STUDIA MATHEMATICA, T. XLIX. (1974)

The joint approximate spectrum of a finite system of elements of a C^*-algebra

by

GH. MOCANU (Bucharest)

Abstract. The aim of this paper is to study the joint approximate spectrum of a finite system of elements of a C^*-algebra A. The main result states that the joint approximate spectrum of a finite system $(x_1, \ldots, x_n)$ of elements of A is exactly the set

$$\{(s(x_1), \ldots, s(x_n)) \mid s \in S_{\{x_1,\ldots,x_n\}}\} = \{(p(x_1), \ldots, p(x_n)) \mid p \in P_{\{x_1,\ldots,x_n\}}\}$$

where $S_{\{x_1,\ldots,x_n\}}$ (resp. $P_{\{x_1,\ldots,x_n\}}$) denotes the set of all states s (resp. pure states p) whith satisfy the following relation

$$s(yx_i) = s(y)s(x_i) \quad (\text{resp. } p(yx_i) = p(y)p(x_i))$$

or any $i \in \{1, \ldots, n\}$ and any $y \in A$.

Throughout this paper A is a C^*-algebra with identity, $S = S(A)$ s the set of all states on A and $P = P(A)$ is the set of all pure states on A. The aim of this paper is to study the joint approximate spectrum of a finite system of elements of A. A state $s \in S$ is called left multiplicative with respect to a subset B of A if

$$s(yx) = s(y)s(x)$$

for any $y \in A$ and any $x \in B$. The set of all left multiplicative states (resp. pure states) with respect to B will be denoted by S_B (resp. P_B).

The main result of this paper (Theorem 4) states that the joint approximate spectrum of a finite system $(x_1, \ldots, x_n)$ of A is exactly the set

$$\{(s(x_1), \ldots, s(x_n)) \mid s \in S_{\{x_1,\ldots,x_n\}}\} = \{(p(x_1), \ldots, p(x_n)) \mid p \in P_{\{x_1,\ldots,x_n\}}\}.$$

Using this theorem we obtain also some results concerning the joint approximate spectrum of a finite system $(x_1, \ldots, x_n)$ of elements of A. These results extend the similar results of Bunce ([4], Proposition 2 and Proposition 3) or Arwerson ([1], Theorem 3.1.2).

THEOREM 1. *Let $(x_1, \ldots, x_n)$ be a finite system of elements of A and let $(\lambda_1, \ldots, \lambda_n)$ be a finite system of complex numbers. Then the following assertions are equivalent:*

1) $\sum_{i=1}^{n} A(x_i-\lambda_i) \neq A$.

2) *For any real number* $\varepsilon > 0$ *there exists a state* s *such that*

$$\sum_{i=1}^{n} s\big((x_i-\lambda_i)^*(x_i-\lambda_i)\big) < \varepsilon.$$

3) *For any real number* $\varepsilon > 0$ *there exists a pure state* p *such that*

$$\sum_{i=1}^{n} p\big((x_i-\lambda_i)^*(x_i-\lambda_i)\big) < \varepsilon.$$

4) *There is no real number* $\varepsilon > 0$ *such that*

$$\sum_{i=1}^{n} (x_i-\lambda_i)^*(x_i-\lambda_i) \geqslant \varepsilon.$$

5) *There exists a sequence* $(u_k)_{k\in N}$ *of elements of* A *such that* $(k \in N \Rightarrow \|u_k\| = 1)$, *and such that*

$$\lim_{k\to\infty} \Big(\sum_{i=1}^{n} \|(x_i-\lambda_i)u_k\|\Big) = 0.$$

1) $\Rightarrow$ 2). Suppose that there exists a real number $\varepsilon > 0$ such that

$$s \in S \Rightarrow \sum_{i=1}^{n} s\big((x_i-\lambda_i)^*(x_i-\lambda_i)\big) \geqslant \varepsilon$$

and therefore

$$\sum_{i=1}^{n} (x_i-\lambda_i)^*(x_i-\lambda_i) \geqslant \varepsilon.$$

Using a standard argument it follows that there exists an element $u \in A$ for which

$$u\Big[\sum_{i=1}^{n} (x_i-\lambda_i)^*(x_i-\lambda_i)\Big] = 1$$

and therefore

$$\sum_{i=1}^{n} A(x_i-\lambda_i) = A.$$

The relations 2) $\Leftrightarrow$ 3) $\Leftrightarrow$ 4) follows from the fact that an element $a \in A$ is positive if and only if for any $s \in S$ (resp. $p \in P$) we have $s(a) \geqslant 0$ (resp. $p(a) \geqslant 0$).

