The joint approximate spectrum of a finite system of elements of a C*-algebra

by

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Abstract. The aim of this paper is to study the joint approximate spectrum of a finite system of elements of a C*-algebra \( A \). The main result states that the joint approximate spectrum of a finite system \( (x_1, \ldots, x_n) \) of elements of \( A \) is exactly the set

\[
\left\{ (s(x_1), \ldots, s(x_n)) \mid s \in S_{(x_1, \ldots, x_n)} \right\} = \left\{ (p(x_1), \ldots, p(x_n)) \mid p \in P_{(x_1, \ldots, x_n)} \right\}
\]

where \( S_{(x_1, \ldots, x_n)} \) (resp. \( P_{(x_1, \ldots, x_n)} \)) denotes the set of all states \( s \) (resp. pure states \( p \)) with the following relation:

\[
x(yz) = s(y) s(z) \quad \text{or any } y, z \in A \text{ and any } y, z \in A.
\]

Throughout this paper \( A \) is a C*-algebra with identity, \( S = S(A) \) the set of all states on \( A \) and \( P = P(A) \) is the set of all pure states on \( A \). The aim of this paper is to study the joint approximate spectrum of a finite system of elements of \( A \). A state \( s \) is called left multiplicative with respect to a subset \( B \) of \( A \) if

\[
x(yz) = s(y) s(x)
\]

for any \( y \in A \) and any \( x \in B \). The set of all left multiplicative states (resp. pure states) with respect to \( B \) will be denoted by \( S_B \) (resp. \( P_B \)).

The main result of this paper (Theorem 4) states that the joint approximate spectrum of a finite system \( (x_1, \ldots, x_n) \) of \( A \) is exactly the set

\[
\left\{ (s(x_1), \ldots, s(x_n)) \mid s \in S_{(x_1, \ldots, x_n)} \right\} = \left\{ (p(x_1), \ldots, p(x_n)) \mid p \in P_{(x_1, \ldots, x_n)} \right\}.
\]

Using this theorem we obtain also some results concerning the joint approximate spectrum of a finite system \( (x_1, \ldots, x_n) \) of elements of \( A \). These results extend the similar results of Bunce ([4], Proposition 2 and Proposition 3) or Arveson ([1], Theorem 3.1.2).

**Theorem 1.** Let \( (x_1, \ldots, x_n) \) be a finite system of elements of \( A \) and let \( (\lambda_1, \ldots, \lambda_n) \) be a finite system of complex numbers. Then the following assertions are equivalent:
1) \( \sum_{i=1}^{n} a(x_{i} - \lambda_{i}) \neq A. \)

2) For any real number \( \varepsilon > 0 \) there exists a state \( \psi \) such that

\[
\sum_{i=1}^{n} \psi^*(x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) \psi < \varepsilon.
\]

3) For any real number \( \varepsilon > 0 \) there exists a pure state \( \rho \) such that

\[
\sum_{i=1}^{n} \rho^*(x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) \rho < \varepsilon.
\]

4) There is no real number \( \varepsilon > 0 \) such that

\[
\sum_{i=1}^{n} (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) \geq \varepsilon.
\]

5) There exists a sequence \( \{u_{k}\}_{k \in \mathbb{N}} \) of elements of \( A \) such that \( \lim_{k \to \infty} \|u_{k}\| = 1 \), and such that

\[
\lim_{k \to \infty} \left( \sum_{i=1}^{n} \left\| (x_{i} - \lambda_{i}) u_{k} \right\| \right) = 0.
\]

1) \( \Rightarrow \) 2). Suppose that there exists a real number \( \varepsilon > 0 \) such that

\[
s \in S \Rightarrow \sum_{i=1}^{n} s (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) \geq \varepsilon
\]

and therefore

\[
\sum_{i=1}^{n} (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) \geq \varepsilon.
\]

Using a standard argument it follows that there exists an element \( u \in A \) for which

\[
u \left( \sum_{i=1}^{n} (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) \right) = 1
\]

and therefore

\[
\sum_{i=1}^{n} a(x_{i} - \lambda_{i}) = A.
\]

The relations 2) \( \Rightarrow \) 3) \( \Rightarrow \) 4) follows from the fact that an element \( a \in A \) is positive if and only if for any \( s \in S \) (resp. \( p \in P \)) we have \( s(a) \geq 0 \) (resp. \( p(a) \geq 0 \)).

