Then there exists a character p on $C^*(\{x_1, \ldots, x_n\})$ such that

$$p(x_i) = \lambda_i, \quad i \in \{1, \ldots, n\}.$$

The assertion follows from the preceding theorem by Corollary 7 and Theorem 4.

Remark. The preceding corollary contains the following theorem of Arwerson [1]: If $x \in B(H)$ and $\lambda \in \partial W(x) \cap \mathrm{Sp}(x)$ then there exists a character p on $C^*(\{x\})$ such that $p(x) = \lambda$. The proof follows from the fact that $\overline{W(x)} = V(x)$ ([2], Theorem 3; [7], Theorem 11).

References

[1] W. B. Arwerson, *Subalgebras of C^*-algebras*, Acta Math. 123 (1969), pp. 141–224.

[2] S. K. Berberian and G. H. Orland, *Closure of the numerical range of an operator*, Proc. Amer. Math. Soc. 18 (1967), pp. 499–503.

[3] J. Bunce, *Characters on single generated C^*-algebras*, Proc. Amer. Math. Soc. 25 (1970), pp. 297–303.

[4] — *The joint spectrum of commuting nonnormal operators*, Proc. Amer. Math. Soc. 29 (1971), pp. 499–505.

[5] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Paris 1964.

[6] S. Hilderbrandt, *Über den numeriche Wertbereich eines Operators*, Math. Ann. 163 (1966), pp. 230–247.

[7] G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. 100 (1961), pp. 29–43.

[8] Gh. Mocanu, *Les fonctionnelles relativement multiplicatives sur des algèbres de Banach*, Rev. Roum. Math. Pures et Appl. 3 (1971), pp. 379–381.

[9] R. Nakamoto and M. Nakamura, *A remark on the approximate spectra of operators*, Proc. Japan. Acad. 48 (1972), pp. 103–107.

[10] G. Orland, *On a class of operators*, Proc. Amer. Math. Soc. 15 (1964), pp. 75–79.

Received October 30, 1972 (610)

STUDIA MATHEMATICA, T. XLIX. (1974)

Normally subregular systems in normed algebras

by

BÉLA BOLLOBÁS (Cambridge, England)

Abstract. The main aim of this note is to give a negative answer to a question about ideals of normed algebras, raised by Arens [3].

Let A be a commutative complex unital normed algebra. $\{a_i\}_1^N \subset A$ is called a *normally subregular system* if there is a commutative algebra $B \supset A$ containing elements $\{b_i\}_1^N$ of norm at most 1 such that $\sum_1^N a_i b_i = 1$. We show that for $N \geqslant 2$ normal subregularity is *not* characterized by the condition

$$\inf\Big\{\sum_1^N \|a_i x\|:\ x \in A,\ \|x\| = 1\Big\} \geqslant 1.$$

The algebras considered in this paper are commutative complex unital normed algebras though our results also hold for real ones. If A is a subalgebra of $B(A \subset B)$, we call B an *isometric extension*, shortly *extension*, of A. An element $a \in A$ is a *topological divisor of zero* if $\inf\{\|ax\|:\ x \in A,\ \|x\| = 1\} = 0$. A well-known result of Shilov [5] states that $a \in A$ has an inverse of norm at most 1 in some extension of A if and only if $\inf\{\|ax\|:\ x \in A,\ \|x\| = 1\} \geqslant 1$. The problem of adjoining inverses of a set of elements was investigated by Arens in [1] and [2]. In [4] I proved that one can always adjoin the inverses of countably many elements which are not topological divisors of zero but this is not necessarily true for uncountably many elements.

A set $\{a_1, \ldots, a_N\} \subset A$ is called a *regular system* if there exist $b_1, \ldots, b_N \in A$ such that $\sum_1^N a_i b_i = 1$. If the elements b_i can be chosen to have norm at most 1 then $\{a_1, \ldots, a_N\}$ is *normally regular*. Finally, $\{a_1, \ldots, a_N\} \subset A$ is *subregular* and *normally subregular*, respectively, if A has an isometric extension B for which the appropriate b_i's can be chosen. These concepts were introduced by Arens [3], mainly in order to pose the following problem. Is normal subregularity characterized by the (obviously necessary) condition

$$\inf \sum_1^N \|a_i x\|:\ x \in A,\ \|x\| = 1\} \geqslant 1? \tag{1}$$

The aim of this paper is to show that *this is not so.*

More precisely, we shall also show that the following *strengthening* of condition (1) is still *insufficient* to ensure that $\{a_1, a_2\}$ is normally subregular:

$$\max\{\|a_1 x\|, \|a_2 x\|\} \geqslant \|x\| \quad \text{for all } x \in A.$$

The subregular systems are closely connected to non-removable ideals, which were investigated by Arens [2], [3] and Żelazko [6]. An ideal $I \subset A$ is a *non-removable ideal* if for every extension B of A $(I)_B \neq B$, where $(I)_B$ is the ideal generated by I in B.

