^n \times \boldsymbol{Z}$ and $S(\bar{f})(x) \leqslant M$. We have also that $\bar{f}(x, k) \to 0$ as $k \to \infty$ since $\bar{f}(x, m)$ has finite support. An easy computation shows that there exists a constant N, depending only on M and q^n, such that the function $\big(\sum_{k=-\infty}^{\infty} |d_k \bar{f}(x)|^2\big)^{\frac12} \leqslant N$ and moreover that it is in $L^2(K^n)$. Hence, by Lemma 2.8 (a) and Proposition 2.1 (a), $\lim_{k\to-\infty} \bar{f}(x, k)$ exists for a.e. $x \in E_M^m$. Thus $D \subseteq A$.

The proof of Theorem 2.6 is completed.

Remark. The "extension" argument in the proof above, namely (2.5), served as the role of the conformal mapping used to prove the corresponding result on the (complex) unit disc, for instance, in [15], Chapter XIV, (and the upper half-plane $\boldsymbol{R}_+^2$) that is not available in the study of $\boldsymbol{R}_+^{n+1}$, $n > 1$.

B. CONJUGATE SYSTEMS OF REGULAR FUNCTIONS

In § 3 we apply the results in Part A to show the main theorem of H^p-spaces. In § 4, we define the conjugate systems which generalize the notion of Hilbert transform in order to have the basic subregularity we need for the study of H^p-spaces. The F. and M. Riesz theorem is treated in § 5.

§ 3.

THEOREM 3.1. *Let $F(x, k) = \big(f_0(x, k), f_1(x, k), \ldots, f_m(x, k)\big)$ be a vector valued function with each component f_j, $j = 0, 1, \ldots, m$, being regular on $K^n \times \boldsymbol{Z}$. Suppose there exists a p_0, $0 < p_0 < 1$, such that $|F(x, k)|^{p_0}$ is subregular where $|F(x, k)| = \big(\sum_{j=0}^{m} |f_j(x, k)|^2\big)^{\frac12}$. Suppose, further, for some $p > p_0$,*

$$(3.2)\qquad \int_{K^n} |F(x, k)|^p\,dx \leqslant A < \infty \quad \textit{for all } k \in \boldsymbol{Z}.$$

Then the limits

$$f_j(x) = \lim_{k\to-\infty} f_j(x, k)$$

exist for a.e. $x \in K^n$ *and*

$$\lim_{k\to-\infty} \int_{K^n} |F(x, k) - F(x)|^p dx = 0$$

where $F(x) = (f_0(x), f_1(x), \ldots, f_m(x))$. *Moreover,* $f_j^*(x) = \sup_{k\in \mathbf{Z}} |f_j(x, k)| \in L^p(K^n)$, $j = 0, 1, \ldots, m$.

Proof. Let $p_1 = \dfrac{p}{p_0} > 1$. We have, for all $k \in \mathbf{Z}$,

$$\int_{K^n} |F(x, k)|^{p_0 \cdot p_1} dx = \int_{K^n} |F(x, k)|^p dx \leqslant A < \infty.$$

By Theorem 1.9, since $|F(x, k)|^{p_0}$ is subregular, $|F(x, k)|^{p_0}$ has the least regular majorant $m(x, k)$ which is the regularization of a function $m \in L^{p_1}(K^n)$. This implies $m^*(x) = \sup_{k\in\mathbf{Z}} |m(x, k)| < \infty$ almost everywhere and $m^* \in L^{p_1}(K^n)$ since the operator $m \to m^*$ is of type (p_1, p_1) with $p_1 > 1$. Now, for a.e. $x \in K^n$, $j = 0, 1, \ldots, m$,

$$f_j^*(x) = \sup_{k\in\mathbf{Z}} |f_j(x, k)| \leqslant \sup_{k\in\mathbf{Z}} |F(x, k)| \leqslant \sup_{k\in\mathbf{Z}} [m(x, k)]^{\frac{1}{p_0}} = [m^*(x)]^{\frac{p_1}{p}}.$$