4) \( \Rightarrow \) 5). Assume that there exists \( \varepsilon > 0 \) such that

\[
u \in A, \|u\| = 1 \Rightarrow \sum_{i=1}^{n} \left\| (x_{i} - \lambda_{i}) u \right\| \geq \varepsilon.
\]

We have

\[
\forall \{1, \ldots, n\} \ni \sum_{i=1}^{n} (x_{i} - \lambda_{i}) u_{i} = \sum_{i=1}^{n} (x_{i} - \lambda_{i}) u_{i} = \sum_{i=1}^{n} (x_{i} - \lambda_{i}) u_{i} = \sum_{i=1}^{n} (x_{i} - \lambda_{i}) u_{i}.
\]

Hence, for any \( u \in A \), \( \|u\| = 1 \), and any \( s \in S \) \( 1 \ldots, n \) such that

\[
\sum_{i=1}^{n} (x_{i} - \lambda_{i}) u_{i} = \sum_{i=1}^{n} (x_{i} - \lambda_{i}) u_{i} = \sum_{i=1}^{n} (x_{i} - \lambda_{i}) u_{i}.
\]

Since, for any \( u \in A \), \( \|u\| = 1 \), we have

\[
\sum_{i=1}^{n} \left\| (x_{i} - \lambda_{i}) u \right\| \geq \frac{1}{n} \left( \sum_{i=1}^{n} \left\| (x_{i} - \lambda_{i}) u \right\| \right) \geq \frac{\varepsilon}{n},
\]

we may find \( s \in S \) \( 1 \ldots, n \) such that

\[
\sum_{i=1}^{n} (x_{i} - \lambda_{i}) u_{i} = \sum_{i=1}^{n} (x_{i} - \lambda_{i}) u_{i}.
\]

Hence, for any \( u \in A \), \( \|u\| = 1 \), we have

\[
\sup_{u \in S} \left( \sum_{i=1}^{n} (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) u_{i} \right) = \sup_{u \in S} \left( \sum_{i=1}^{n} (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) u_{i} \right) = \sum_{i=1}^{n} (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) u_{i} = \frac{\varepsilon^2}{n},
\]

\[
\sum_{i=1}^{n} (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) u_{i} = \frac{\varepsilon^2}{n}.
\]

Since \( \sum_{i=1}^{n} (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) \) is a positive element of \( A \), the preceding relation implies the inequality

\[
\sum_{i=1}^{n} (x_{i} - \lambda_{i})^* (x_{i} - \lambda_{i}) \geq \frac{\varepsilon^2}{n}.
\]

5) \( \Rightarrow \) 1). Assume that

\[
\sum_{i=1}^{n} a(x_{i} - \lambda_{i}) = A
\]

and let \( u_{1}, \ldots, u_{n} \in A \) be such that

\[
\sum_{i=1}^{n} u_{i} (x_{i} - \lambda_{i}) = 1.
\]
We have, for any $u \in A$, $\|u\| = 1$,
\[
1 = \left\| \sum_{i=1}^{n} u_{i}(x_{i} - \lambda)u \right\| \leq \sum_{i=1}^{n} \|u_{i}\| \|\lambda(x_{i} - \lambda)u\| \\
\leq a \left\| (x_{i} - \lambda)u \right\| \leq \frac{1}{a} \sum_{i=1}^{n} \| (x_{i} - \lambda)u \|
\]
where
\[
a = \sup \{|w_{i}|, \ldots, |w_{n}|\}.
\]

Remarks. a) Let $S_{n}$ be a subset of $S$ such that for any $x \in A$ we have $x \geq 0 \iff \{s \in S_{n} \mid s(x) \geq 0\}$.

