Then there exists a character \( p \) on \( C^*([x_1, \ldots, x_n]) \) such that
\[
p(x_i) = \lambda_i, \quad i \in \{1, \ldots, n\}.
\]

The assertion follows from the preceding theorem by Corollary 7 and Theorem 4.

Remark. The preceding corollary contains the following theorem of Arveson [1]: If \( x \in B(H) \) and \( \lambda \in \partial W(x) \cap \text{Sp}(x) \) then there exists a character \( p \) on \( C^*(\lambda) \) such that \( p(x) = \lambda \). The proof follows from the fact that \( W(\lambda) = \text{Sp}(x) \) ([3], Theorem 3; [4], Theorem 1).

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Normally subregular systems in normed algebras

by

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Abstract. The main aim of this note is to give a negative answer to a question about ideals of normed algebras, raised by Arveson [1].

Let \( A \) be a commutative complex unital normed algebra. \( \{a_1\}^N \subseteq A \) is called a normally subregular system if there is a commutative algebra \( B \supseteq A \) containing elements \( \{b_1\}^N \) of norm at most 1 such that \( \sum_{I=1}^N a_i b_i = 1 \). We show that for \( N \geq 2 \) normal subregularity is not characterized by the condition

\[
\inf \sum_{I=1}^N \|a_i\|: \|\varepsilon A, \|\leq 1 \} \geq 1.
\]

The algebras considered in this paper are commutative complex unital normed algebras though our results also hold for real ones. If \( A \) is a subalgebra of \( B(A \subseteq B) \), we call \( B \) an isometric extension, shorty extension, of \( A \). An element \( a \in A \) is a topological divisor of zero if \( \inf \|a_{\infty}\|: \|\varepsilon A, \|\leq 1 \} = 0 \). A well-known result of Shilov [5] states that \( \varepsilon A \) has an inverse of norm at most 1 in some extension of \( A \) if and only if \( \inf \|a_{\infty}\|: \|\varepsilon A, \|\leq 1 \} \geq 1 \). The problem of adjoining inverses of a set of elements was investigated by Arveson in [1] and [2]. In [4] I proved that one can always adjoin the inverses of uncountably many elements which are not topological divisors of zero but this is not necessarily true for uncountably many elements.

A set \( \{a_1, \ldots, a_N\} \subseteq A \) is called a regular system if there exist \( b_1, \ldots, b_N \in A \) such that \( \sum_{I=1}^N a_i b_i = 1 \). If the elements \( b_i \) can be chosen to have norm at most 1 then \( \{a_1, \ldots, a_N\} \) is normally regular. Finally, \( \{a_1, \ldots, a_N\} \subseteq A \) is subregular and normally subregular, respectively, if \( A \) has an isometric extension \( B \) for which the appropriate \( b_i \)'s can be chosen. These concepts were introduced by Arveson [3], mainly in order to pose the following problem: Is normal subregularity characterized by the (obviously necessary) condition

\[
\inf \sum_{I=1}^N \|a_i\|: \|\varepsilon A, \|\leq 1 \} \geq 1
\]
The aim of this paper is to show that this is not so.

More precisely, we shall also show that the following strengthening of condition (1) is still insufficient to ensure that \( \{a_1, a_2\} \) is normally subregular:

\[
\max \{\|a_1 x\|, \|a_2 x\|\} \geq \|x\| \quad \text{for all } x \in A.
\]

The subregular systems are closely connected to non-renovable ideals, which were investigated by Arens [2], [3] and Zelazko [6]. An ideal \( I \subset A \) is a non-renovable ideal if for every extension \( B \) of \( A \), \( I_B \cong B \), where \( (I)_B \) is the ideal generated by \( I \) in \( B \).

I am grateful to Professor B. E. Johnson of Newcastle and Professor W. Zelazko of Warsaw for drawing my attention to the problem discussed in this note.

