and similarly,

(6) \[ |a_0| + 10 |a_1| + 90 \sum |a_k| \leq 1. \]

The required inequality clearly follows from inequalities (4)-(6).

Suppose now that there exist \( B = A \) and \( b_1, b_2 \in B \) such that \( a_1 b_1 + a_2 b_2 = 1 \). Then

\[ \|b_1\| = \|a_1 + a_2 b_2\| \leq |a_1| + \|b_2\| = 1. \]

This contradiction proves that \( \{A, a_1, a_2\} \) has property (ii). Thus the proof of the theorem is complete.

References


INTRODUCTION

Stein and Weiss [10] have developed a theory of \( H^p \)-spaces for Banach spaces \( E \) of functions \( f \) of conjugate harmonic functions on euclidean half-spaces \( R^{n+1}_+ \) satisfying

\[ \int_{R^n} |f(x, y)|^p dx dy \leq A < \infty \quad \text{for all } y > 0. \]

Coffman and Weiss [2] extended the theory to Generalized Cauchy–Riemann systems. The basic result needed, common to all these systems, is the existence of a positive \( p_0 < 1 \) such that \( |f|^{p_0} \) is subharmonic. It is our main objective in this paper to construct conjugate systems on local fields such that the analogue of the above basic result is valid which enable us to develop a theory of \( H^p \)-spaces on local fields.

Let \( K \) be a local field. That is, \( K \) is a locally compact, non-discrete, complete, totally disconnected field. Such a field is a \( p \)-adic field, a finite algebraic extension of a \( p \)-adic field, or a field of formal Laurent series over a finite field. See [8] for details. Various aspects of harmonic analysis on \( K \) and \( K^* \), the \( n \)-dimensional vector spaces over \( K \), have been studied in [4], [8], [22], [13], [14], [6], [7], and [5]. In particular, from [14], [6], and [7] we have the notion of singular integral operators and multipliers; from [13] we have the notion of regular functions on \( K \times K \) which play the role of harmonic functions on \( R^{n+1}_+ \).

In Part I, we study the theory of regular functions, including subregular functions and the Laplace area function. Conjugate systems of
regular functions are introduced in Part B, so that we have the theory of $L^p$-spaces. The F. and M. Riesz theorem is also treated.

Most of the corresponding results in euclidean spaces were being anticipated by Beppo Levi [8] and [9]. The local field variants of the work of Fefferman and Stein [3] will be studied in a sequel to this paper.

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Notation and preliminaries. In general we follow the notation of [12]. The materials in [8]; § 2 and [12]; § 1 serve well as our preliminaries. However, we shall repeat them briefly as follows.

Let $K$ be a fixed local field and let $dx$ be a Haar measure on $K^+$. There is a natural non-archimedian norm on $K$ such that $\|ax\| = |a|d\|x\|$, $|x + y| \leq |x| + |y|$ ($\equiv \max |[x], |y]|$) and $|x^{|y}| = |x| + |y|$ if $x \neq y$. The set $\theta = (\bar{x} \in K^*: |x| \leq 1)$ is the ring of integers in $K$. Haar measure is normalized such that the measure of $\theta$ is 1, i.e., $|\theta| = \int d\theta = 1$. The set $\theta = (\bar{x} \in K^*: |x| < 1)$ is the (unique) maximal ideal in $\theta$. $\theta \forall \theta \theta \theta \theta \theta$ where $q$ is some prime power. Let $\rho_n$ be a generator of $\theta^n$. Then $|\rho_n| = q^{-n}$ and for any $x \in K$, either $|x| = 0$ (when $x = 0$) or $|x| = q^{n}$ for some $k \in \mathbb{Z}$. The set $\theta^n = (\bar{x} \in K^*: |x| \leq q^n)$ has measure $q^{-n}$. The collection $\{\theta^n\}_{n=0}^{\infty}$ is a neighborhood basis for the identity in $K^+$. Cosets of $\theta^n$ are called spheres. $\theta^n = \{x + \theta^n\}$ is the sphere with center $x$ and radius $q^n$. Every point in a sphere is its center. For any two spheres either they are disjoint or one contains the other. We note the existence of a nontrivial additive character $x$ such that $x$ is trivial on $\theta$ but is not trivial on $\theta^n$. The Fourier transform for $f \in L^1(K)$ is defined as $\hat{f}(x) = \int f(y) \chi(x, y) dy$.

Let $K^n$ be the $n$-dimensional vector space over $K$.

$$K^n = \{x = (x_1, x_2, \ldots, x_n): x_i \in K, i = 1, 2, \ldots, n\}.$$ The norm on $K^n$ defined by $|x| = \max |x_i|$, $x \in K^n$, is such that $|x + y| \leq |x| + |y|$ and $|x + y| = |x| + |y|$ when $|x| \neq |y|$ for $x, y \in K^n$. A Haar measure is given by $d\theta^n = d\theta_1 d\theta_2 \ldots d\theta_n$, where $d\theta$ is the (additive) Haar measure on $K$ as the $i$th coordinate space of $K^n$, $d(\theta) = |\|\theta^n\| = 1$ and $|\|\theta^n\| = q^{-n}$. For $a, y \in K^n$, let $a \cdot y = a_1 y_1 + a_2 y_2 + \ldots + a_n y_n$. The Fourier transform for $f \in L^1(K^n)$ is defined by $\hat{f}(a) = \int f(y) \chi(a, y) dy$.

Let $\mathcal{S}$ be the space of test functions on $K^n$, i.e., those are constant on the cosets of some $\theta^n$ and supported on some $\theta^n \mathcal{S}$; the topological dual of $\mathcal{S}$, called the space of distributions. For every $f \in \mathcal{S}$, the Fourier transform of $f$ is in $\mathcal{S}'$ and is defined by $\hat{f}(y) = \int f(y) \chi(a, y) dy$.

Let $\mathcal{S} = \langle \phi \rangle$ be the characteristic function of $\theta^n$. Then for $x = (x_1, x_2, \ldots, x_n) \in K^n$, $\Phi(x) \equiv \Phi_1(x_1) \Phi_2(x_2) \ldots \Phi_n(x_n)$ is the characteristic function of $\theta^n$. We also denote it by $\phi_n$.

The notation is used for $\mathcal{A}$, $\mathcal{A}'$ denotes the complement of $\mathcal{A}$, $\mathcal{N} \mathcal{A} \mathcal{B} = \mathcal{A} \mathcal{A} \mathcal{B}$, $\mathcal{B}$ and $\mathcal{B} = \mathcal{A} \mathcal{B} \mathcal{B}$ for $\mathcal{A} \mathcal{B} \mathcal{B}$. In the latter case we say that the two sets $\mathcal{A}$ and $\mathcal{B}$ are equivalent. We denote $\mathcal{Z}$ for the non-negative integers and $\mathcal{Z}_-$, the non-positive ones. For a sequence of real numbers $\{a_i\}_{i=1}^{\infty}$, we write $a_{n}= a_{i}$ as $n \to \infty$ if $a_n \leq a_i$ for $n \leq i$ and $a_{n}= a_{i}$ as $n \to -\infty$.