Hence, $f_j^*(x) < \infty$ almost everywhere and $f_j^* \in L^p(K^n)$, $j = 0, 1, \ldots, m$. By Theorem 2.6, the limits $f_j(x) = \lim_{k\to-\infty} f_j(x, k)$ exist almost everywhere. That is, $\lim_{k\to-\infty} F(x, k) = F(x)$ for a.e. $x \in K^n$. Note that

$$|F(x, k) - F(x)|^p \leqslant 2^p (|F(x, k)|^p + |F(x)|^p) \leqslant 2^{p+1} [m^*(x)]^{p_1} \in L^1.$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\lim_{k\to-\infty} \int_{K^n} |F(x, k) - F(x)|^p dx = 0.$$

This completes the proof.

From Theorem 3.1, we know that it is desirable to define "conjugate systems" of regular functions, $F(x, k) = (f_0(x, k), f_1(x, k), \ldots \ldots, f_m(x, k))$, such that $|F(x, k)|^{p_0}$ is subregular for some positive $p_0 < 1$. This will be our main task in § 4.

§ 4. We restrict our attention to (the one dimensional case) K and exclude the case with even q.

Let $\mathcal{O}^* = \mathcal{O} \setminus \mathcal{P}$ be the group of units in K and $\mathcal{O}/\mathcal{P} = \{0, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{q-1} = 1\}$ $(\cong GF(q))$. Let $\pi \in \widehat{K^*}$ be a (multiplicative) unitary character in K^*. Denote $A_0 = \mathcal{O}^*$, $A_1 = 1 + \mathcal{P}$, $A_h = 1 + \mathcal{P}^h$; $h \geqslant 1$. If π is trivial on A_0, we say that π is unramified. If π is trivial on A_h, but not on A_{h-1} $(h \geqslant 1)$, we say that π is ramified of degree h. See [8] for details.

We consider those $\pi \in \widehat{K^*}$ such that π is ramified of degree 1 and is homogeneous of degree 0 (i.e., $\pi(\varrho^s x) = \pi(x)$, for all $s \in \mathbf{Z}$). It follows that π is constant on the cosets of $\mathcal{P}$ in $\mathcal{O}^*$ and, since π is a nontrivial character on the compact group $\mathcal{O}^*$, that $\int_{\mathcal{O}^*} \pi(x)\,dx = 0$. Thus π takes values on U_{q-1}, the cyclotomic group of order $q-1$ (the group of all $(q-1)$th roots of unity). The set Π of all such π's forms a cyclic group of order $q-1$. Any π, such that $\pi(\varepsilon) = \zeta$ with ζ a primitive root, is a generator. $\Pi = \langle \pi \rangle = \{\pi, \pi^2, \ldots, \pi^{q-1} = 1\}$. Note that $\pi^l(\varepsilon^j) = \zeta^{lj}$ and $\pi(-x) = -\pi(x)$ for all $x \in K^*$, i.e., π is odd.

Let $Q^l(x) = c_l \dfrac{\pi^l(x)}{|x|}$; $l = 1, 2, \ldots, q-2$ where $\dfrac{1}{c_l} = \Gamma(\pi^l) = \text{p.v.} \int_K \chi(x) \pi^l(x) |x|^{-1} dx$. (See [8] for details about Γ-function.) Denote $Q_k^l(x) = Q^l(x, k)$. It is easy to see that $Q_k^l(x) = Q^l(x)[1 - \Phi_{-k}(x)]$.

Define, for a "nice" function f on K, $T_l f$ ap $T_l' f$ by

$$T_l f(x) = \lim_{k\to-\infty} Q_k^l * f(x);$$

$$(T_l' f)^\wedge(u) = \pi^{-l}(u) \hat{f}(u).$$

T_l is a singular integral transform and T_l' is a multiplier transform, by the results in [7] and [14], §4 respectively, we know that they are of type (p, p), $1 < p < \infty$ and of weak-type $(1, 1)$.