Then each of the assertions 1) - 5) is equivalent to the following one:

2') For any real number $\varepsilon > 0$ there exists $x \in S_{n}$ such that
\[
\sum_{i=1}^{n} \varepsilon [(x_{i} - \lambda)^{n} (x_{i} - \lambda)] < \varepsilon.
\]

In the case when $A$ is equal to $B(H)$, the algebra of all bounded linear operators on a complex Hilbert space $H$, we may take instead of $S_{n}$ the set of states $p_{N}$ on $A$ defined by
\[
p_{N}(x) = \langle p_{N}, h \rangle
\]
where $p_{N}, \|h\| = 1$. In this particular case the assertion 2') is exactly the following:

2'') For any real number $\varepsilon > 0$ there exists $x \in H_1$ such that
\[
\sum_{i=1}^{n} \| (x_{i} - \lambda)h \|^2 < \varepsilon.
\]

b) In the case when $A$ is a $W^{*}$-algebra (in particular, if $A = B(H)$) each of the assertions 1) - 5) is equivalent to the following:

3') There exists a sequence $(u_{i})_{i \in \mathbb{N}}$ of elements of $A$ which are projections such that
\[
\lim_{k \to \infty} \sum_{i=1}^{n} \| (x_{i} - \lambda)u_{i} \| = 0.
\]

The proof follows from the fact that if $x$ is a positive element of $A$ and $\varepsilon$ is a real number $> 0$ then the relation
\[
\|w_{i}u\| \geq \varepsilon \quad \text{for any projection } w \in A
\]
implies the relation $x \geq \varepsilon$. e) The preceding theorem contains Theorems 1, 4 from [9]. The relations 1) - 5) $\Rightarrow$ 5') for the case $A = B(H)$ is the solution of a problem stated in [9].

DEFINITION. Let $(x_{1}, \ldots, x_{n})$ be a finite system of elements of $A$. A finite system $(\lambda_{1}, \ldots, \lambda_{n})$ of complex numbers is called a joint approximate proper value of $(x_{1}, \ldots, x_{n})$ if one of the assertions 1) - 5) of Theorem 1 holds. The set of all joint approximate proper values of a system $(x_{1}, \ldots, x_{n})$ is called the joint approximate spectrum and is denoted by
\[
\pi_{A}(x_{1}, \ldots, x_{n}) = \pi(x_{1}, \ldots, x_{n}).
\]

THEOREM 2 (Bunce [4]). Let $(x_{1}, \ldots, x_{n})$ be a finite system of elements of $A$ such that $x_{i}x_{j} = x_{j}x_{i}$ for any $i, j \in \{1, \ldots, n\}$ and let $(\lambda_{1}, \ldots, \lambda_{n})$ be an element of $\pi(x_{1}, \ldots, x_{n})$. Then there exists $\eta_{n+1} \in \pi(x_{n+1})$ such that
\[
(\lambda_{1}, \ldots, \lambda_{n}, \eta_{n+1}) \in \pi(x_{1}, \ldots, x_{n}, x_{n+1}).
\]

From this theorem follows the fact that for any finite system $(x_{1}, \ldots, x_{n})$ of elements of $A$ with $x_{i}x_{j} = x_{j}x_{i}$ for $i, j \in \{1, \ldots, n\}$, the joint approximate spectrum of $(x_{1}, \ldots, x_{n})$ is non-empty.

We recall the following:

DEFINITION. Let $(x_{1}, \ldots, x_{n})$ be a finite system of elements of $A$.

The set
\[
V(x_{1}, \ldots, x_{n}) := \{s(x_{1}), \ldots, s(x_{n}) | s \in S\}
\]

is called the joint numerical range of $(x_{1}, \ldots, x_{n})$.