4) $\Rightarrow$ 5). Assume that there exists $\varepsilon > 0$ such that

$$u \in A,\ \|u\| = 1 \Rightarrow \sum_{i=1}^{n} \|(x_i-\lambda_i)u\| \geqslant \varepsilon.$$

We have

$$i \in \{1, \dots, n\} \Rightarrow \|(x_i-\lambda_i)u\|^2 = \|u^*(x_i-\lambda_i)^*(x_i-\lambda_i)u\|.$$

Hence, for any $u \in A$, $\|u\| = 1$, and any $i \in \{1, \dots, n\}$, we may find $s_{i,u} \in S$ such that

$$s_{i,u}\big(u^*(x_i-\lambda_i)^*(x_i-\lambda_i)u\big) = \|(x_i-\lambda_i)u\|^2.$$

Since, for any $u \in A$, $\|u\| = 1$,

$$\sum_{i=1}^{n} \|(x_i-\lambda_i)u\|^2 \geqslant \frac{1}{n}\Big(\sum_{i=1}^{n} \|(x_i-\lambda_i)u\|\Big)^2 \geqslant \frac{\varepsilon^2}{n},$$

we may find $i_u \in \{1, \dots, n\}$ such that

$$\|(x_{i_u}-\lambda_{i_u})u\|^2 \geqslant \frac{\varepsilon^2}{n^2}.$$

Hence, for any $u \in A$, $\|u\| = 1$, we have

$$\sup_{s\in S}\Big(s\Big(u^*\Big[\sum_{i=1}^{n}(x_i-\lambda_i)^*(x_i-\lambda_i)\Big]u\Big)\Big) = \sup_{s\in S}\sum_{i=1}^{n} s\big(u^*(x_i-\lambda_i)^*(x_i-\lambda_i)u\big)$$

$$\geqslant s_{i_u,u}\big(u^*(x_{i_u}-\lambda_{i_u})^*(x_{i_u}-\lambda_{i_u})u\big) = \|(x_{i_u}-\lambda_{i_u})u\|^2 \geqslant \frac{\varepsilon^2}{n^2},$$

$$\Big\|u^*\Big[\sum_{i=1}^{n}(x_i-\lambda_i)^*(x_i-\lambda_i)\Big]u\Big\| \geqslant \frac{\varepsilon^2}{n^2}.$$

Since $\sum_{i=1}^{n}(x_i-\lambda_i)^*(x_i-\lambda_i)$ is a positive element of A, the preceding relation implies the inequality

$$\sum_{i=1}^{n} (x_i-\lambda_i)^*(x_i-\lambda_i) \geqslant \frac{\varepsilon^2}{n^2}.$$

5) $\Rightarrow$ 1). Assume that

$$\sum_{i=1}^{n} A(x_i-\lambda_i) = A$$

and let $u_1, \dots, u_n \in A$ be such that

$$\sum_{i=1}^{n} u_i(x_i-\lambda_i) = 1.$$

We have, for any $u \in A$, $\|u\| = 1$,

$$1 = \Big\| \sum_{i=1}^{n} u_i(x_i - \lambda_i)u \Big\| \leqslant \sum_{i=1}^{n} \|u_i\| \, \|(x_i-\lambda_i)u\|$$
$$\leqslant a\|(x_i-\lambda_i)u\|, \quad \frac{1}{a} \leqslant \sum_{i=1}^{n} \|(x_i-\lambda_i)u\|$$

where

$$a = \sup(\|u_1\|, \ldots, \|u_n\|).$$

Remarks. a) Let S_0 be a subset of S such that for any $x \in A$ we have

$$x \geqslant 0 \Leftrightarrow \big(s \in S_0 \Rightarrow s(x) \geqslant 0\big).$$