Moreover, they are actually the same operator as follows from the following lemma of Sally–Taibleson [8]:

Lemma. $\left(\dfrac{\pi(x)}{|x|}\right)^\wedge(u) = \Gamma(\pi)\pi^{-1}(u)$, *for all* $\pi \in \widehat{K^*}$.

Indeed, $(Q_k^l * f)^\wedge(u) = \hat{Q}_k^l(u)\hat{f}(u) = \hat{Q}^l(u)\Phi_k(u)\hat{f}(u) = \pi^{-l}(u)\Phi_k(u)\hat{f}(u)$ and for $f \in L^2$, by Plancherel's theorem, $T_l f = \lim_{k\to-\infty} Q_k^l * f(x)$ has Fourier transform $\pi^{-l}\hat{f}$. Thus, $T_l f = T_l' f$ almost everywhere.

Note that $(R_k * T_l f)^\wedge = \Phi_k \pi^{-l} \hat{f}$. Hence, we have $(T_l f)(x, k) = Q_k^l * f(x) = T_l(f(x, k))$.

Setting $\varepsilon_m^j = \varrho^{-(m+1)} \varepsilon^j$ for $m \in \mathbf{Z}$, we have $\mathcal{P}^{-(m+1)}/\mathcal{P}^{-m} = \{0, \varepsilon_m, \varepsilon_m^2, \ldots, \varepsilon_m^{q-1}\}$. Let us compute $T_l f(x, k)$:

$$T_l f(x, k) = Q_k^l * f(x) = c_l \int_{|t| > q^k} f(x-t) \frac{\pi^l(t)}{|t|} dt \tag{4.1}$$

$$= c_l \sum_{m=k}^{\infty} q^{-(m+1)} \int_{|t| = q^{m+1}} f(x-t) \pi^l(t)\, dt$$

$$= c_l \sum_{m=k}^{\infty} q^{-(m+1)} \sum_{i=1}^{q-1} \pi^l(\varepsilon_m^i) \int_{\varepsilon_m^i + \mathscr{P}^{-m}} f(x-t)\,dt$$

$$= c_l q^{-1} \sum_{m=k}^{\infty} \sum_{i=1}^{q-1} \pi^l(\varepsilon_m^i) f(x-\varepsilon_m^i, m).$$

Denote $d_k f(x) = f(x, k) - f(x, k+1)$. Hence $T_l d_k f(x) = T_l f(x, k) - T_l f(x, k+1) = d_k T_l f(x)$. And from (4.1), we have

$$T_l d_k f(x) = d_k T_l f(x) = c_l q^{-1} \sum_{i=1}^{q-1} \pi^l(\varepsilon_k^i) f(x-\varepsilon_k^i, k) \tag{4.2}$$

$$= c_l q^{-1} \sum_{i=1}^{q-1} \pi^l(\varepsilon_k^i)\, d_k f(x-\varepsilon_k^i).$$

For a fixed coset $y + \mathscr{P}^{-(k+1)}$, write $\varepsilon_k^0 = 0$, let

$$a_0 = f(y, k+1), \quad a_0^i = d_k f(y+\varepsilon_k^i); \tag{4.3}$$

$$a_l = T_l f(y, k+1), \quad a_l^i = T_l d_k f(y+\varepsilon_k^i),$$

for $i = 0, 1, \ldots, q-1$; $l = 1, 2, \ldots, q-2$. Then, for $x \in y + \varepsilon_k^j + \mathscr{P}^{-k}$, after a change of variable (4.2) becomes

$$a_l^j = c_l q^{-1} \sum_{i=0}^{q-1} \pi^l(\varepsilon_k^j - \varepsilon_k^i)\, d_k f(y+\varepsilon_k^i) \tag{4.4}$$

$$= c_l q^{-1} \sum_{i=0}^{q-1} \pi^l(\varepsilon^j - \varepsilon^i)\, a_0^i$$

with the convention $\pi(0) = 0$. Let $\|a_l\| = \left(\sum_{i=0}^{q-1} |a_l^i|^2\right)^{\frac{1}{2}}$, $l = 0, 1, \ldots, q-2$. With this notation, we have

PROPOSITION 4.5. (a) $\sum_{i=0}^{q-1} a_l^i = 0$, $l = 0, 1, \ldots, q-2$;

(b) $\|a_l\| = \|a_0\|$, $l = 1, 2, \ldots, q-2$;

(c) $\sum_{i=0}^{q-1} a_j^i a_l^i = 0$ *whenever* $j+l$ *is odd*, $j, l = 0, 1, \ldots, q-2$.