A. THE THEOREM OF REGULAR FUNCTIONS

In § 1, we define regular functions and subregular functions on a domain in $K^n \times \mathbb{Z}$ and show that they behave very much like harmonic and subharmonic functions on euclidean spaces. We also prove the theorem on regular remainders of subregular functions. In § 2, we show that, for a regular function, the non-tangential convergence is equivalent to the radial convergence and also, locally, equivalent to the radial limit existence and to the existence of the Linus area function.

§ 1. We write $(\mathcal{S}^n, \mathcal{L}) = \langle (y, l) \in K^n \times \mathbb{Z}: y \in \theta^n \mathcal{S} \rangle$ where $\theta^n \mathcal{S} = \theta + \theta^n \mathcal{S}$. A set $\mathcal{S} \subset K^n \times \mathbb{Z}$ is called a domain in $K^n \times \mathbb{Z}$ if

(i) $(s, k) \in \mathcal{S}$ implies $(s, k-1) \in \mathcal{S}$;

(ii) $(s, k) \in \mathcal{S}$ and $(s, k-1) \in \mathcal{S}$ imply $(s, k-2) \in \mathcal{S}$.

A domain $\mathcal{S}$ in $K^n \times \mathbb{Z}$ is bounded if there exists a $k \in \mathbb{Z}$ such that $k > k_0$ for all $(s, k) \in \mathcal{S}$. For a domain $\mathcal{S}$ in $K^n \times \mathbb{Z}$, let $\mathcal{D} = \{s \in \mathcal{S}: (s, k-1) \mathcal{S} \}$ and let $\mu(\mathcal{D}) = \mu(\mathcal{D}) \subset \mathcal{S} = \mu(\mathcal{D}) \subset \text{fin sup}(\mathcal{S})$. A domain in $K^n \times \mathbb{Z}$ is said to be simple provided that $(s, k) \in \mathcal{S}$ implies $(s, k) \notin \mathcal{S}$ for all $l < k$ and that $\mu(\mathcal{S}) > -\infty$.

DEFINITION. A function $f(s, k)$ defined on a domain $\mathcal{S} = K^n \times \mathbb{Z}$ is said to be regular on $\mathcal{S}$, for all $(s, k) \in \mathcal{S} \mathcal{S} \mathcal{S} \mathcal{S}$, $f(s, k)$ is constant on $(\mathcal{S}^{n-1})$ and

$$f(s, k) = \frac{1}{\mu(\mathcal{S})} \int f(y, k-1) dy.$$
A function \( f(x, k) \) is subregular (superregular, respectively) on \( \mathcal{B} \) if it is real-valued and "\( \leq \)" ("\( \geq \)", respectively).

Note that the defining property (1.1) of a regular function is the analogue of the mean value property of a harmonic function. For \( \mathcal{B} = K^n \times Z \), this is the same as the definition of regularity in [13].

**Proposition 1.3 (Maximum and Minimum Principles).** (a) If \( f(x, k) \) is subregular on a bounded domain \( \mathcal{B} \subseteq K^n \times Z \), then

\[
\sup_{(x, k) \in \mathcal{B}} f(x, k) = \sup_{(x, k) \in \partial \mathcal{B}} f(x, k).
\]

(b) If \( f(x, k) \) is superregular on a bounded domain \( \mathcal{B} \subseteq K^n \times Z \), then

\[
\inf_{(x, k) \in \mathcal{B}} f(x, k) = \inf_{(x, k) \in \partial \mathcal{B}} f(x, k).
\]

Proof. Since domain \( \mathcal{B} \) is a union of the sets \( \{ p, k \} \) and \( \partial \mathcal{B} \) is bounded, it suffices to consider the special case where \( \mathcal{B} = \{ p, k \} \) and \( \partial \mathcal{B} = \{ p, k \} \). But in this case, the conclusions follow immediately from the subregularity or the superregularity of \( f \).

**Corollary 1.3 (Uniqueness of Boundary Values of Regular Functions).** If \( f \) and \( g \) are regular functions on a bounded domain \( \mathcal{B} \subseteq K^n \times Z \) and agree on \( \partial \mathcal{B} \), then \( f(x, k) = g(x, k) \) for all \( (x, k) \in \mathcal{B} \).

Proof. Apply Proposition 1.2 to the real and the imaginary parts of the regular function \( f - g \).

**Proposition 1.4.** (a) The linear combination of regular functions on \( \mathcal{B} \) is regular on \( \mathcal{B} \). If \( f \) and \( g \) are regular on \( \mathcal{B} \), then \( af + bg \) is regular on \( \mathcal{B} \).

(b) If \( f \) is subregular on \( \mathcal{B} \) and \( \varphi \) is a non-decreasing convex function defined on an interval containing the range of \( f \). Then the composition \( \varphi \circ f \) is subregular on \( \mathcal{B} \).

Proof. (a) Immediate.

(b) For \( (x, k) \in \mathcal{B} \setminus \partial \mathcal{B} \),

\[
\varphi \circ f(x, k) \leq \varphi \left( \frac{1}{|\varphi|_{x_k}} \int_{x_k} f(y, k-1) dy \right)
\]

\[
\leq \frac{1}{|\varphi|_{x_k}} \int_{x_k} \varphi \circ f(y, k-1) dy
\]

as follows from Jensen’s inequality. Therefore \( \varphi \circ f \) is subregular on \( \mathcal{B} \).

Remark. A useful consequence of Proposition 1.4 (b) is that if \( f(x, k) \) is regular on \( \mathcal{B} \) and \( p \geq 1 \), then \( |f(x, k)|^p \) is subregular on \( \mathcal{B} \). This is no longer true if \( p < 1 \). It should be noted that if \( f(x, k) \geq 0 \) is regular on \( \mathcal{B} \) and \( p \leq 1 \), then \( |f(x, k)|^p \) is superregular on \( \mathcal{B} \) as follows from the inequality

\[
\frac{1}{m} \sum_{i=1}^{m} a_i^p \geq \frac{1}{m} \sum_{i=1}^{m} a_i \quad \text{for} \quad 0 < p \leq 1, \quad a_i \geq 0, \quad i = 1, 2, \ldots, m.
\]

The following being defined in [13] generalizes the notion of Poisson kernel and Poisson integral:

**Definition.** \( R(x, k) = R_k(x) = g_{-k}(x) \) is called the regularization kernel. For \( f \in \mathcal{D}' \), the space of distributions, let \( f(x, k) = R_k * f(x) \) and is said to be the regularization (integral) of \( f \).