Then each of the assertions 1) – 5) is equivalent to the following one:

2′) *For any real number $\varepsilon > 0$ there exists $s \in S_0$ such that*

$$\sum_{i=1}^{n} s[(x_i-\lambda_i)^*(x_i-\lambda_i)] < \varepsilon.$$

In the case when A is equal to $B(H)$, the algebra of all bounded linear operators on a complex Hilbert space H, we may take instead of S_0 the set of states p_h on A defined by

$$p_h(x) = \langle xh, h\rangle$$

where $h \in H$, $\|h\| = 1$. In this particular case the assertion 2′) is exactly the following:

2″) *For any real number $\varepsilon > 0$ there exists $h \in H$, $\|h\| = 1$ such that*

$$\sum_{i=1}^{n} \|(x_i-\lambda_i)h\|^2 < \varepsilon.$$

b) In the case when A is a W^*-algebra (in particular, if $A = B(H)$) each of the assertions 1) – 5) is equivalent to the following:

5′) *There exists a sequence $(u_k)_{k\in N}$ of elements of A which are projections such that*

$$\lim_{k\to\infty}\Big(\sum_{i=1}^{n} \|(x_i-\lambda_i)u_k\|\Big) = 0.$$

The proof follows from the fact that if x is a positive element of A and ε is a real number > 0 then the relation

$$\|uxu\| \geqslant \varepsilon \quad \text{for any projection } u \in A$$

implies the relation $x \geqslant \varepsilon$.

c) The preceding theorem contains Theorems 1,4 from [9]. The relations 1) $\Leftrightarrow$ 5) $\Leftrightarrow$ 5′) for the case $A = B(H)$ is the solution of a problem stated in [9].

Definition. Let $(x_1, \ldots, x_n)$ be a finite system of elements of A. A finite system $(\lambda_1, \ldots, \lambda_n)$ of complex numbers is called a *joint approximate proper value* of $(x_1, \ldots, x_n)$ if one of the assertions 1) – 5) of Theorem 1 holds. The set of all joint approximate proper values of a system $(x_1, \ldots, x_n)$ is called the *joint approximate spectrum* and is denoted by

$$\pi_A(x_1, \ldots, x_n) = \pi(x_1, \ldots, x_n).$$

Theorem 2 (Bunce [4]). *Let $(x_1, \ldots, x_n)$ be a finite system of elements of A such that $x_ix_j = x_jx_i$ for any $i, j \in \{1, \ldots, n\}$ and let $(\lambda_1, \ldots, \lambda_n)$ be an element of $\pi(x_1, \ldots, x_n)$. Then there exists $\lambda_{n+1} \in \pi(x_{n+1})$ such that*

$$(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) \in \pi(x_1, \ldots, x_n, x_{n+1}).$$

From this theorem follows the fact that for any finite system $(x_1, \ldots, x_n)$ of elements of A with $x_ix_j = x_jx_i$ for $i, j \in \{1, \ldots, n\}$, the joint approximate spectrum of $(x_1, \ldots, x_n)$ is non-empty.

We recall the following:

Definition. Let $(x_1, \ldots, x_n)$ be a finite system of elements of A. The set

$$V(x_1, \ldots, x_n) =: \{(s(x_1), \ldots, s(x_n)) \mid s \in S\}$$

is called the *joint numerical range* of $(x_1, \ldots, x_n)$.

Theorem 3. *Let $(\lambda_1, \ldots, \lambda_n) \in V(x_1, \ldots, x_n)$ be such that*

$$i \in \{1, \ldots, n\} \Rightarrow |\lambda_i| = \|x_i\|.$$

Then

$$(\lambda_1, \ldots, \lambda_n) \in \pi(x_1, \ldots, x_n).$$

Assume that $(\lambda_1, \ldots, \lambda_n) \notin \pi(x_1, \ldots, x_n)$. Then there exists a real number $\varepsilon > 0$ for which

$$\sum_{i=1}^{n} (x_i-\lambda_i)^*(x_i-\lambda_i) \geqslant \varepsilon.$$

We have

$$\sum_{i=1}^{n} x_i^*x_i + \sum_{i=1}^{n} |\lambda_i|^2 \geqslant \sum_{i=1}^{n} (\lambda_i x_i^* + \bar{\lambda}_i x_i) + \varepsilon.$$