THEOREM 3. Let $(\lambda_{1}, \ldots, \lambda_{n}) \in V(x_{1}, \ldots, x_{n})$ be such that
\[
i \in \{1, \ldots, n\} \iff |\lambda_{i}| = |\lambda_{i}|.
\]

Then
\[
(\lambda_{1}, \ldots, \lambda_{n}) \in \pi(x_{1}, \ldots, x_{n}).
\]

Assume that $(\lambda_{1}, \ldots, \lambda_{n}) \in \pi(x_{1}, \ldots, x_{n})$. Then there exists a real number $\epsilon > 0$ for which
\[
\sum_{i=1}^{n} |x_{i} - \lambda_{i}| \geq \epsilon.
\]

We have
\[
\sum_{i=1}^{n} |x_{i} - \lambda_{i}| \geq \sum_{i=1}^{n} |\lambda_{i}x_{i} + \lambda_{i}x_{i}| + \epsilon.
\]

Let us denote by $s_{n}$ a state such that
\[
s_{n}(x_{i}) = \lambda_{i}.
\]
We have
\[
\sum_{i=1}^{n} s(x_i) + \sum_{i=1}^{n} |x_i|^2 \geq \sum_{i=1}^{n} [s(x_i) + \bar{s}(x_i)] + \epsilon = 2 \sum_{i=1}^{n} |x_i|^2 + \epsilon,
\]
\[
\sum_{i=1}^{n} s(x_i) + \sum_{i=1}^{n} |x_i|^2 \geq \sum_{i=1}^{n} [s(x_i) + \bar{s}(x_i)] + \epsilon = \sum_{i=1}^{n} |x_i|^2 + \epsilon.
\]

Remark. From this theorem follows the following theorem of Winter-\-Hildebrandt \cite{6}, Ornold \cite{10}: If \(x \in B(H), \lambda \in \mathbb{W}(a) \) where \(\mathbb{W}(a) = \{x \lambda, \lambda \} \) \(\lambda \in H, ||\lambda|| = 1\) and \(|\lambda| = |\lambda|\), then \(\lambda\) is an approximate proper value.

For the proof it is sufficient to see that \(\mathbb{W}(a) \subseteq \mathbb{V}(a)\).

Let \(B\) be a subset of \(A\). A state \(\sigma\) on \(A\) is called left \(B\)-multiplicative if
\[
\sigma \in B, \; y \in A \mapsto \sigma(y a) = \sigma(y) \sigma(a).
\]

From \(\mathbb{B}_A\) (Theorem 1) it follows that a state \(\sigma\) is left \(B\)-multiplicative if and only if
\[
\sigma \in B \Rightarrow \sigma(a \sigma(x)) = \sigma(a) \sigma(x).
\]

We denote by \(S_B\) (resp. \(P_B\)) the set of all states \(\sigma\) (resp. pure states \(\sigma\)) which are left \(B\)-multiplicative.

**Theorem 4.** For any finite system \((x_1, \ldots, x_n)\) of \(A\) we have
\[
\pi(x_1, \ldots, x_n) = \{(s(x_1), \ldots, s(x_n)) \in S_{B_{[x_1, \ldots, x_n]}}\}
\]
\[
= \{(p(x_1), \ldots, p(x_n)) \in P_{B_{[x_1, \ldots, x_n]}}\}.
\]

Let \(s \in S_{B_{[x_1, \ldots, x_n]}}\) and let us denote, for any \(i \in \{1, \ldots, n\}, \lambda_i = s(x_i)\).