**Theorem.** There is a commutative unital Banach algebra \( A \) which contains two elements, \( a_1 \) and \( a_2 \), such that

(i) \( \max \{\|a_1 x\|, \|a_2 x\|\} \geq \|x\| \) for all \( x \in A \),

(ii) there is no extension \( B \) of \( A \) which contains elements \( b_1, b_2 \) for which

\[
a_1 b_1 + a_2 b_2 = 1
\]

and

\[
\|b_1\|, \|b_2\| = 1.
\]

**Proof.** Denote by \( S \) the set consisting of the following elements:

\[
1, g_1, a_1 a_2^j, \quad i, j = 0, 1, \ldots, \quad i = 0, 1, 2, \quad j = 0, 1, 2, \quad j + k = 2.
\]

Put

\[
\|a_1\| = \|a_2\| = 1, ~ \|a_1 a_2\| = \|g_1 a_2 - g_2\| = 1, ~ \|g_2 a_1 - g_2\| = 5, ~ \|g_1 a_2^2\| = 10^{1/3}.
\]

Let \( L_1 \) be the Banach space with basis \( S \), consisting of the formal sums

\[
a = \sum_{i,j} \lambda_i a_i a_2^j, \quad \sum_{i,j} \lambda_i \|a_i\| = \infty, \quad \lambda_i \in C.
\]

with norm

\[
\|a\| = \sum_{i,j} \lambda_i \|a_i\|.
\]

Let \( L_2 \) be the Banach space with basis \( \{g_1, g_2\} \) and norm

\[
\|g_1 + \mu g_2\| = \max \{10 \|g_1\|, 5 \|\mu g_2\|\}, \quad \lambda, \mu \in C.
\]

and let \( L_3 \) be the Banach space with basis \( \{a_1, a_2, a_3 a_4\} \) and with maximal norm under which

\[
\|a_1 g_1\| = \|a_2 g_1\| = 10, \quad \|a_1 g_2 + a_2 g_2\| = 1.
\]

Put \( L = L_1 + L_2 + L_3 \), where

\[
\|x\| = \|a_1 x + a_2 x + a_3 a_4 x\| = \|a_1 x\| + \|a_2 x\| + \|a_3 a_4 x\|.
\]

In \( L \) we shall use the notations \( g_1 a_1 - g_2, g_2 = g_2 a_1, g_1 = g_1 a_2, g_1 a_2 - g_2 + a_2 = g_1 a_2 \).

Equipped with the formal commutative multiplication (for example \( g_1 a_1 - g_2 ) a_1 a_2 = g_1 a_1 a_2 - g_2 a_1 a_2 \), together with the relations

\[
g_1 g_j = 0, \quad i, j = 0, 1, 2,
\]

\( L \) is easily seen to be a commutative complex unital Banach algebra, which we denote by \( A \).

We claim that \( A \) and \( a_1, a_2 \in A \) satisfy the requirements of the theorem.

Let us show first that (i) is satisfied, i.e. for any

\[
\|a\| = \sum_{i,j} \lambda_i a_i a_2^j, \quad a_1 \in S, a_2 \in L_2, \quad a_3 \in L_3,
\]

we have

\[
\max \{\|a_1\|, \|a_2\|\} \geq \|a\|.
\]

The definition of the norm implies that, in proving (3), we can suppose without loss of generality that the following elements do not occur among the \( a_i \):

\[
1, a_1 a_2^i, i + j \geq 1, \quad g_2 a_1 a_2, g_1 a_2 a_2^i, k + l \geq 2.
\]

Then we have

\[
\|a\| \geq \|a_1 a_2^i + g_2 a_1 a_2 + a_2 a_2^i\| + \|a_1 a_2^i + \lambda a_1 a_2 + \lambda a_1 a_2 a_2^i + \lambda a_1 a_2 a_2 a_2^i\| + \|a_1 a_2^i + \lambda a_1 a_2 + \lambda a_1 a_2 a_2^i + \lambda a_1 a_2 a_2 a_2^i + \lambda a_1 a_2 a_2 a_2^i + \ldots\|
\]

\[
\leq \max \{10 \|a\|, [5] \|a\| + 5[\|a\| + 10[\|a\| + 10[\|a\| + 10[\|a\| + 10[\|a\| + \ldots]\}
\]