Let us denote by s a state such that

$$s(x_i) = \lambda_i.$$

We have

$$\sum_{i=1}^{n} s(x_i^* x_i) + \sum_{i=1}^{n} |\lambda_i|^2 \geqslant \sum_{i=1}^{n} \left(\lambda_i s(x_i^*) + \bar{\lambda}_i s(x_i)\right) + \varepsilon = 2\sum_{i=1}^{n} |\lambda_i|^2 + \varepsilon,$$

$$\sum_{i=1}^{n} \|x_i\|^2 \geqslant \sum_{i=1}^{n} s(x_i^* x_i) \geqslant \sum_{i=1}^{n} |\lambda_i|^2 + \varepsilon = \sum_{i=1}^{n} \|x_i\|^2 + \varepsilon.$$

Remark. From this theorem follows the following theorem of Winter–Hildebrand [6], Orland [10]: If $x \in B(H)$, $\lambda \in \overline{W(x)}$ where $W(x) = \{\langle xh, h\rangle | h \in H, \|h\| = 1\}$ and $|\lambda| = \|x\|$, then λ is an approximate proper value. For the proof it is sufficient to see that $\overline{W(x)} \subset V(x)$.

Let B be a subset of A. A state s on A is called *left B-multiplicative* if

$$x \in B,\ y \in A \Rightarrow s(yx) = s(y)s(x).$$

From [8], (Theorem 1) it follows that a state s is left B-multiplicative if and only if

$$x \in B \Rightarrow s(x^*x) = s(x^*)s(x).$$

We denote by S_B (resp. P_B) the set of all states s (resp. pure states p) which are left B-multiplicative.

THEOREM 4. *For any finite system $(x_1, \ldots, x_n)$ of A we have*

$$\pi(x_1, \ldots, x_n) = \{(s(x_1), \ldots, s(x_n)) |\ s \in S_{\{x_1,\ldots,x_n\}}\}$$
$$= \{(p(x_1), \ldots, p(x_n)) |\ p \in P_{\{x_1,\ldots,x_n\}}\}.$$

Let $s \in S_{\{x_1,\ldots,x_n\}}$ and let us denote, for any $i \in \{1, \ldots, n\}$, $\lambda_i = s(x_i)$. We have immediately

$$i \in \{1, \ldots, n\} \Rightarrow s\left((x_i - \lambda_i)^*(x_i - \lambda_i)\right) = 0$$

and therefore

$$\sum_{i=1}^{n} s\left((x_i - \lambda_i)^*(x_i - \lambda_i)\right) = 0.$$

Hence, by Theorem 1, it follows that

$$(\lambda_1, \ldots, \lambda_n) \in \pi(x_1, \ldots, x_n).$$

Let now $(\lambda_1, \ldots, \lambda_n) \in \pi(x_1, \ldots, x_n)$. Since

$$1 \notin \sum_{i=1}^{n} A(x_i - \lambda_i),$$

the set $L = \sum_{i=1}^{n} A(x_i - \lambda_i)$ is a proper left ideal of A. We deduce that

$$d(1, L) = 1$$

where

$$d(1, L) = \inf_{y \in L} \|1 - y\|.$$

Using the Hahn–Banach theorem, we may find a state s such that $s_{|L} = 0$. Let us denote by S_0 the set of all states s such that $s_{|L} = 0$. By the preceding observation, S_0 is a non-empty set. Obviously, S_0 is a compact convex subset of S. We are going to show that S_0 is a face of S. Indeed, let $s_0 \in S_0$ and $s_1, s_2 \in S$, $t \in (0, 1)$ such that

$$s_0 = (1-t)s_1 + ts_2.$$

Since

$$i \in \{1, \ldots, n\} \Rightarrow (x_i - \lambda_i)^*(x_i - \lambda_i) \in L$$

and $s_0 \in S_0$, we have, for any $i \in \{1, \ldots, n\}$,

$$0 = (1-t)s_1\left((x_i - \lambda_i)^*(x_i - \lambda_i)\right) + ts_2\left((x_i - \lambda_i)^*(x_i - \lambda_i)\right)$$

and thus

$$s_1\left((x_i - \lambda_i)^*(x_i - \lambda_i)\right) = s_2\left((x_i - \lambda_i)^*(x_i - \lambda_i)\right) = 0.$$