We have immediately
\[
i \in \{1, \ldots, n\} \mapsto s(x_i - \lambda_i)(x_i - \lambda_i) = 0.
\]

and therefore
\[
\sum_{i=1}^{n} s(x_i - \lambda_i)(x_i - \lambda_i) = 0.
\]

Hence, by Theorem 1, it follows that
\[
(\lambda_1, \ldots, \lambda_n) \in \pi(x_1, \ldots, x_n).
\]

Let now \((\lambda_1, \ldots, \lambda_n) \in \pi(x_1, \ldots, x_n)\). Since
\[
1 \neq \sum_{i=1}^{n} A(x_i - \lambda_i),
\]
the set \(L = \sum_{i=1}^{n} A(x_i - \lambda_i)\) is a proper left ideal of \(A\). We deduce that
\[
d(1, L) = 1.
\]

\[
d(1, L) = \inf_{\nu \in L} ||\nu - y||.
\]

Using the Hahn-Banach theorem, we may find a state \(\sigma\) such that \(\sigma_L = 0\).

We denote by \(S_\sigma\) the set of all states \(\sigma\) such that \(\sigma_L = 0\). By the preceding observation, \(S_\sigma\) is a non-empty set. Obviously, \(S_\sigma\) is a compact convex subset of \(S\). We are going to show that \(S_\sigma\) is a face of \(S\). Indeed, let \(\sigma, \sigma_1, \sigma_2 \in S, t \in (0, 1)\) such that
\[
\sigma = (1-t)\sigma_1 + t\sigma_2.
\]

Since
\[
i \in \{1, \ldots, n\} \mapsto (x_i - \lambda_i) \sigma(x_i - \lambda_i) \in L
\]
and \(\sigma_1, \sigma_2 \in S_\sigma\), we have, for any \(i \in \{1, \ldots, n\},
\]
\[
0 = (1-t) \varepsilon_{1}((x_i - \lambda_i)(x_i - \lambda_i)) + t\varepsilon_{2}((x_i - \lambda_i)\sigma(x_i - \lambda_i))
\]

and thus
\[
\varepsilon_{1}((x_i - \lambda_i)(x_i - \lambda_i)) = \varepsilon_{2}((x_i - \lambda_i)(x_i - \lambda_i)) = 0.
\]

From the Schwarz inequality
\[
|\langle x, y \rangle |^2 \leq \|x\| \|y\| y
\]

which is satisfied for any \(\sigma \in S\) and any \(x, y \in A\), we deduce that
\[
y \in A \mapsto \varepsilon_{1}((x_i - \lambda_i)) = \varepsilon_{2}((x_i - \lambda_i)) = 0
\]

and so \(\sigma_1, \sigma_2 \in S_\sigma\). It follows that \(S_\sigma\) contains an extremal point \(p_0\) of \(S\).

We have, for any \(i \in \{1, \ldots, n\},
\]
\[
p_0(x_i - \lambda_i) = 0,
\]

and thus
\[
p_0((x_i - \lambda_i)^*) = 0.
\]

Hence
\[
(\lambda_1, \ldots, \lambda_n) \in \pi(x_1, \ldots, x_n).
\]

**Corollary 5.** For any finite system \((x_1, \ldots, x_n)\) of elements of \(A\) the set \(\pi(x_1, \ldots, x_n)\) is a compact subset of \(G\).

**Corollary 6.** For any finite system \((x_1, \ldots, x_n)\) of pairwise commuting elements of \(A\) the set
\[
P_{[x_1, \ldots, x_n]} \neq \emptyset.
\]

**Corollary 7.** For any finite system \((x_1, \ldots, x_n)\) of \(A\) we have
\[
Sp(x_1, \ldots, x_n) = \pi(x_1, \ldots, x_n) \cup \pi^*(x_1^*, \ldots, x_n^*),
\]

\[s = \text{Studia Mathematica XIII}.\]
where $\text{Sp}(x_1, \ldots, x_n)$ is the joint spectrum of the system $(x_1, \ldots, x_n)$ and

$$\pi^*(y_1, \ldots, y_n) = \{(\lambda_1, \ldots, \lambda_n) \mid (\lambda_1, \ldots, \lambda_n) \in \pi(y_1, \ldots, y_n)\}.$$ 