Note that \( R(x, k) \in \mathcal{D}' \) for all \( k \) and \( R_k * f(x) \) is well-defined for \( f \in \mathcal{D}' \). Also, \( R(x, k) \) and \( R_k * f(x) \), with \( f \in \mathcal{D}' \), are regular on \( K^n \times Z \). Moreover, regular functions on \( K^n \times Z \) stand in one-to-one correspondence with distributions on \( K^n \) (13, Lemma 1).

**Proposition 1.5.** (a) If \( f(x, k) \) is regular on \( K^n \times Z \), then

\[
R_k * f(x, 1)(x) = f(x, k + 1).
\]

(b) If \( f(x, k) \) is subregular on \( K^n \times Z \), then

\[
R_k * f(x, 1)(x) \geq f(x, k + 1).
\]

Proof.

\[
R_k * f(x, 1)(x) = \frac{1}{|x_k|} \int f(y, 1) dy.
\]

If \( f(x, k) \) is regular, this is just \( f(x, k + 1) \). If \( f(x, k) \) is subregular, then

\[
\frac{1}{|x_k|} \int f(y, 1) dy \geq f(x, k + 1)
\]

Thus, \( R_k * f(x, 1)(x) \geq f(x, k + 1) \).

The following two results can be found in [13]:

**Proposition 1.6.** Let \( f \in L^p(K^n) \) with \( 1 < p < \infty \) and \( f(x, k) = R_k * f(x) \).

(a) \( f(x, k) \to f(x) \) a.e. as \( k \to -\infty \).

(b) \( \| f(x, k) \|_{L^p} \to \| f(x) \|_{L^p} \) as \( k \to -\infty \), \( 1 < p < \infty \).

(c) \( f(x, k) \to f(x) \) in the \( \mathcal{M}' \)-topology as \( k \to -\infty \) if \( p = \infty \).

II. If \( \mu \) is a finite Borel measure with total variation \( |\mu| \), then \( |\mu(x, k)|\sim |\mu| \) as \( k \to -\infty \), and \( \mu(x, k) \to \mu(x) \) in the \( \mathcal{M}' \)-topology as \( k \to -\infty \) where \( \mu(x, k) = R_k(x-y) \delta(y) \).

**Proposition 1.7.** Suppose \( f(x, k) \) is regular on \( K^n \times Z \) and \( \sup_{x \in K^n} |f(x, k)|^p \leq A < \infty \) where \( 1 < p < \infty \).
(a) If \(1 < p < \infty\), \(f(x, k)\) is the regularization of a function in \(L^p(K^n)\).
(b) If \(p = 1\), \(f(x, k)\) is the regularization of a finite Borel measure on \(K^n\).

The following is an immediate consequence (compare with Lemma 11 in [13]).

**Corollary 1.8.** If \(f(x, k)\) is regular on \(K^n \times Z\), \(\lim_{k \to \infty} f(x, k) = 0\) a.e., and \(f(x, k)\) is bounded, then \(f\) must be identically zero.

Note that Corollary 1.8 can be regarded as a result on the uniqueness of regular functions on the unbounded domain \(K^n \times Z\). The result is not true unless a restriction, such as boundedness, is imposed on \(f\). \(R(x, k)\) is such an example.

If a regular function \(m(x, k)\) majorizes the function \(f(x, k)\) on a domain in \(K^n \times Z\), we say that \(m\) is a regular majorant of \(f\). If \(m \leq h\) whenever \(h\) is another regular majorant of \(f\), \(m\) is called the least regular majorant of \(f\).

**Theorem 1.9.** If \(f(x, k)\) is a non-negative subregular function on \(K^n \times Z\) and \(\sup_{x \in Z} \|f(x, k)\|_p \leq A < \infty\) where \(1 \leq p < \infty\). Then \(f(x, k)\) has the least regular majorant \(m(x, k)\). Moreover,

(a) \(1 \leq p \leq \infty\), \(m(x, k)\) is the regularization of a function in \(L^p(K^n)\); (b) \(p = 1\), \(m(x, k)\) is the regularization of a finite Borel measure on \(K^n\).

**Proof.** For fixed \(k \in Z\), let \(m_0(x, k) = R_0 f:\{1, 1\}(x)\). Since \(f(x, k)\) is subregular, \(m_0(x, k) \leq R_0 f:\{1, 1\}(k - 1) = m_0(x, k)\) for \(k \leq k_0\). Thus \(m_0(x, k) = m(x, k)\), say (as \(k \to \infty\)). By Proposition 1.5, for \(k \leq k_0\), we have \(f(x, k) \leq m(x, k) \leq m(x, k)\). Moreover, applying the monotone convergence theorem, we have

\[
m(x, k) = \lim_{k \to \infty} m(x, k) = \lim_{k \to \infty} \frac{1}{|\sigma^n_k|} \int_{\sigma^n_k} m(x, k - 1) dx = \frac{1}{|\sigma^n_k|} \int_{\sigma^n_k} m(x, k - 1) dx.
\]

That is, \(m(x, k)\) is regular on \(K^n \times Z\).

Now, from Proposition 1.6, we know that \(\|m(\cdot, k)\|_p \leq \|f(\cdot, 1)\|_p \leq A\). If \(1 \leq p < \infty\), by the monotone convergence theorem,

\[
\int_{K^n} m^p(x, k) dx = \lim_{k \to \infty} \int_{K^n} m^p(x, k) dx \leq A^p;
\]

that is, \(\sup_{k \in Z} \|m(\cdot, k)\|_p \leq A\). For the case \(p = \infty\), \(\|m(\cdot, k)\|_\infty \leq A\) for all \(k \in Z\), is obvious. Therefore, by Proposition 1.7,

(a) \(1 < p < \infty\), \(m(x, k)\) is the regularization of a function in \(L^p(K^n)\); (b) \(p = 1\), \(m(x, k)\) is the regularization of a finite Borel measure on \(K^n\).

It remains to show that the majorant \(m\) is in fact the least one. Suppose \(h(x, k)\) is a regular majorant of \(f(x, k)\), then for \(k \leq k_0\),

\[
m_k(x, k) = R_0 f:\{1, 1\}(x) \leq R_0 h:\{1, 1\}(x) = h(x, k + 1) = h(x, k).
\]

Letting \(k \to \infty\), we thus have \(m(x, k) \leq h(x, k)\). This completes the proof.

**§ 2.** We identify \(K^n\) with \(K^n \times (-\infty)\). For \(\ell \in Z^+\) and \(x \in K^n\), let \(\ell = (\ell, k) \subseteq K^n \times Z\). If \(f(x, k)\) is defined on \(K^n \times Z\), we say that it has a nonregular limit \(L\) at \(x \in K^n\) if, for each \(\ell \in Z^+\), \(\lim_{\ell \to \infty} f(x, \ell) = L\) as \((x, k)\) tends to \((x, \ell)\). We write, simply, n.t., \(\lim_{\ell \to \infty} f(x, \ell) = L\).