From the Schwarz inequality

$$|s(zy)|^2 \leqslant s(zz^*)s(y^*y)$$

which is satisfied for any $s \in S$ and any $z, y \in A$, we deduce that

$$y \in A \Rightarrow s_1\left(y(x_i - \lambda_i)\right) = s_2\left(y(x_i - \lambda_i)\right) = 0$$

and so $s_1, s_2 \in S_0$. It follows that S_0 contains an extremal point p_0 of S.

We have, for any $i \in \{1, \ldots, n\}$,

$$p_0(x_i - \lambda_i) = 0,$$
$$p_0\left((x_i - \lambda_i)^*(x_i - \lambda_i)\right) = 0$$

and thus

$$p_0(x_i) = \lambda_i, \quad p_0(x_i^* x_i) = p_0(x_i^*)p_0(x_i).$$

Hence

$$(\lambda_1, \ldots, \lambda_n) \in \{(p(x_1), \ldots, p(x_n)) |\ p \in P_{\{x_1,\ldots,x_n\}}\}.$$

COROLLARY 5. *For any finite system $(x_1, \ldots, x_n)$ of elements of A the set $\pi(x_1, \ldots, x_n)$ is a compact subset of C^n.*

COROLLARY 6. *For any finite system $(x_1, \ldots, x_n)$ of pairwise commuting elements of A the set*

$$P_{\{x_1,\ldots,x_n\}} \neq \emptyset.$$

COROLLARY 7. *For any finite system $(x_1, \ldots, x_n)$ of A we have*

$$\mathrm{Sp}(x_1, \ldots, x_n) = \pi(x_1, \ldots, x_n) \cup \pi^*(x_1^*, \ldots, x_n^*),$$

where $\mathrm{Sp}(x_1, \ldots, x_n)$ *is the joint spectrum of the system* $(x_1, \ldots, x_n)$ *and*

$$\pi^*(y_1, \ldots, y_n) = \{(\bar\lambda_1, \ldots, \bar\lambda_n) \mid (\lambda_1, \ldots, \lambda_n) \in \pi(y_1, \ldots, y_n)\}.$$

Moreover, if $x_1, \ldots, x_n$ *are hyponormal elements then*

$$\mathrm{Sp}(x_1, \ldots, x_n) = \pi^*(x_1^*, \ldots, x_n^*);$$

if $x_1, \ldots, x_n$ *are normal elements then*

$$\mathrm{Sp}(x_1, \ldots, x_n) = \pi(x_1, \ldots, x_n).$$

THEOREM 8. *For any finite system* $(x_1, \ldots, x_n)$ *of hyponormal elements of* A *if* $s \in S_{\{x_1,\ldots,x_n\}}$ *then* $s|_{C^*(\{x_1,\ldots,x_n\})}$ *is a character, where* $C^*(\{x_1, \ldots, x_n\})$ *is the* C^**-algebra generated by* $1, x_1, \ldots, x_n$.

Let $s \in S_{\{x_1,\ldots,x_n\}}$ and write

$$\lambda_i = s(x_i), \quad i \in \{1, \ldots, n\}.$$

We have

$$s(x_i^* x_i) = s(x_i^*)s(x_i), \quad s\big((x_i - \lambda_i)^*(x_i - \lambda_i)\big) = 0.$$

Since $x_1, \ldots, x_n$ are hyponormal, then $x_1 - \lambda_1, \ldots, x_n - \lambda_n$ are hyponormal and therefore, for any $i \in \{1, \ldots, n\}$, we have

$$0 \leqslant s\big((x_i - \lambda_i)(x_i - \lambda_i)^*\big) \leqslant s\big((x_i - \lambda_i)^*(x_i - \lambda_i)\big) = 0,$$
$$s(x_i x_i^*) = s(x_i)s(x_i^*).$$

Hence

$$s \in S_{\{x_1^*,\ldots,x_n^*\}}.$$

Consequently for any $z \in C^*(\{x_1, \ldots, x_n\})$ and any element y of A we have

$$s(zy) = s(yz) = s(z)s(y)$$

and thus the element

$$s|_{C^*(\{x_1,\ldots,x_n\})}$$

is a character.