Proof. (a) Follows immediately from the regularity.

(b) Let $g(x)$ be the restriction of $d_k f(x)$ on $y + \mathscr{P}^{-(k+1)}$. We see from (4.2) that $T_l g(x)$ is also supported on $y + \mathscr{P}^{-(k+1)}$. By the Plancherel's theorem, since $|\pi^l| = 1$ for all l, we have

$$\|T_l g\|_2 = \|\widehat{T_l g}\|_2 = \|\pi^{-l}\hat{g}\|_2 = \|\hat{g}\|_2 = \|g\|_2.$$

(Note that $\hat{g}(0) = 0$.) That is, $\|a_l\| = \|a_0\|$ for all l.

(c) Since $T_j T_k = T_{j+k}$, we may assume that $j = 0$ and l is odd. Thus π^l is odd. From (4.4) we then have

$$\sum_{i=0}^{-1} a_0^i a_l^i = c_l q^{-1} \sum_{i=1}^{q-1} a_0^i \sum_{k=0}^{q-1} \pi^l(\varepsilon^i - \varepsilon^k)\, a_0^k$$

$$= c_l q^{-1} \sum_{i \neq k} \pi^l(\varepsilon^i - \varepsilon^k)\, a_0^i a_0^k$$

$$= c_l q^{-1} \sum_{i > k} [\pi^l(\varepsilon^i - \varepsilon^k) + \pi^l(\varepsilon^k - \varepsilon^i)]\, a_0^i a_0^k = 0.$$

Remark. The introduction of the function $g(x)$ in the above proof, in particular $g(x) = d_{-1} f(x) \Phi_0(x)$, can be used to study "Fourier analysis" on $GF(q)$. $\widehat{GF(q)} \cong GF(q)$. (b) is, as we have seen, just Plancherel's theorem and (c) follows from the Parseval formula.

Consider now a $q \times (m+1)$ matrix (a_j^i) with complex entries. Let

$$a_j = (a_j^0, a_j^1, \ldots, a_j^{q-1}) \in C^q \quad \text{with} \quad \|a_j\| = \left(\sum_{i=0}^{q-1} |a_j^i|^2\right)^{\frac{1}{2}}, \; j = 0, 1, \ldots, m;$$

$$a^i = (a_0^i, a_1^i, \ldots, a_m^i) \in C^{m+1} \quad \text{with} \quad |||a^i||| = \left(\sum_{j=0}^{m} |a_j^i|^2\right)^{\frac{1}{2}}, \; i = 0, 1, \ldots, q-1.$$

Also $a = (a_0, a_1, \ldots, a_m) \in C^{m+1}$. Suppose $\{0, 1, \ldots, m\} = D \cup E$, where D and E are non-empty, disjoint.

THEOREM 4.6. *If*

$$\sum_{i=0}^{q-1} a_j^i = 0, \qquad j = 0, 1, \ldots, m; \tag{4.7}$$

$$\|a_j\| = \|a_0\|, \qquad j = 1, 2, \ldots, m; \tag{4.8}$$

$$\sum_{i=0}^{q-1} a_j^i a_k^i = 0, \quad \text{whenever } j \in D \text{ and } k \in E, \tag{4.9}$$

then there exists a p_0, $0 < p_0 < 1$, *such that*

$$|||a|||^p \leq \frac{1}{q} \sum_{i=0}^{q-1} |||a + a^i|||^p \tag{4.10}$$

for all $p \geq p_0$, *where* p_0 *is independent of* a *and* (a_j^i).