The nonregular convergence is obviously stronger than the "radial" convergence, i.e., \(\lim_{\ell \to \infty} f(x, \ell) = L\). We shall show that for regular functions they are equivalent. Let us first consider the case for regularizations:

**Proposition 2.1.** (a) If \(f\) is locally integrable on \(K^n\), then \(f(x, k) \to f(x)\) as \(k \to \infty\) for a.e. \(x \in K^n\).

(b) If \(f\) is locally integrable on \(K^n\), then \(f(x, k) \to f(x)\) as \((x, k)\) tends nonregularly for a.e. \(x \in K^n\).

**Proof.** (a) See [13] for a proof.

(b) It follows from (a) that

\[
\frac{1}{|\sigma^n_k|} \int_{\sigma^n_k} [f(t) - f(x)] dt \to 0 \quad \text{as} \quad k \to \infty \quad \text{for a.e.} \quad x \in K^n.
\]

We claim, moreover, that

\[
\frac{1}{|\sigma^n_k|} \int_{\sigma^n_k} |f(t) - f(x)| dt \to 0 \quad \text{as} \quad k \to \infty \quad \text{for a.e.} \quad x \in K^n.
\]

The desired result follows from the above claim. In fact, for \((x, k) \in I(\ell)\), we have \(|\sigma^n_k| = \sigma^n_{k-1} + |\sigma^n_{k-1}|\). Let \(x \in K^n\) be a point such that (2.3) is valid. Then, for \((x, k) \in I(\ell)\),
\[
|f(x, k) - f(x)| = \frac{1}{|x|^s} \int_{|x|^s} |\int (f(t) - f(x))\,dt| \leq \frac{1}{|x|^s} \int_{|x|^s} \int |f(t) - f(x)|\,dt \leq \frac{q^d}{|x|^s} \int_{|x|^s} \int |f(t) - f(x)|\,dt \to 0 \quad \text{as} \quad k \to -\infty.
\]

Thus \(f(x, k) - f(x)\) as \((x, k)\to\) non-tangentially.

It remains to show (2.3). Note that \(|f(t) - q|\) is locally integrable for any rational (complex) number \(q\). Hence, by (2.2),

\[
(2.4) \quad \frac{1}{|x|^s} \int_{|x|^s} [\int (|f(t)| - |q| - |f(x)| - |q|)\,dt \to 0 \quad \text{a.e. as} \quad k \to -\infty.
\]

Let \(E\) be the exceptional set for (2.4), \(E = \bigcup E_\varepsilon\) has measure zero. If \(x \not\in E\), for any \(\varepsilon > 0\), let \(q\) be such that \(|f(x) - q| < \varepsilon/2\). Then

\[
\frac{1}{|x|^s} \int_{|x|^s} [\int (|f(t) - q| - |f(x)| - |q|)\,dt \to 0 < \varepsilon
\]

for \(-k\) large enough as follows from (2.3). This completes the proof of (2.3) and (b).

Before proceeding the theorem on equivalence, we introduce the following:

**Definition.** Let \(\mathcal{G}\) be a simple domain in \(K^s \times Z\) and \(m = m(\mathcal{G}) = \inf \sup \{k: (x, k) \in \mathcal{G}\} < -\infty\). See §1. For \(f(x, k)\) regular on \(\mathcal{G}\), we define \(f(x, k)\) on \(K^s \times Z\) as follows:

\[
(2.5) \quad \tilde{f}(x, k) = \begin{cases} f(x, k) & \text{if} \ (x, k) \in \mathcal{G} \\ f(x, l) & \text{if} \ (x, l) \in \partial \mathcal{G} \text{ and } k \leq l \end{cases}
\]

for other values of \((x, k)\), \(\tilde{f}(x, k) = 0\) if \(k \leq m\) and, finally, \(\tilde{f}(x, k) = \tilde{f}_k f(x, m)\) if \(k > m\). \(f(x, k)\) which is obviously well-defined and regular on \(K^s \times Z\) is called the extension of \(f(x, k)\) on \(K^s \times Z\).

For \(l \in Z^+\) and \(l \in Z\), let \(\Gamma_l(z)\) denote the truncated cone \(\Gamma_l(z) = \{(x, k) \in K^s \times Z: |x-z| < 2^{l+1}, k < l\} = \bigcup_{k=2^{l+1}}^{2^l} \Gamma_{k}\). The height \(k\) of the truncation is not essential. For a regular function \(f(x, k)\) on \(K^s \times Z\), let \(\delta_2 f(x, k) = f(x, k) - f(x, k+1)\). The *Lusin area function* of \(f\) with respect to the fixed cone \(\Gamma^{l}_l(z)\) is given by

\[
S^0 (f(z)) = \left(\sum |\delta_2 f(z)|^2 \right)^{1/2}
\]

where the summation runs over distinct \((\Phi^2_{k, l}, k) \in \Gamma^{l}_l(z)\). We single out the cone \(\Gamma^{l}_l(z)\) and \(S^0 (f(z)) = S^0 (f(z)) = \left(\sum |\delta_2 f(z)|^2 \right)^{1/2}\) is just a truncated Littlewood–Paley function.

We are now going to prove the following version of the Fatou–Calderón–Stein theorem:

**Theorem 2.6.** If \(f(x, k)\) is regular on \(K^s \times Z\), then the following sets are equivalent:

\[
\begin{align*}
A & = \{ x \in K^s: \lim_{k \to -\infty} f(x, k) \text{ exists} \}; \\
B & = \{ x \in K^s: \lim_{k \to -\infty} f(x, k) \text{ exists} \}; \\
C & = \{ x \in K^s: \sup_{k \to -\infty} |f(x, k)| < \infty \}; \\
D & = \{ x \in K^s: S^0 (f(x)) < \infty \}; \\
L & = \{ x \in K^s: S^0 (f(x)) < \infty \}.
\end{align*}
\]

We need the following lemmas:

**Lemma 2.7.** Let \(\{(x_j, k_j)\}_{j=1}^{\infty} \subset \Gamma_l(z)\), \(l \in Z^+,\) be such that \((x_j, k_j) \to (z, -\infty)\) as \(j \to \infty\). If \(x\) is a point of density of \(F\), then \((x_j, k_j) \to (z, -\infty)\) as \(j \to \infty\) for all \(z\) large enough.

**Proof.** We first note that if \(F\) is the characteristic function of a (measurable) set \(E \subset K^s\), then \(x\) is a point of density of \(E\) if

\[
\frac{1}{|x|^s} \int \xi_F(x)\,dx \to 1 \quad \text{as} \quad m \to \infty.
\]

Observe that Proposition 2.1 (a) implies that almost every point in \(E\) is a point of density of \(F\).