COROLLARY 9 (Bunce [4]). *For any finite system* $(x_1, \ldots, x_n)$ *of hyponormal elements of* A *we have*

$$\pi(x_1, \ldots, x_n) = \{\big(p(x_1), \ldots, p(x_n)\big) \mid \ p \in \mathrm{Car}\big(C^*(\{x_1, \ldots, x_n\})\big)\}$$

where $\mathrm{Car}\big(C^*(\{x_1, \ldots, x_n\})\big)$ *is the set of all characters on* $C^*(\{x_1, \ldots, x_n\})$.

THEOREM 10. *Let* $(x_1, \ldots, x_n)$ *be a finite system of* A *and let* $s_0 \in S_{\{x_1,\ldots,x_n\}}$ *be such that*

$$s_0(x_i) \in \partial V(x_i) \quad \text{for } i \in \{1, \ldots, n\}.$$

Then the restriction of s_0 *to* $C^*(\{x_1, \ldots, x_n\})$ *is a character.*

Write $\lambda_i = s_0(x_i)$, $i \in \{1, \ldots, n\}$. Since $V(x_i)$ is a compact convex subset of C and $\lambda_i \in \partial V(x_i)$, there exists a complex number a_i, $a_i \neq 0$ such that

$$s \in S \Rightarrow \mathrm{Re}\, a_i \lambda_i \leqslant \mathrm{Re}\, a_i s(x_i).$$

From this fact we deduce that

$$\mathrm{Re}(a_i x_i - a_i \lambda_i) \geqslant 0.$$

For the proof of the fact that $s_0|_{C^*(\{x_1,\ldots,x_n\})}$ is a character it is sufficient to show that

$$s_0(x_i x_i^*) = s_0(x_i)s_0(x_i^*), \quad i \in \{1, \ldots, n\},$$

i.e. to show that

$$s_0\big((a_i x_i - a_i\lambda_i)(a_i x_i - a_i \lambda_i)^*\big) = 0, \quad i \in \{1, \ldots, n\}.$$

This assertion follows from the preceding considerations and from the following lemma:

Let y *be an element of* A *such that*

$$\mathrm{Re}\, y \geqslant 0$$

and let $s \in S$ *be such that* $s(y^*y) = 0$, $s(y) = 0$. *Then*

$$s(yy^*) = 0.$$

We have

$$y^* + y = 2\,\mathrm{Re}\, y,$$
$$yy^* = 2y\,\mathrm{Re}\, y - y^2,$$
$$s(yy^*) \leqslant 2\,|s(y\,\mathrm{Re}\, y)| + |s(y^2)|.$$

Since

$$|s(y^2)|^2 \leqslant s(yy^*)s(y^*y) = 0$$

and

$$|s(y\,\mathrm{Re}\, y)|^2 \leqslant s(yy^*)s\big((\mathrm{Re}\, y)^2\big),$$

we have

$$s\big((\mathrm{Re}\, y)^2\big) = s\big((\mathrm{Re}\, y)^{\frac12}\mathrm{Re}\, y(\mathrm{Re}\, y)^{\frac12}\big) \leqslant \|\mathrm{Re}\, y\|\, s(\mathrm{Re}\, y)$$

and

$$s(\mathrm{Re}\, y) = \mathrm{Re}\, s(y) = 0.$$

We deduce that

$$s(yy^*) = 0.$$

COROLLARY 11. *Let* $(x_1, \ldots, x_n)$ *be a finite system of elements of* A *and let* $(\lambda_1, \ldots, \lambda_n) \in \mathrm{Sp}(x_1, \ldots, x_n)$ *be such that*

$$\lambda_i \in \partial V(x_i), \quad i \in \{1, \ldots, n\}.$$

Then there exists a character p on $C^*(\{x_1, \ldots, x_n\})$ such that

$$p(x_i) = \lambda_i, \quad i \in \{1, \ldots, n\}.$$

The assertion follows from the preceding theorem by Corollary 7 and Theorem 4.