This theorem generalizes Theorem 2 of [1] where the case $m = 1$ was treated. Before proceeding to the proof, we shall study the statement further. Notice that (4.10) is the local subregularity we need. Thus it suffices to define a "conjugate system" satisfying (4.7), (4.8) and (4.9). Namely:

DEFINITION. Suppose $f_0, f_1, \ldots, f_m$ are regular functions defined on a domain $\mathscr{D} \subset K \times \mathbf{Z}$. For any fixed $(y, k+1) \in \mathscr{D} \setminus \partial\mathscr{D}$ (thus, $(\mathscr{P}_y^{-(k+1)}, k+1) \cup (\mathscr{P}_y^{-(k+1)}, k) \subset \mathscr{D}$), denote

$a_j = f_j(y, k+1)$; $a_j^i = d_k f_j(y+\varepsilon_k^i)$, $\quad j = 0, 1, \ldots, m$; $i = 0, 1, \ldots, q-1$,

where $\varepsilon_k^i = p^{-(k+1)}\varepsilon^i$ and $\varepsilon_k^0 = 0$. If (4.7), (4.8) and (4.9) are satisfied, then $F(x, k) = \big(f_0(x, k), f_1(x, k), \ldots, f_m(x, k)\big)$ is called a *conjugate system* on $\mathscr{D}$.

Thus once Theorem 4.6 is established we have immediately the following:

THEOREM 4.11. *If $F = (f_0, f_1, \ldots, f_m)$ is a conjugate system of regular functions on a domain $\mathscr{D} \subset K \times \mathbf{Z}$, then there exists a p_0, $0 < p_0 < 1$, p_0 independent of F, such that $|F(x, k)|^p$ is subregular on $\mathscr{D}$ for all $p \geqslant p_0$.*

We provide some examples of conjugate systems:

(i) Suppose $\{\pi^{l_j}\}_{j=1}^m$ is a subset of $\Pi = \langle\pi\rangle$ which contains at least one even l_j and at least one odd l_j. Then for a regular function f, $F = (T_{l_1}f, \ldots, T_{l_m}f)$ is a conjugate system as follows from Proposition 4.5 by letting the sets D and E in (4.9) be the odd and the even integers, respectively. In particular, $(f, T_1 f)$ and $(f, T_1 f, \ldots, T_{q-2}f)$ are conjugate systems.

(ii) $F = (f, T_l f)$, with odd l, is a conjugate system. If $\dfrac{q-1}{2}$ happens to be odd, then π^l, with $l = \dfrac{q-1}{2}$, takes only ± 1 as its values. With π^l substituting the sgn function, T_l is the "exact" analogue of the Hilbert transform. The case $q = 3$ has been studied in [1]. Note that, in this case $(q = 3)$, if we take $\chi(x) = e^{-2\pi i\sigma(x)}$ where for $x = (x_m)_{m=-k}^{\infty}$, $x_m \in \{0, 1, 2\}$, $\sigma(x) = \sum_{m=-k}^{-1} 3^m x_m$ on the 3-adic field; $\sigma(x) = 3^{-1}x_{-1}$ on the 3-series field as used by Phillips in [6], then by an easy computation we have

$$\Gamma(\pi) = \int_K \chi(x)\frac{\pi(x)}{|x|}\,dx = -\frac{i}{\sqrt{3}}.$$

Hence (4.4) takes the following simple form

$$a_1^j = \frac{i}{\sqrt{3}}(a_0^{j+1} - a_0^{j-1}), \qquad j \in \mathbf{Z}_3.$$

This simplest case often plays a suggestive role.