I am grateful to Professor B. E. Johnson of Newcastle and Professor W. Żelazko of Warsaw for drawing my attention to the problem discussed in this note.

THEOREM. *There is a commutative unital Banach algebra A which contains two elements, a_1 and a_2, such that*

(i) $\max\{\|a_1 x\|, \|a_2 x\|\} \geqslant \|x\|$ *for all* $x \in A$,

(ii) *there is no extension B of A which contains elements b_1, b_2 for which*

$$a_1 b_1 + a_2 b_2 = 1$$

and

$$\|b_1\|, \|b_2\| = 1.$$

Proof. Denote by S the set consisting of the following elements:

$$1, q_2, \quad a_1^i a_2^j, \quad i, j = 0, 1, \ldots, \quad i+j \geqslant 1,$$
$$q_0 a_1 - q_1, \quad q_0 a_2 - q_2, \quad q_1 a_1, \quad q_2 a_2,$$
$$q_i a_1^j a_2^k, \quad i = 0, 1, 2, \quad j, k = 0, 1, \ldots, \quad j+k \geqslant 2.$$

Put

$$\|1\| = \|q_2\| = 1, \quad \|a_1^i a_2^j\| = 100^{i+j},$$
$$\|q_1 a_1\| = \|q_2 a_2\| = \|q_0 a_2 - q_2\| = 1, \quad \|q_0 a_1 - q_1\| = 5,$$
$$\|q_i a_1^j a_2^k\| = 10^{j+k}.$$

Let L_1 be the Banach space with basis S, consisting of the formal sums

$$x_1 = \sum_{s \in S} \lambda_s s, \quad \sum_{s \in S} |\lambda_s| \|s\| < \infty, \quad \lambda_s \in \boldsymbol{C},$$

with norm

$$\|x_1\| = \sum_{s \in S} |\lambda_s| \|s\|.$$

Let L_2 be the Banach space with basis $\{q_0, q_1\}$ and norm

$$\|\lambda q_0 + \mu q_1\| = \max\{10|\lambda|, 5|\mu|\}, \quad \lambda, \mu \in \boldsymbol{C},$$

and let L_3 be the Banach space with basis $\{a_1 q_2, a_2 q_1\}$ and with maximal norm under which

$$\|a_1 q_2\| = \|a_2 q_1\| = 10,$$
$$\|a_1 q_2 + a_2 q_1\| = 1.$$

Put $L = L_1 + L_2 + L_3$, where

$$\|x\| = \|(x_1, x_2, x_3)\| = \|x_1\| + \|x_2\| + \|x_3\|.$$

In L we shall use the notations $(q_0 a_1 - q_1) + q_1 = q_0 a_1$, $(q_0 a_2 - q_2) + q_2 = q_0 a_2$.

Equipped with the formal *commutative* multiplication (for example $(q_0 a_1 - q_1) a_2 a_1 = q_0 a_1^2 a_2 - q_1 a_1 a_2$), *together with* the relations

$$q_i q_j = 0, \quad i, j = 0, 1, 2,$$

L is easily seen to be a commutative complex unital Banach algebra, which we denote by A.

We claim that A and $a_1, a_2 \in A$ satisfy the requirements of the theorem. Let us show first that (i) is satisfied, i.e. for any

$$x = \sum_1^n \lambda_i s_i + x_2 + x_3, \quad s_i \in S, x_2 \in L_2, \quad x_3 \in L_3, \tag{2}$$

we have

$$\max\{\|a_1 x\|, \|a_2 x\|\} \geqslant \|x\|. \tag{3}$$

The definition of the norm implies that, in proving (3), we can suppose without loss of generality that the following elements do not occur among the s_i's:

$$1, \quad a_1^i a_2^j, \quad i+j \geqslant 1, \quad q_1 a_1, \quad q_2 a_2, \quad q_j a_1^k a_2^l, \quad k+l \geqslant 2.$$

Then we have

$$\|x\| = \Big\| \sum_0^2 \lambda_j q_j + \lambda_3 (q_0 a_1 - q_1) + \lambda_4 (q_0 a_2 - q_2) + \lambda_5 a_1 q_2 + \lambda_6 a_2 q_1 \Big\| \tag{4}$$
$$\leqslant \max\{|10\lambda_0|, |5\lambda_1|\} + |\lambda_2| + 5|\lambda_3| + |\lambda_4| + 10|\lambda_5| + 10|\lambda_6|.$$