Now, for \((x, k) \in \bigcup \Gamma_l(y)\), we have \(x \in \Gamma^{l+1}(y)\) and \(x \in \Gamma^{l}(y)\). Let \(E = \Gamma^{l+1}(y) \cap \Gamma^{l}(y)\). Then

\[
\frac{1}{|x|^s} \int \xi_E(x)\,dx = \frac{|\Gamma^{l+1}(y) \cap E|}{|\Gamma^{l}(y)\cap E|} \leq \frac{|E|}{q^{|l-\alpha||x|^s|}} = 1 - q^{-2^l}.
\]

Suppose the conclusion is not true, then there exists a subsequence \((x_j, k_j) \in \{(x_j, k_j)\}\) such that \((x_j, k_j) \to (z, -\infty)\) as \(j \to \infty\) and
\( (a_i, b_i) \cup \bigcup \Gamma_k(y) \). But then
\[
\lim_{t \to 0^+} \frac{1}{|P_{a_i-b_i}|} \int_{P_{a_i-b_i}} |f_t(y)|dy \leq 1 - q^{-m} < 1.
\]
This contradicts the fact that \( z \) is a point of density of \( F \). The proof of Lemma 2.7 is therefore completed.

\textbf{Lemma 2.8.} (a) Suppose \( f(x, k) \) is regular on \( K^x \times Z \) such that \( f(x, k) \to 0 \)
\( k \to \infty \) for each \( x \) and \( \sum_{b=m}^{\infty} |d_k f(a_i)|^p \leq L^p(K^x) \) for some \( 1 < p < \infty \). Then \( f(x, k) \) is the regularization of a function \( F \in L^p(K^x) \).

(b) If \( f \in L^p(K^x) \) with \( 1 < p < \infty \), then \( \sum_{b=m}^{\infty} |d_k f(a_i)|^p \) exists for a.e. \( x \in K^x \) and is in \( L^p(K^x) \).

(c) If \( f \in L^p(K^x) \) and \( S_0(f)(x) \) is the Luzin area function with respect to \( T_n^x \), then \( S_0(f)(x) \) exists for a.e. \( x \in K^x \) and \( \|S_0(f)(x)\|^p = q^p \sum_{b=m}^{\infty} |d_k f(a_i)|^p dx < \infty \).

Proof. (a) and (b) are known. See [13] for a proof.

(c) For fixed \( k \in Z \),
\[
\sum_{b=m}^{\infty} |d_k f(a_i)|^p dx = q^p \sum_{b=m}^{\infty} |d_k f(a_i)|^p dx < \infty,
\]
where \( z \) are such that \( \mathscr{S}_{u-z} \) being distinct costs in \( \mathscr{S}_{u-z} \).

\textbf{Proof of Theorem 2.6.} The fact that \( B \subset A \subset C \subset \bar{D} \subset D \) is trivial. We start by showing that \( C \subset B \subset C \subset L \).

For \( M \in Z^* \), let \( E_M = \{ x \in K^x : \text{sup} |f(x, k)| \leq M \} \cap \mathcal{E} \). We observe \( x \)
that it suffices to consider \( E_M \) instead of \( C \). For \( m = 0 \) \( \in \mathbb{Z} \), let \( E_0 = \{ x \in E_M : \text{sup} |f(x, k)| \leq M \} \cap \mathcal{E} \). We observe \( z \)
that it suffices to consider the set \( E_0 \).

\textbf{Remark.} The “extension” argument in the proof above, namely \( (2.5), \) served as the role of the conformal mapping used to prove the corresponding result on the (complex) unit disc, for instance, in [15], Chapter XIV, (and the upper half-plane \( \mathbb{R}_+^2 \)) that is not available in the study of \( \mathbb{R}_+^2 \).}

\section{B. Conjugate Systems of Regular Functions}

In § 3 we apply the results in Part A to show the main theorem of \( HP \)-spaces. In § 4, we define the conjugate systems which generalize the notion of Hilbert transform in order to have the basic subregularity we need for the study of \( HP \)-spaces. The F. and M. Riesz theorem is treated in § 5.

\textbf{Theorem 3.1.} \( LF(x, k) = (f_1(x, k), f_2(x, k), \ldots, f_m(x, k)) \) be a vector valued function with each component \( f_j, j = 1, \ldots, m \), being regular on \( K^x \times Z \).

Suppose there exists a \( p_0, 0 < p_0 < 1 \), such that \( |LF(x, k)|^p \) is subregular \( \text{for some} \ p > p_0 \). \( (3.2) \)
\[
\int_{K^x \times Z} |LF(x, k)|^p dx < A < \infty \text{ for all } k \in Z.
\]

Then the limits
\[
f_j(x) = \lim_{b \to \infty} f_j(x, k)
\]
exist for a.e. \(x \in K^a\) and
\[
\lim_{b \to 0} \int_{K^a} |F(x, y) - F(x, y) f| \, dy = 0
\]
where \(F(x) = (f_1(x), f_2(x), \ldots, f_n(x))\). Moreover, \(f_j^*(x) = \sup_{k \in K^a} |f_j(x, k)|\in L^{p}(K^a), j = 0, 1, 2, \ldots, m\).

Proof. Let \(p_1 = \frac{p}{p_0} > 1\). We have, for all \(k \in Z\),
\[
\int_{K^a} |F(x, k)|^p \, dy = \int_{K^a} |F(x, k)|^p \, dy < A < \infty.
\]
By Theorem 1.6, since \(|F(x, k)|^p_0\) is subregular, \(|F(x, k)|^p_0\) has the least regular majorant \(m(x, k)\) which is the regularization of a function \(m \in L^p(K^a)\).
This implies \(m^*(x) = \sup_{k \in Z} |m(x, k)| < \infty\) almost everywhere and \(m^* \in L^p(K^a)\) since the operator \(m \mapsto m^*\) is of type \(p_1, p_2\) with \(p_1 > 1\). Now, for a.e. \(x \in K^a\),
\[
f_j^*(x) = \sup_{k \in Z} |f_j(x, k)| \leq \sup_{k \in Z} |F(x, k)|^p_0 = \left[ m^*(x) \right]^p \leq \frac{1}{p_1} p_1.
\]
Hence, \(f_j^*(x) < \infty\) almost everywhere and \(f_j \in L^{p}(K^a), j = 0, 1, 2, \ldots, m\).
By Theorem 3.5, the limits \(f_j = \lim_{b \to 0} f_j(x, k)\) exist almost everywhere.
That is, \(\lim_{b \to 0} F(x, k) = F(x)\) for a.e. \(x \in K^a\).
Note that
\[
|F(x, k) - F(x, k)|^p \leq 2^p |f_j(x, k)|^p + |F(x, k)|^p \leq 2^p \left[ m^*(x) \right]^p \in L^p.
\]
Therefore, by the Lebesgue dominated convergence theorem,
\[
\lim_{b \to 0} \int_{K^a} |F(x, k) - F(x, k)|^p \, dy = 0.
\]
This completes the proof.

From Theorem 3.1, we know that it is desirable to define "conjugate systems" of regular functions, \(F(x, k) = (f_1(x, k), f_2(x, k), \ldots, f_n(x, k))\), such that \(|F(x, k)|^p_0\) is subregular for some positive \(p_0 < 1\).
This will be our main task in § 4.