We now give a proof of Theorem 4.6. The proof follows very closely that of Theorem 2 in [1]. We need the following lemma:

LEMMA 4.12. *Let a and (a_j^i) be as in the theorem; that is,* (4.7), (4.8) *and* (4.9) *are valid. Given p_1, $p_1 > \dfrac{m-1}{m}$, there exists a constant $A_{p_1} > 0$ such that* (4.10) *holds for all $p \geqslant p_1$ provided $\|a_0\| \leqslant A_{p_1}|||a|||$.*

Proof. We may assume that $|||a||| \neq 0$ and $0 < p_1 \leqslant p \leqslant 1$.

$$(4.13)\qquad \sum_{i=0}^{q-1} |||a + a^i|||^p = \sum_{i=0}^{q-1}\Big\{\sum_{j=0}^{m} |a_j + a_j^i|^2\Big\}^{\frac{p}{2}}$$
$$= \sum_i \Big\{\sum_j |a_j|^2 + 2\,\mathrm{Re}\sum_j \bar{a}_j a_j^i + \sum_j |a_j^i|^2\Big\}^{\frac{p}{2}}$$
$$= |||a|||^p \sum_i \Big\{1 + \frac{2\,\mathrm{Re}\sum_j \bar{a}_j a_j^i}{|||a|||^2} + \frac{|||a^i|||^2}{|||a|||^2}\Big\}^{\frac{p}{2}}.$$

By using (4.9) we have the following estimate:

$$(4.14)\qquad \sum_i \Big(\mathrm{Re}\sum_j \bar{a}_j a_j^i\Big)^2 \leqslant \sum_i \Big|\sum_{j\in D} \bar{a}_j a_j^i + \sum_{j\in E} a_j \bar{a}_j^i\Big|^2$$
$$= \sum_i \Big|\sum_{j\in D} \bar{a}_j a_j^i\Big|^2 + \sum_i \Big|\sum_{j\in E} a_j \bar{a}_j^i\Big|^2 + 2\,\mathrm{Re}\sum_{\substack{j\in D\\ k\in E}} \bar{a}_j \bar{a}_k \sum_i a_j^i a_k^i$$
$$\leqslant \sum_i \Big(\sum_{j\in D} |a_j|\,|a_j^i|\Big)^2 + \sum_i \Big(\sum_{j\in D} |a_j|\,|a_j^i|\Big)^2$$
$$\leqslant \sum_i \Big(\sum_{j\in D} |a_j|^2\Big)\Big(\sum_{j\in D} |a_j^i|^2\Big) + \sum_i \Big(\sum_{j\in E} |a_j|^2\Big)\Big(\sum_{j\in E} |a_j^i|^2\Big)$$
$$\leqslant m\|a_0\|^2\Big(\sum_{j\in D} |a_j|^2\Big) + m\|a_0\|^2\Big(\sum_{j\in E} |a_j|^2\Big)$$
$$= m\|a_0\|^2\,|||a|||^2.$$

In particular, $|\mathrm{Re}\sum_j \bar{a}_j a_j^i| \leqslant \sqrt{m}\|a_0\|\cdot|||a|||^2$. Thus, assuming $\|a_0\| \leqslant (3\sqrt{m})^{-1}|||a|||$, we have

$$\left|\frac{2\,\mathrm{Re}\sum_j \bar{a}_j a_j^i}{|||a|||^2} + \frac{|||a^i|||^2}{|||a|||^2}\right| \leqslant 2\sqrt{m}\frac{\|a_0\|}{|||a|||} + (m+1)\frac{\|a_0\|^2}{|||a|||^2}$$
$$\leqslant \frac{2}{3} + \frac{m+1}{9m} \leqslant \frac{8}{9} < 1.$$

Hence with the binomial expansion of each summand in (4.13), we have

$$(4.15)\qquad \sum_{i=0}^{q-1} |||a+a^i|||^p = |||a|||^p \sum_i \Big\{1+\frac{p\,\mathrm{Re}\sum_j \bar a_j a_j^i}{|||a|||^2}+\frac{p}{2}\cdot\frac{|||a^i|||^2}{|||a|||^2} -$$
$$-\frac{p(2-p)}{8|||a|||^4}\Big[4\Big(\mathrm{Re}\sum_j \bar a_j a_j^i\Big)^2+4|||a^i|||^2\Big(\mathrm{Re}\sum_j \bar a_j a_j^i\Big)+|||a^i|||^4\Big]+R_{3i}\Big\}$$

where R_{3i} are the third Taylor remainders.