Furthermore

\[
\|a\| \geq \|a_1 a_2^i + g_2 a_1 a_2 + a_2 a_2^i\| + \|a_1 a_2^i + \lambda a_1 a_2 + \lambda a_1 a_2 a_2^i + \lambda a_1 a_2 a_2 a_2^i + \lambda a_1 a_2 a_2 a_2 a_2^i\| + \|a_1 a_2^i + \lambda a_1 a_2 + \lambda a_1 a_2 a_2^i + \lambda a_1 a_2 a_2 a_2^i + \lambda a_1 a_2 a_2 a_2 a_2^i + \lambda a_1 a_2 a_2 a_2 a_2 a_2^i + \ldots\|
\]

\[
\geq 10[\|a\| + [\|a\| + 10[\|a\| + 90 \sum \lambda_i]\}
\]
and similarly,

\[ |a_0 x| \geq 2 |a_0| + 10 |b_1| + |b_2| + 90 \sum_1^4 |a_i| . \]

The required inequality clearly follows from inequalities (4)–(6).
Suppose now that there exist \( A = a_1 \) and \( b_1, b_2 \in B \) such that \( a_1 b_1 + + a_2 b_2 = 1 \). \( \|b_1\|, \|b_2\| < 1 \). Then

\[
10 = \|x_0\| - \|x_0 + (a_0 + a_1 b_1 + a_2 b_2) x_0 (a_1 b_1 + a_2 b_2 - 1)\|
\leq \|a_0 x_0 - x_0 a_0 x_0\| + \|a_1 x_0 - x_0 a_1 x_0\| + + \|a_2 x_0 - x_0 a_2 x_0\| + \|a_2 x_0 - x_0 a_2 x_0\| + \|a_3 x_0 - x_0 a_3 x_0\|
\leq 5 + 1 + 1 + 1 = 9,
\]

and this contradiction proves that \( \{ A, a_1, a_2 \} \) has property (ii). Thus, the proof of the theorem is complete.

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H^p-spaces of conjugate systems on local fields

by

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Abstract. Properties of regular functions and subregular functions, analogous
to harmonic functions and subharmonic functions, are studied. The local field variant
of the Fatou-Calderón-Stein theorem on harmonic function and its Lusin area function
is proved. Conjugate systems of regular functions are defined. The theory of H^p-spaces
of conjugate systems in the sense of Stein-Weiss is presented. The F. and M. Riesz
theorem is also treated.

INTRODUCTION

Stein and Weiss [10] have developed a theory of H^p-spaces for
M. Riesz systems \( F(x, y) = (f_1(x, y), f_2(x, y), \ldots, f_n(x, y)) \) of conjugate
harmonic functions on euclidean half-spaces \( R^n_{+} \) satisfying

\[
\int_{R^n} |F(x, y)|^p dx dy \prec A < \infty \quad \text{for all } y > 0.
\]

The basic result needed, common to all these systems, is the existence of a positive \( p \) such that \( F^p \) is subharmonic. It is our main objective in this paper to construct conjugate systems on
local fields such that the analogue of the above basic result is valid which
enable us to develop a theory of H^p-spaces on local fields.

Let \( K \) be a local field. That is, \( K \) is a locally compact, non-discrete, complete, tally disconnected field. Such a field is a p-adic field, a finite
algebraic extension of a p-adic field, or a field of formal Laurent series over a finite field. See [8] for details. Various aspects of harmonic analysis
on K and \( K^* \), the n-dimensional vector spaces over \( K \), have been studied
in [4], [8], [12], [13], [14], [6], [7], and [5]. In particular, from [14], [6], and [7] we have the notion of singular integral operators and multipliers; from [13] we have the notion of regular functions on \( K^n \times Z \) which play the role of harmonic functions on \( R^n_{+} \).

In Part A, we study the theory of regular functions, including sub-
regular functions and the Lusin area function. Conjugate systems of