§ 4. We restrict our attention to the one-dimensional case \(K\) and exclude the case with \(q\).

Let \(m^* = \mathcal{O} \cdot \mathcal{D}\) be the group of units in \(K\) and \(\mathcal{O} = \{0, 1, \omega, \ldots, \omega^{q-1}\} = GF(q)\). Let \(\pi \in \hat{K}^\times\) be a (multiplicative) unitary character in \(K^a\).
Denote \(A_0 = \mathcal{O}, A_1 = 1 + \mathcal{D}, A_h = 1 + \mathcal{D}, h \gg 1\).
If \(\pi\) is trivial on \(A_h\),
we say that \(\pi\) is unramified. If \(\pi\) is trivial on \(A_h\), but not on \(A_{h-1}(h \gg 1)\),
we say that \(\pi\) is ramified of degree \(h\). See [8] for details.

We consider those \(\pi \in \hat{K}^\times\) such that \(\pi\) is ramified of degree 1 and is homogeneous of degree 0 (i.e., \(\pi(\omega^m) = \pi(\omega)\) for all \(m \in Z\)). It follows that \(\pi\) is constant on the cosets of \(\mathcal{D}\) in \(\mathcal{O}\) and, since \(\pi\) is a nontrivial character on the compact group \(\mathcal{O}^\times\), \(\hat{\pi}(\mathcal{O}^\times) = 0\). Thus \(\pi\) takes values on \(U_{q-1}\), the cyclic subgroup of order \(q - 1\) (the group of all \((q - 1)\)th roots of unity). The set \(H\) of all such \(\pi\)'s forms a cyclic group of order \(q - 1\). Any \(\pi\), such that \(\pi(\omega) = \xi \in \mathcal{I}\) is a primitive root, is a generator. \(\pi(x) = \pi(\omega) = \omega^0 = 1\). Note that \(\pi(x) = \omega^0\) and \(\pi(x) = \omega^0\) for all \(\pi \in \mathcal{O}^\times\), i.e., \(\pi\) is odd.

Let \(Q(p) = \frac{1}{p_1} \pi(x)\); \(1, 1, 2, \ldots, q - 2\) where \(\frac{1}{p_1} = \Gamma(p)\) is p.v.
\(\frac{1}{p_1} \pi(x) \pi(x)\left|d\pi(x)\right| = 0\).

Let \(Q(p) = \pi(x)\left|d\pi(x)\right| = 0\).

Define, for a "nice" function \(f\) on \(K\), \(T_f \mapsto T_{\pi} f\) by
\[
T_f(x) = \lim_{b \to 0} Q_{\pi} f(x);
\]
\(T_{\pi} f(x) = \pi^{-1}(u) f(x)\).

Theorem 4.1. \(T_{\pi} f(x) = \pi^{-1}(u) f(x)\).

Moreover, they are actually the same operator as follows from the following lemma of Sully–Tableman [8]:

**Lemma.** \(L^p \bigg\{ \frac{\pi(x)}{|x|} \bigg\} (u) = \int_{K^a} \pi(x) \pi^{-1}(u) f(x) d\mu(x)
\]

Indeed, \(L^p \bigg\{ \pi(x) \bigg\} (u) = \frac{1}{p_1} \int_{K^a} \pi(x) \pi^{-1}(u) f(x) d\mu(x)
\]
and for \(f \in L^p\), by Plancherel's theorem, \(T_{\pi} f = \lim_{b \to 0} Q_{\pi} f(x)\) has Fourier transform \(\pi^{-1}(u) f(x)\). Thus, \(T_{\pi} f = \pi^{-1}(u) f(x)\).

Note that \(T_f \mapsto T_{\pi} f = \pi^{-1}(u) f(x)\).

Setting \(v_{\pi}(x, k) = \pi^{-1}(u) f(x) = \pi^{-1}(u) f(x, k)\) for \(u \in Z\), we have \(\mathcal{O} = \mathcal{O}^\times = \{0, \pi(u), \pi(u)\omega, \ldots, \pi(u)\omega^{q-2}\}\).
Let us compute \(T_{\pi} f(x, k)\):

\[
T_{\pi} f(x, k) = \int_{[\mathcal{O} \cdot \mathcal{D}]} \int_{\mathcal{O} \cdot \mathcal{D}} f(t, t, t) \pi^{-1}(u) \delta(t) \delta(t) dt
\]

where \(e_{m, 0}, e_{m, 1}, \ldots, e_{m, q-1}\).

Compute \(T_{\pi} f(x, k)\):

\[
T_{\pi} f(x, k) = \int_{[\mathcal{O} \cdot \mathcal{D}]} \int_{\mathcal{O} \cdot \mathcal{D}} f(t, t, t) \pi^{-1}(u) \delta(t) \delta(t) dt
\]

where \(e_{m, 0}, e_{m, 1}, \ldots, e_{m, q-1}\).
\[ a = \sum_{l=0}^{\infty} a_l e^{i\theta_l} \]

Proposition 4.5. (a) \( \sum_{l=-\infty}^{\infty} e^{i\phi_l} = 0 \), \( l = 0, 1, \ldots, q-2 \);
(b) \( \|a\| = |a_0|, \quad l = 1, 2, \ldots, q-2 \);
(a) \( \sum_{l=0}^{\infty} q_l a_l = 0 \) whenever \( j + l \) is odd, \( j, l = 0, 1, \ldots, q-2 \).

Proof. (a) Follows immediately from the regularity.
(b) Let \( g(x) \) be the restriction of \( d_0(x) \) on \( y + \mathbb{R}^{(0,0)} \). We see from (4.2) that \( T_q g(x) \) is also supported on \( y + \mathbb{R}^{(0,0)} \). By the Plancherel's theorem, since \( \|e^{i\theta_l}\| = 1 \) for all \( l \), we have

\[ \|T_q g\|_2 = \|T_q g\|_2 = \|\mathbb{R}^{(0,0)}\| = \|g\|_2. \]

(Note that \( \tilde{g}(0) = 0 \).) That is, \( |a_0| = \|a_0\| \) for all \( l \).

(c) Since \( T_j T_k = T_{j+k} \), we may assume that \( j = 0 \) and \( l \) is odd. Thus \( \pi' \) is odd. From (4.6) we then have

\[ \begin{align*}
\sum_{l=0}^{\infty} a_l q^l a_l & = q^{-1} \sum_{l=0}^{\infty} \pi'(q^{-1} a_l) a_l \\
& = q^{-1} \sum_{l=0}^{\infty} \pi'(q^{-1} a_l) a_l \\
& = q^{-1} \sum_{l=0}^{\infty} \pi'(a_l) a_l \\
& = q^{-1} \sum_{l=0}^{\infty} \pi'(a_l) a_l \\
& = 0.
\end{align*} \]

Remark. The introduction of the function \( g(x) \) in the above proof, in particular \( g(x) = \tilde{d}_0 f(x) \), can be used to study "Fourier analysis" on \( GF(q) \), \( GF(q) \). (b) is, as we have seen, just Plancherel's theorem and (c) follows from the Parseval formula.