Observe that

$$\sum_i \frac{p\,\mathrm{Re}\sum_j \bar a_j a_j^i}{|||a|||^2} = 0, \quad \text{by (4.7)},$$

$$\frac{p}{2}\sum_i \frac{|||a^i|||^2}{|||a|||^2} = \frac{(m+1)p}{2}\cdot\frac{\|a_0\|^2}{|||a|||^2}, \quad \text{by (4.8)},$$

$$\frac{p(2-p)}{8|||a|||^4}\sum_i 4\Big(\mathrm{Re}\sum_j \bar a_j a_j^i\Big)^2 \leqslant \frac{p(2-p)m}{2}\cdot\frac{\|a_0\|^2}{|||a|||^2}, \quad \text{by (4.14)}$$

and that the remaining terms in (4.15) are bounded by $B\dfrac{\|a_0\|^3}{|||a|||^3}$, with positive B independent of p. Therefore, with these estimates, (4.15) gives us

$$\sum_i |||a+a^i|||^p$$
$$\geqslant q|||a|||^p+\frac{\|a_0\|^2}{|||a|||^2}\Big[\frac{(m+1)p}{2}-\frac{mp(2-p)}{2}-B\frac{\|a_0\|}{|||a|||}\Big]|||a|||^p$$
$$\geqslant q|||a|||^p+\frac{\|a_0\|^2}{|||a|||^2}\Big[\frac{mp_1}{2}\Big(p_1-\frac{m-1}{m}\Big)-B\frac{\|a_0\|}{|||a|||}\Big]|||a|||^p\geqslant q|||a|||^p$$

provided $p\geqslant p_1>\dfrac{m-1}{m}$ and $\dfrac{\|a_0\|}{|||a|||}\leqslant A_{p_1}=\min\Big[\dfrac{1}{3\sqrt{m}},\dfrac{p_1(mp_1-m+1)}{2B}\Big]$.

This completes the proof of Lemma 4.12.

Proof of Theorem 4.6. We may assume that, for any fixed p_1, $p_1>\dfrac{m-1}{m}$,

$$(4.16)\qquad \frac{1}{q}\sum_i |||a+a^i||| = 1 \quad\text{and}\quad \|a_0\|\geqslant A_{p_1}|||a|||,$$

by the homogeneity of the first expression and Lemma 4.12.

The collection $\mathscr{B}$ of all vectors $\beta=\{a+a^i\}_{i=0}^{q-1}$ satisfying (4.7), (4.8), (4.9) and (4.16) forms a compact set (in $C^{q(m+1)}$).

We claim that there exists a δ, $0<\delta<1$, such that

$$(4.17)\qquad |||a|||\leqslant \delta\frac{1}{q}\sum_i |||a+a^i||| \quad \text{for all } \beta\in\mathscr{B}.$$

If this is the case, then (4.10) holds for all $\beta\in\mathscr{B}$, $p\geqslant p_2=\Big(1+\Big[\log\Big(\dfrac{1}{\delta}\Big)/\log q\Big]\Big)^{-1}$. Thus (4.10) is valid for all $p\geqslant p_0=\max(p_1,p_2)$.

It remains to show the claim. Suppose there does not exist a δ, $0<\delta<1$ such that (4.17) is valid, then by the compactness of $\mathscr{B}$, there is a $\beta=\{a+a^i\}_{i=0}^{q-1}\in\mathscr{B}$ such that

$$|||a||| = \frac{1}{q}\sum_i |||a+a^i|||.$$

Hence there exist real λ_i; $i=0,1,\ldots,q-1$ such that

$$a+a^i=\lambda_i a,\qquad i=0,1,\ldots,q-1.$$

That is, $a_j^i=(\lambda_i-1)a_j$, $i=0,1,\ldots,q-1$; $j=0,1,\ldots,m$. From (4.9) we have, for $j\in D$ and $k\in E$,

$$0=\sum_i a_j^i a_k^i=\Big[\sum_i(\lambda_i-1)^2\Big]a_j a_k.$$

Thus, $\lambda_i=1$, $i=0,1,\ldots,q-1$ or $a_j=0$ or $a_k=0$. But each of these three cases implies $\|a_0\|=0$, a contradiction.