Remark. The preceding corollary contains the following theorem of Arwerson [1]: If $x \in B(H)$ and $\lambda \in \partial W(x) \cap \mathrm{Sp}(x)$ then there exists a character p on $C^*(\{x\})$ such that $p(x) = \lambda$. The proof follows from the fact that $\overline{W(x)} = V(x)$ ([2], Theorem 3; [7], Theorem 11).

References

[1] W. B. Arwerson, *Subalgebras of C^*-algebras*, Acta Math. 123 (1969), pp. 141–224.

[2] S. K. Berberian and G. H. Orland, *Closure of the numerical range of an operator*, Proc. Amer. Math. Soc. 18 (1967), pp. 499–503.

[3] J. Bunce, *Characters on single generated C^*-algebras*, Proc. Amer. Math. Soc. 25 (1970), pp. 297–303.

[4] — *The joint spectrum of commuting nonnormal operators*, Proc. Amer. Math. Soc. 29 (1971), pp. 499–505.

[5] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Paris 1964.

[6] S. Hilderbrandt, *Über den numeriche Wertbereich eines Operators*, Math. Ann. 163 (1966), pp. 230–247.

[7] G. Lumer, *Semi-inner-product spaces*, Trans. Amer. Math. Soc. 100 (1961), pp. 29–43.

[8] Gh. Mocanu, *Les fonctionnelles relativement multiplicatives sur des algèbres de Banach*, Rev. Roum. Math. Pures et Appl. 3 (1971), pp. 379–381.

[9] R. Nakamoto and M. Nakamura, *A remark on the approximate spectra of operators*, Proc. Japan. Acad. 48 (1972), pp. 103–107.

[10] G. Orland, *On a class of operators*, Proc. Amer. Math. Soc. 15 (1964), pp. 75–79.

Received October 30, 1972 (610)

STUDIA MATHEMATICA, T. XLIX. (1974)

Normally subregular systems in normed algebras

by

BÉLA BOLLOBÁS (Cambridge, England)

Abstract. The main aim of this note is to give a negative answer to a question about ideals of normed algebras, raised by Arens [3].

Let A be a commutative complex unital normed algebra. $\{a_i\}_1^N \subset A$ is called a *normally subregular system* if there is a commutative algebra $B \supset A$ containing elements $\{b_i\}_1^N$ of norm at most 1 such that $\sum_1^N a_i b_i = 1$. We show that for $N \geqslant 2$ normal subregularity is *not* characterized by the condition

$$\inf\Big\{\sum_1^N \|a_i x\|:\ x \in A,\ \|x\| = 1\Big\} \geqslant 1.$$

The algebras considered in this paper are commutative complex unital normed algebras though our results also hold for real ones. If A is a subalgebra of $B (A \subset B)$, we call B an *isometric extension*, shortly *extension*, of A. An element $a \in A$ is a *topological divisor of zero* if $\inf\{\|ax\|: x \in A,\ \|x\| = 1\} = 0$. A well-known result of Shilov [5] states that $a \in A$ has an inverse of norm at most 1 in some extension of A if and only if $\inf\{\|ax\|: x \in A,\ \|x\| = 1\} \geqslant 1$. The problem of adjoining inverses of a set of elements was investigated by Arens in [1] and [2]. In [4] I proved that one can always adjoin the inverses of countably many elements which are not topological divisors of zero but this is not necessarily true for uncountably many elements.

A set $\{a_1, \ldots, a_N\} \subset A$ is called a *regular system* if there exist $b_1, \ldots, b_N \in A$ such that $\sum_1^N a_i b_i = 1$. If the elements b_i can be chosen to have norm at most 1 then $\{a_1, \ldots, a_N\}$ is *normally regular*. Finally, $\{a_1, \ldots, a_N\} \subset A$ is *subregular* and *normally subregular*, respectively, if A has an isometric extension B for which the appropriate b_i's can be chosen. These concepts were introduced by Arens [3], mainly in order to pose the following problem. Is normal subregularity characterized by the (obviously necessary) condition

$$\inf \sum_1^N \|a_i x\|:\ x \in A,\ \|x\| = 1\} \geqslant 1? \tag{1}$$