Consider now a \( q \times (m+1) \) matrix \( (a') \) with complex entries. Let \( \alpha_j = \langle a'_j, a'_{j-1} \rangle \in \mathbb{C}^q \) with \( \|a_j\| = \left( \sum_{l=k}^{\infty} |a_l|^2 \right)^{1/2} \), \( j = 0, 1, \ldots, m \);
\[ \alpha' = \langle a'_j, a'_{j-1} \rangle \in \mathbb{C}^{m+1} \] with \( \|a'_j\| = \left( \sum_{l=k}^{\infty} |a'_l|^2 \right)^{1/2} \), \( i = 0, 1, \ldots, q-1 \). Also \( a = (a_0, a_1, \ldots, a_q) \in \mathbb{C}^{m+1} \). Suppose \( 0, 1, \ldots, m \) is \( D \cup E \), where \( D \) and \( E \) are non-empty, disjoint.

Theorem 4.6. If

\[ \sum_{l=0}^{m} a_l q^l = 0, \quad j = 0, 1, \ldots, m; \]
\[ \sum_{l=0}^{m} |a_l| = |a_0|, \quad j = 1, 2, \ldots, m; \]
\[ \sum_{l=0}^{m} a_l q^l = 0, \quad \text{whenever } j \notin D \text{ and } k \in E, \]
then there exists a \( p_n \), \( 0 < p_n < 1 \), such that

\[ \|a|_p^p = \frac{1}{q} \sum_{l=0}^{m} \|a_l| \|a'_l\|^p \]

for all \( p \geq p_n \), where \( p_n \) is independent of \( a \) and \( (a') \).

This theorem generalizes Theorem 2 of [1] where the case \( m = 1 \) was treated. Before proceeding to the proof, we shall study the statement further. Notice that (4.10) is the local subregularity we need. Thus it suffices to define a "conjugate system" satisfying (4.7), (4.8) and (4.9). Namely:
Definition. Suppose \( f_1, f_2, \ldots, f_m \) are regular functions defined on a domain \( \mathcal{D} \subset \mathbb{C} \times \mathbb{C} \). For any fixed \( (y, k+1) \in \mathcal{D} \), denote
g_j = f_j(y, k+1); f_j(y, k+1) \in \mathcal{D}, j = 0, 1, \ldots, m; j = 0, 1, \ldots, m, \) where \( \phi = e^{-i(\frac{\pi}{2})} \) and \( \phi = e^{-i\frac{\pi}{2}} \). If (4.7), (4.8) and (4.9) are satisfied, then \( F(z, k) = (f_1(z, k), f_2(z, k), \ldots, f_m(z, k)) \) is called a conjugate system on \( \mathcal{D} \).

Thus once Theorem 4.6 is established we have immediately the following:

Theorem 4.11. If \( F = (f_1, f_2, \ldots, f_m) \) is a conjugate system of regular functions on a domain \( \mathcal{D} \subset \mathbb{C} \times \mathbb{C} \), then there exists a \( p, 0 < p < 1, \) \( p \) independent of \( F, \) such that \( |F(a, b)|^p \) is subregular on \( \mathcal{D} \) for all \( p \geq p_0. \)

We provide some examples of conjugate systems:

(i) Suppose \( \{a_j\}_{j=1}^m \) is a subset of \( \mathcal{D} \). Suppose \( \sum \) which contains at least one even \( l \) and at least one odd \( l \). Then for a regular function \( f \neq (T_1 f, \ldots, T_m f) \) is a conjugate system as follows from Proposition 4.5 by letting the sets \( D \) and \( B \) in (4.9) be the odd and the even integers, respectively. In particular, \( (f, T_1 f) \) and \( (f, T_1 f, \ldots, T_m f) \) are conjugate systems.

(ii) \( F = (f_i, T_i f) \), with odd \( l \), is a conjugate system. If \( \frac{q-1}{2} \) is odd, then \( n_l \), with \( l = \frac{q-1}{2} \), takes only \( \pm 1 \) as its values. With \( \phi \) substituting the \( \phi \) function, \( T_l \) is the \( \mu \)-analogue of the Hilbert transform. The case \( q = 3 \) has been studied in [1]. Note that, in this case \( \mu = 3 \), if we take \( \phi(x) = e^{-i \phi(x)} \) where \( x = (x_n) \in \mathbb{R}^n \), \( \sigma \) is \( \sum \) on the \( \mathbb{R} \) series and \( \sigma \) is \( \sum \) on the \( \mathbb{R} \) series as used by Phillips in [6], then by an easy computation we have
\[
P'(x) = \int x \pi(x) \chi(x) \frac{dx}{x} = \frac{i}{\sqrt{3}}.
\]

Hence (4.4) takes the following simple form
\[
a_j = \frac{i}{\sqrt{3}} (a_j^{(k+1)} - a_j^{(k)}), \quad j \in \mathbb{Z}.
\]

This simplest case often plays a suggestive role.

We now give a proof of Theorem 4.6. The proof follows very closely that of Theorem 2 in [1]. We need the following lemma:

**Lemma 4.12.** Let \( a \) and \( (a) \) be as in the theorem; that is, (4.7), (4.8) and (4.9) are valid. Given \( p_1, p_2 > \frac{m-m}{m} \), there exists a constant \( A \) such that (4.10) holds for all \( p \geq p_0. \)

**Proof.** We may assume that \( ||a|| = 0 \) and \( 0 < p \leq 1. \)

\[
(4.13) \quad \sum_{l=0}^{m-1} ||a + a||^p = \sum_{l=0}^{m} \left[ \sum_{j=0}^{m} |a_j + a_j|^p \right]^\frac{p}{p} = \sum_{l=0}^{m} \left[ \sum_{j=0}^{m} |a_j|^p + 2 \sum_{j=0}^{m} a_j a_j^* \right]^\frac{p}{p} = \sum_{l=0}^{m} \left[ 1 + \frac{2 \sum_{j=0}^{m} a_j a_j^*}{||a||^p} \right].
\]

By using (4.9) we have the following estimate:

\[
(4.14) \quad \sum_{l=0}^{m} \left[ \sum_{j=0}^{m} |a_j|^p \right]^2 \leq \sum_{l=0}^{m} \left[ \sum_{j=0}^{m} a_j a_j^* + \sum_{j=0}^{m} a_j a_j^* \right]^2 \leq \sum_{l=0}^{m} \left[ \sum_{j=0}^{m} |a_j|^p \right]^2 + \sum_{l=0}^{m} \left[ \sum_{j=0}^{m} a_j a_j^* \right]^2 \leq \sum_{l=0}^{m} \left( \sum_{j=0}^{m} |a_j|^p \right)^2 + \sum_{l=0}^{m} \left( \sum_{j=0}^{m} |a_j|^p \right)^2 \leq 2 \sqrt{m} ||a||^2 \left( \sum_{j=0}^{m} |a_j|^2 \right) \leq 2 \sqrt{m} ||a||^2 ||a||^2 = 2 \sqrt{m} (m+1) ||a||^2.
\]