The proof of Theorem 4.6 is completed.

§ 5. Theorem 3.1 provides information about the convergence in L^p-norm for some $p\leqslant 1$. The most interesting case $p=1$ is included. The following, corollary to Theorem 3.1 for $p=1$, is a version of the F. and M. Riesz theorem.

THEOREM 5.1. *Suppose $\mu_0,\mu_1,\ldots,\mu_m$ are bounded Borel measures on K. If $F(x,k)=(\mu_0(x,k),\mu_1(x,k),\ldots,\mu_m(x,k))$ forms a conjugate system (where $\mu_j(x,k)$ is the regularization of μ_j). Then each μ_j, $j=0,1,\ldots,m$, is absolutely continuous.*

Proof. Let $\|\mu_j\|$ be the total variation of μ_j, $j=0,1,\ldots,m$.

$$\int_K |F(x,k)|\,dx=\int_K\Big(\sum_{j=0}^m|\mu_j(x,k)|^2\Big)^{\frac12}dx$$
$$\leqslant\int_K\sum_{j=0}^m|\mu_j(x,k)|\,dx=\sum_{j=0}^m\|\mu_j(\cdot,k)\|_1$$
$$\leqslant\sum_{j=0}^m\|\mu_j\| \quad\text{for all } k\in Z.$$

Hence, by Theorem 3.1 and the definition of conjugate system, there exists $F(x) = (f_0(x), f_1(x), \ldots, f_m(x))$ with each component $f_j \in L^1$ such that $F(x, k) \to F(x)$ as $k \to -\infty$ almost everywhere and in L^1-norm. Therefore $d\mu_j = f_j dx$, that is, μ_j is absolutely continuous, $j = 0, 1, \ldots, m$.

Now let π be a generator of Π, the group of (unitary) multiplicative character which are ramified of degree 1 and homogeneous of degree 0. T_l is the operator such that $(T_l f)^{\hat{}} = \pi^{-l}\hat{f}$ as in § 4. We shall give another version of the F. and M. Riesz theorem in terms of the Fourier transform.

THEOREM 5.2. *Suppose μ is a finite Borel measure on K. If there exist a set $A \subset K$ and an odd l, $1 \leqslant l \leqslant q-1$, such that π^l is constant on A and $\hat{\mu}$ is supported on A, then μ is absolutely continuous.*

Proof. Let $\pi^l(t) = \zeta^j$ on A for some $\zeta^j \in \boldsymbol{U}_{q-1}$. Then $\widehat{T_l\mu}(t) = \zeta^{-j}\hat{\mu}(t)$ for $t \in A$ and $\widehat{T_l\mu}(t) = 0$ for $t \notin A$. Thus $T_l\mu = \zeta^{-j}\mu$ is also a finite Borel measure. By Theorem 5.1, since $F(x, k) = (\mu(x, k), T_l\mu(x, k))$, with l odd, forms a conjugate system, we have that μ is absolutely continuous.

The following two corollaries are immediate consequences of Theorem 5.2:

COROLLARY 5.3. *Suppose μ is a finite Borel measure on K such that $\hat{\mu}$ is supported on A where A is a "cone", i.e., $A = \bigcup_{k=-\infty}^{\infty} \mathfrak{p}^{-k}(\varepsilon^i + \mathscr{P})$ for some $\varepsilon^i + \mathscr{P} \in \mathcal{O}/\mathscr{P}$. Then μ is absolutely continuous.*

In the case when $\frac{q-1}{2} = l$ is odd (thus $\pi^l(t) = \pm 1$), let $W = \{t \in K: \pi^l(t) = 1\}$. Hence $K^* = -W \cup W$. We thus have:

COROLLARY 5.4. *If $\frac{q-1}{2}$ is odd and μ is a finite Borel measure on K whose Fourier transform is supported on W, then μ is absolutely continuous.*

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