Moreover, if $x_1, \ldots, x_n$ are hyponormal elements then

$$\text{Sp}(x_1, \ldots, x_n) = \pi^*(x_1^*, \ldots, x_n^*);$$

if $x_1, \ldots, x_n$ are normal elements then

$$\text{Sp}(x_1, \ldots, x_n) = \pi(x_1, \ldots, x_n).$$

**Theorem 8.** For any finite system $(x_1, \ldots, x_n)$ of hyponormal elements of $A$ if $s \in \text{Sp}(x_1, \ldots, x_n)$ then $s(\text{Car}(x_1, \ldots, x_n))$ is a character, where $\text{Car}(\{x_1, \ldots, x_n\})$ is the $C^*$-algebra generated by $1, x_1, x_2, \ldots, x_n$.

Let $s \in \text{Sp}(x_1, \ldots, x_n)$ and write

$$\lambda_i = s(x_i), \quad i \in \{1, \ldots, n\}.$$ 

We have

$$s(x_i^* x_i) = s(x_i^*) s(x_i) = s(\lambda_i^* (\lambda_i - \lambda_i^*)) = 0.$$ 

Since $x_1, \ldots, x_n$ are hyponormal, then $x_i - \lambda_i, x_n - \lambda_n$ are hyponormal and therefore, for any $i \in \{1, \ldots, n\}$, we have

$$0 \leq s((x_i - \lambda_i)(x_i - \lambda_i^*)) \leq s((x_i - \lambda_i^*)(x_i - \lambda_i)) = 0,$$

Hence

$$s \in \text{S}(x_1, \ldots, x_n).$$

Consequently for any $s \in \text{G}(\{x_1, \ldots, x_n\})$ and any element $y$ of $A$ we have

$$s(y) = s(y x_i) = s(x_i) s(y)$$

and thus the element

$$s([y, x_1, \ldots, x_n])$$

is a character.

**Corollary 9 (Bunce [4]).** For any finite system $(x_1, \ldots, x_n)$ of hyponormal elements of $A$ we have

$$\pi(x_1, \ldots, x_n) = \{(p(x_1), \ldots, p(x_n)) \mid p \in \text{Car}(\text{G}(\{x_1, \ldots, x_n\}))\}$$

where $\text{Car}(\text{G}(\{x_1, \ldots, x_n\}))$ is the set of all characters on $\text{G}(\{x_1, \ldots, x_n\})$.

**Theorem 10.** Let $(x_1, \ldots, x_n)$ be a finite system of $A$ and let $s \in \text{S}(x_1, \ldots, x_n)$ be such that

$$s(x_i) \in \partial V(x_i) \quad \text{for} \quad i \in \{1, \ldots, n\}.$$ 

Then the restriction of $s$ to $\text{G}(\{x_1, \ldots, x_n\})$ is a character.

Write $\lambda_i = s(x_i), \quad i \in \{1, \ldots, n\}$. Since $V(x_i)$ is a compact convex subset of $G$ and $\lambda_i \in \partial V(x_i)$, there exists a complex number $a_i, a_i \neq 0$ such that

$$s \in S = \text{Re} \lambda_i \leq \text{Re} a_i s(x_i).$$

From this fact we deduce that

$$\text{Re}(a_i \lambda_i - a_i \lambda_i^*) \geq 0.$$ 

For the proof of the fact that $s(\text{Car}(x_1, \ldots, x_n))$ is a character it is sufficient to show that

$$s_i(x_i x_i^*) = s_i(x_i) s_i(x_i^*), \quad i \in \{1, \ldots, n\},$$

i.e. to show that

$$s_i((a_i \lambda_i - a_i \lambda_i^*)(a_i \lambda_i - a_i \lambda_i^*)) = 0, \quad i \in \{1, \ldots, n\}.$$ 

This assertion follows from the preceding considerations and from the following lemma:

**Lemma.** Let $y$ be an element of $A$ such that

$$\text{Re} y \geq 0$$

and let $s \in S$ be such that $s(y^* y) = 0, s(y) = 0$. Then

$$s(y y^*) = 0.$$ 

We have

$$y^* + y = 2 \text{Re} y,$$

$$yy^* = 2y \text{Re} y - y^*.$$ 

Since

$$|s(y^* y)^2 \leq s(y^* y) s(y^* y),$$

we have

$$s(|y|^2) = s((|y|^2) \text{Re} (|y|^2)) \leq ||y|| s(|y|).$$

We deduce that

$$s(y y^*) = 0.$$ 

**Corollary 11.** Let $(x_1, \ldots, x_n)$ be a finite system of elements of $A$ and let $(\lambda_1, \ldots, \lambda_n) \in \text{Sp}(x_1, \ldots, x_n)$ be such that

$$\lambda_i \in \partial V(x_i), \quad i \in \{1, \ldots, n\}.$$
Then there exists a character \( p \) on \( C^*([x_1, \ldots, x_n]) \) such that

\[ p(x_i) = \lambda_i, \quad i \in \{1, \ldots, n\}. \]

The assertion follows from the preceding theorem by Corollary 7 and Theorem 4.

Remark. The preceding corollary contains the following theorem of Arveson [1]: If \( r \in B(H) \) and \( \lambda \in \partial W(\lambda) \cap \text{Sp}(\lambda) \) then there exists a character \( p \) on \( C^*([\lambda]) \) such that \( p(\lambda) = \lambda \). The proof follows from the fact that \( W(\lambda) = Y(\lambda) \) ([2], Theorem 3; [7], Theorem 11).

References


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Normally subregular systems in normed algebras

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Abstract. The main aim of this note is to give a negative answer to a question about ideals of normed algebras, raised by Arveson [3].

Let \( A \) be a commutative complex unital normed algebra. \( \{a_i\}_1^N \subset A \) is called a normally subregular system if there is a commutative algebra \( B = A \) containing elements \( \{b_i\}_1^N \) of norm at most 1 such that

\[ \sum_{i=1}^N a_i b_i = 1. \]

We show that for \( N > 2 \) normal subregularity is not characterized by the condition

\[ \inf \left\{ \sum_{i=1}^N \|a_i b_i\| : x \in A, \|x\| = 1 \right\} > 1. \]

The algebras considered in this paper are commutative complex unital normed algebras though our results also hold for real ones. If \( A \) is a subalgebra of \( B(A \subset B) \), we call \( B \) an isometric extension, shortly an extension, of \( A \). An element \( a \in A \) is a topological divisor of zero if

\[ \inf \{\|a x\| : x \in A, \|x\| = 1\} = 0. \]

A well-known result of Shilov [5] states that \( a \in A \) has an inverse of norm at most 1 in some extension of \( A \) if and only if

\[ \inf \{\|a x\| : x \in A, \|x\| = 1\} > 1. \]

The problem of adjoining inverses of a set of elements was investigated by Arveson in [1] and [2]. In [4] I proved that one can always adjoin the inverses of countably many elements which are not topological divisors of zero but this is not necessarily true for uncountably many elements.

A set \( \{a_1, \ldots, a_N\} \subset A \) is called a regular system if there exist \( b_1, \ldots, b_N \in A \) such that

\[ \sum_{i=1}^N a_i b_i = 1. \]

If the elements \( b_i \) can be chosen to have norm at most 1 then \( \{a_1, \ldots, a_N\} \) is normally regular. Finally, \( \{a_1, \ldots, a_N\} \subset A \) is subregular and normally subregular, respectively, if \( A \) has an isometric extension \( B \) for which the appropriate \( b_i \) can be chosen. These concepts were introduced by Arveson [3], mainly in order to pose the following problem. Is normal subregularity characterized by the (obviously necessary) condition

\[ \inf \left\{ \sum_{i=1}^N \|a_i b_i\| : x \in A, \|x\| = 1 \right\} > 1? \]