In particular, \( A = \sqrt{m} (m+1) ||a||^2. \)

\[
(3y)^{m-1} ||a||^p \leq \frac{2 \text{Re} \sum |a_j|^p}{||a||^p} \quad \text{and} \quad 2 (3y)^{m-1} ||a||^p \leq \text{Re} \sum |a_j|^p \quad \text{and} \quad 2 (3y)^{m-1} ||a||^p \leq \text{Re} \sum |a_j|^p.
\]

Hence with the binomial expansion of each summand in (4.13), we have

\[
2 \leq \frac{m+1}{9m} \leq \frac{8}{9} < 1.
\]
We claim that there exists a \( \delta, 0 < \delta < 1 \), such that

\[
|||a||| \leq \frac{\delta}{q} \sum_{i} |||a + a^i||| \quad \text{for all } \beta \in \mathcal{B}.
\]

If this is the case, then (4.10) holds for all \( \beta \in \mathcal{B} \), \( p \gg p_2 = \left(1 + \left(\frac{\log \frac{1}{q}}{\log q}\right)^{-1}\right)^{-1} \). Thus (4.10) is valid for all \( p \gg p_2 = \max(p_1, p_2) \).

It remains to show the claim. Suppose there does not exist a \( \delta, 0 < \delta < 1 \) such that (4.17) is valid, then by the compactness of \( \mathcal{B} \), there is a \( \beta = (a + a^i)^{i \in [m]} \in \mathcal{B} \) such that

\[
|||a||| = \frac{1}{q} \sum_{i} |||a + a^i|||.
\]

Hence there exist real \( \lambda_i; i = 0, 1, \ldots, q - 1 \) such that

\[
a + a^i = \lambda_i a, \quad i = 0, 1, \ldots, q - 1.
\]

That is, \( a^i = (\lambda_i - 1) a, \quad i = 0, 1, \ldots, q - 1 \); \( f = 0, 1, \ldots, m \). From (4.9) we have, for \( f \in D \) and \( k \in \mathbb{N} \)

\[
0 \leq \sum_{i} |a|^2 = \frac{1}{q} \sum_{i} |||a + a^i|||.
\]

Thus, \( \lambda_i = 1, \quad i = 0, 1, \ldots, q - 1 \) or \( a^i = 0 \) or \( a = 0 \). But each of these three cases implies \( |||a||| = 0 \), a contradiction.

The proof of Theorem 4.6 is completed.

§ 5. Theorem 3.1 provides information about the convergence in \( L^p \)-norm for some \( p \leq 1 \). The most interesting case \( p = 1 \) is included.

The following, corollary to Theorem 3.3 for \( p = 1 \), is a version of the F. and M. Riesz theorem.

**Theorem 5.3.** Suppose \( \mu_1, \mu_2, \ldots, \mu_m \) are bounded Borel measures on \( K \). If \( \{\mu_j(x, k) \}_{j=1}^{m} \) forms a conjugate system (where \( \mu_j(x, k) \) is the regularization of \( \mu_j \)). Then each \( \mu_j, j = 0, 1, \ldots, m \), is absolutely continuous.

**Proof.** Let \( \|\mu_j\| \) be the total variation of \( \mu_j \), \( j = 0, 1, \ldots, m \).

\[
\int_{K} |\mu_j(x, k)| \, dx \leq \int_{K} \left( \sum_{j=1}^{m} |\mu_j(x, k)|^{2} \right)^{1/2} \, dx
\]

\[
\leq \sum_{j=1}^{m} \left| \int_{K} |\mu_j(x, k)| \, dx \right|^{1/2} \leq \sum_{j=1}^{m} |||\mu_j|||, \quad k \in \mathbb{K}.
\]

by the homogeneity of the first expression and Lemma 4.12.

The collection \( \mathcal{B} \) of all vectors \( \beta = (a + a^i)^{i \in [m]} \) satisfying (4.7), (4.8), (4.9) and (4.10) forms a compact set in \( C^{\infty}(\mathbb{C}) \).
Hence, by Theorem 3.1 and the definition of conjugate system, there exists $F(x) = f_0(x), f_1(x), \ldots, f_m(x)$ with each component $f_j \in L^1$ such that $F(a, b) \to F(x)$ as $a \to b \to \infty$ almost everywhere and in $L^1$-norm. Therefore $dj \to f_j dx$, that is, $j \in \mathbb{R}$ is absolutely continuous, $j = 0, 1, \ldots, m$.

Now let $\pi$ be a generator of $H$, the group of (unitary) multiplicative character which are ramified of degree 1 and homogeneous of degree 0. $T_i$ is the operator such that $(T_i f)(t) = \pi^{-i} \hat{f}(t)$ as in § 3. We shall give another version of the F. and M. Riesz theorem in terms of the Fourier transform.

**Theorem 5.2.** Suppose $\mu$ is a finite Borel measure on $X$. If there exist a set $A \subset X$ and an odd $1 < \alpha < q - 1$, such that $\alpha^\prime$ is constant on $A$ and $\mu$ is supported on $A$, then $\mu$ is absolutely continuous.

**Proof.** Let $\pi^\prime(t) = \pi(t)$ for some $\pi^\prime \in U_{q_0}$. Then $\left( T_i \mu \right)(t) := \alpha^\prime \hat{\mu}(t)$ for $t \in A$ and $T_i \mu = 0$ for $t \notin A$. Thus $T_i \mu = \alpha^\prime \mu$ is also a finite Borel measure. By Theorem 5.1, since $F(x, h) = \mu(x, h), T_i \mu(x, h)$, with $i$ odd, forms a conjugate system, we have that $\mu$ is absolutely continuous.

The following two corollaries are immediate consequences of Theorem 5.2:

**Corollary 5.3.** Suppose $\mu$ is a finite Borel measure on $X$, such that $\hat{\mu}$ is supported on $A$ where $A$ is a "cone", i.e., $A = \bigcup_{\theta \in \theta} \{ x \in \mathbb{R}^n : \theta \cdot x = \theta \cdot a \}$ for some $a \in \mathbb{R}^n$. Then $\mu$ is absolutely continuous.

In the case when $\frac{q - 1}{2} = \frac{1}{2}$ is odd, let $W = \{ x \in X : \pi^\prime(t) = \pm 1 \}$. Hence $X = X \setminus W$. We thus have:

**Corollary 5.4.** If $\frac{q - 1}{2}$ is odd and $\mu$ is a finite Borel measure on $X$, $\mu$ is absolutely continuous on $W$, then $\mu_0$ is also absolutely continuous.

References