Furthermore

$$\|a_1 x\| = \|\lambda_0 q_0 a_1 + (\lambda_1 - \lambda_3) q_1 a_1 + (\lambda_2 - \lambda_4) q_2 a_1 + \tag{5}$$
$$+ \lambda_3 q_0 a_1^2 + \lambda_4 q_0 a_1 a_2 + \lambda_5 q_2 a_1^2 + \lambda_6 q_1 a_1 a_2\|$$
$$= 10|\lambda_0| + |\lambda_1 - \lambda_3| + 10|\lambda_2 - \lambda_4| + 100 \sum_3^6 |\lambda_i|$$
$$\geqslant 10|\lambda_0| + |\lambda_1| + 10|\lambda_2| + 90 \sum_3^6 |\lambda_1|,$$

and similarly,

$$\|a_2 x\| \geqslant 2|\lambda_0| + 10|\lambda_1| + |\lambda_2| + 90 \sum_{3}^{6} |\lambda_i|. \tag{6}$$

The required inequality clearly follows from inequalities (4)–(6).

Suppose now that there exist $B \supset A$ and $b_1, b_2 \in B$ such that $a_1 b_1 + a_2 b_2 = 1$, $\|b_1\|$, $\|b_2\| \leqslant 1$. Then

$$\begin{aligned} 10 = \|q_0\| &= \|q_0 + (q_0 + q_1 b_1 + q_2 b_2)(a_1 b_1 + a_2 b_2 - 1)\| \\ &\leqslant \|b_1(q_1 - q_0 a_1)\| + \|b_2(q_2 - q_0 a_2)\| + \\ &\qquad + \|b_1 b_2 (q_1 a_2 + q_2 a_1)\| + \|b_1^2 q_1 a_1\| + \|b_2^2 q_2 a_2\| \\ &\leqslant 5+1+1+1+1 = 9, \end{aligned}$$

and this contradiction proves that $\{A, a_1, a_2\}$ has property (ii). Thus the proof of the theorem is complete.

References

[1] R. Arens, *Inverse-producing extensions of a normed algebra*, Trans. Amer. Math. Soc. 88 (1958), pp. 536–548.

[2] — *Extensions of Banach algebras*, Pacific J. Math. 10 (1960), pp. 1–16.

[3] — *Ideals in Banach algebra extensions*, Studia Math. 31 (1968), pp. 29–34.

[4] B. Bollobás, *Adjoining inverses to Banach algebras*, to appear in Trans. Amer. Math. Soc.

[5] Г. Е. Шилов, *О нормированных кольцах с одной образующей*, Мат. Сборник 21 (63) (1947), pp. 25–37.

[6] W. Żelazko, *On a certain class of non-removable ideals in Banach algebras*, Studia Math. 44 (1972), pp. 87–92.

UNIVERSITY OF CAMBRIDGE

Received February 10, 1972 (646)

STUDIA MATHEMATICA T. XLIX. (1974)

H^p-spaces of conjugate systems on local fields

by

JIA-ARNG CHAO (Austin, Tex.)

Abstract. Properties of regular functions and subregular functions, analogous to harmonic functions and subharmonic functions, are studied. The local field variant of the Fatou–Calderón–Stein theorem on harmonic function and its Lusin area function is proved. Conjugate systems of regular functions are defined. The theory of H^p-spaces of conjugate systems in the sense of Stein–Weiss is presented. The F. and M. Riesz theorem is also treated.

INTRODUCTION

Stein and Weiss [10] have developed a theory of H^p-spaces for M. Riesz systems $F(x, y) = (f_0(x, y), f_1(x, y), \ldots, f_n(x, y))$ of conjugate harmonic functions on euclidean half-spaces $\boldsymbol{R}_+^{n+1}$ satisfying

$$\int_{\boldsymbol{R}^n} |F(x, y)|^p dx \leqslant A < \infty \quad \text{for all } y > 0.$$

Coifman and Weiss [2] extended the theory to Generalized Cauchy–Riemann systems. The basic result needed, common to all these systems, is the existence of a positive $p_0 < 1$ such that $|F|^{p_0}$ is subharmonic. It is our main objective in this paper to construct conjugate systems on local fields such that the analogue of the above basic result is valid which enable us to develop a theory of H^p-spaces on local fields.

Let K be a local field. That is, K is a locally compact, non-discrete, complete, totally disconnected field. Such a field is a p-adic field, a finite algebraic extension of a p-adic field, or a field of formal Laurent series over a finite field. See [8] for details. Various aspects of harmonic analysis on K and K^n, the n-dimensional vector spaces over K, have been studied in [4], [8], [12], [13], [14], [6], [7], and [5]. In particular, from [14], [6], and [7] we have the notion of singular integral operators and multipliers; from [13] we have the notion of regular functions on $K^n \times \boldsymbol{Z}$ which play the role of harmonic functions on $\boldsymbol{R}_+^{n+1}$.

In Part A, we study the theory of regular functions, including subregular functions and the Lusin area function. Conjugate systems of