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Interpolation by cones

by

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Abstract. A dual formulation of the assertion that a closed cone in a Banach space of functions interpolates as well as the whole space is given, and applied to some results of R. E. Edwards, E. Hewitt and K. Ross on Fatou–Zygmund sets.

1. Let G be a compact abelian group and E a symmetric subset of the discrete dual Γ. Evidently $L_1^R(G)\hat{}$, the space of Fourier transforms of the real integrable functions on G, consists of hermitian symmetric C_0 functions on Γ, and thus the set of restrictions $L_1^R(G)\hat{}\,|E \subset C_{0h}(E)$, the hermitian symmetric C_0 functions on E. Recently R. E. Edwards, E. Hewitt and K. Ross [3] considered those sets E (which they call Fatou–Zygmund sets or FZ(G) sets) with the property that all trigonometric series constructed from E with (certain) partial sums bounded from below are absolutely convergent; they showed in particular that when $0 \notin E$ these sets are precisely those for which $L_1^{R+}(G)\hat{}\,|E = C_{0h}(E)$ (where $L_1^{R+}(G)$ is the non-negative cone in $L_1^R(G)$). Thus the Fatou–Zygmund sets are special Sidon sets: those for which the non-negative cone interpolates as well as its linear span, indeed as well as possible. (Whether all Sidon sets are Fatou–Zygmund is a still open question which was raised earlier in [6], p. 67, 5.3.)

The purpose of the present note is mainly to point out the dual formulation of the property that a closed cone interpolate as well as the entire space; more precisely, the dual formulation of the fact that a closed cone in a Banach space have the same image under a linear map into another Banach space as the full domain space. This provides a slightly different interpretation of some of the equivalences of [3], and allows application in several (related and unrelated) settings; it differs from the functional analytic tools developed in [3] which were designed to relate to sequences of partial sums. Our result is wholly unsuited to deal with such matters, but has the merit of showing the interpolation phenomena occurring are not at all peculiar to $L_1(G)$: for example, if μ is a prob-

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ability measure on G with dense support and with $\hat{\mu} \in C_0(\Gamma)$ then

$$(L_1^{R_+}(\mu)\cdot\mu)\hat{}\,|E = C_{0h}(E)$$

exactly when E is Fatou–Zygmund.

As indicated we shall frequently use the superscripts R or R_+ to denote the function space or space of measures with the obvious restriction: thus $M^{R_+}(G)$ denotes the space of non-negative measures on G. We shall also define the Fourier or Fourier–Stieltjes transform without the usual conjugation, simply for convenience.

2. The basic lemma. Our basic result is closely related to the fact ([2], p. 488) that a bounded linear map from one Banach space into another has closed range iff its adjoint has; in fact the result has a proof suggested by a proof of the known result shown to the author long ago by K. de Leeuw. Although it will be stated for real Banach spaces it of course applies to complex spaces by taking the adjoint space to be space of *real* linear continuous functionals.

THEOREM 2.1. *Let X and Y be real Banach spaces with unit balls B_X, B_Y, and let T be a continuous linear map of X onto Y. Suppose P is a closed cone in X. Then $TP = Y$ if and only if*

$$\exists c > 0 \ni \|T^*y^*\| \leqslant c\sup\langle B_X \cap P, T^*y^*\rangle, \quad y^* \in Y^*. \tag{2.1}$$

Finally, if (2.1) holds there is a constant k for which, for each y, there is an x in P with $Tx = y$ and $\|x\| \leqslant k\|y\|$.

Proof. Suppose that (2.1) holds. To see $TP = Y$ it suffices to show there is an n with $B_Y \subset (T(nB_X \cap P))^-$, exactly as in the proof of the open mapping theorem: indeed then for $y \in B_Y$ we have an $x_1 \in nB_X \cap P$ for which $\|y - Tx_1\| < \frac{1}{2}$ and thus an $x_2 \in \frac{1}{2}(nB_X \cap P) = \left(\frac{n}{2}B_X\right) \cap P$ for which $\|y - Tx_1 - Tx_2\| < \frac{1}{4}$, and continuing we obtain a sequence $\{x_j\}$ in P with $\|x_j\| \leqslant 2^{-j+1}n$, $y = \sum_1^\infty Tx_j$, so that $x = \sum_1^\infty x_j \in 2nB_X \cap P$, since P is a closed cone, and $y = Tx$ since T is continuous. (Note that once we know such an n exists we have $B_Y \subset T(2nB_X \cap P)$, which yields the final assertion.)

If no such n exists, then for each n we have a $y_n \in B_Y \setminus (T(nB_X \cap P))^-$, and so a $y_n^* \in Y^*$ for which $\langle y_n, y_n^*\rangle \geqslant 1 \geqslant \sup\langle T(nB_X \cap P)\, y_n^*\rangle$. That implies $\|y_n^*\| \geqslant 1$ since $y_n \in B_Y$, and also that $1 \geqslant n\sup\langle T(B_X \cap P), y_n^*\rangle$, again since P is a cone, so $\frac{1}{n} \geqslant \sup\langle B_X \cap P, T^*y_n^*\rangle \geqslant c^{-1}\|T^*y_n^*\|$. Thus $\|y_n^*\| \geqslant 1$ and $\|T^*y_n^*\| \to 0$ so that T^* cannot be topological. On the other hand it must be: T^* is 1-1 since $TX = Y$, and for the same reason T^* has closed range by the theorem on adjoints cited earlier; thus T^* is open by the open mapping theorem, and we have our contradiction.

Now suppose $TP = Y$. Then for some n, $(T(nB_X \cap P))^{-\circ}$ is non-void, where $^\circ$ denotes interior. Let $y_0 \in (T(nB_X \cap P))^{-\circ}$ and choose an x_0 in P for which $-y_0 = Tx_0$. Then

$$0 \in (T(nB_X \cap P))^{-\circ} + Tx_0 = (T(nB_X \cap P + x_0))^{-\circ} \subset [T((n + \|x_0\|)B_X \cap P)]^{-\circ}$$

so if εB_Y lies in the last set then for any y^*,

$$\varepsilon\|y^*\| = \sup\langle \varepsilon B_Y, y^*\rangle \leqslant \sup\langle T((n + \|x_0\|)B_X \cap P), y^*\rangle$$

and

$$\varepsilon\|T^*y^*\| \leqslant \|T^*\|(n + \|x_0\|)\sup\langle B_X \cap P, T^*y^*\rangle$$

as desired.

Clearly the sufficiency of (2.1) is the deeper half of 2.1. As we shall see later, in several of our applications involving translation invariant cones it can be replaced by a simple appeal to Hahn–Banach ([1]).

We can restate our result in a couple of ways.

COROLLARY 2.2. *Let N be a closed subspace and P a closed cone in a Banach space X. Then $P + N = X$ iff*

$$\exists c > 0 \ni \|x^*\| \leqslant c\sup\langle B_X \cap P, x^*\rangle, \; x^* \in N^\perp, \tag{2.2}$$

where $N^\perp$ is the subspace of X^* orthogonal to N.

Here Y is the quotient Banach space X/N and T the canonical map, so that T^*: $(X/N)^* = N^\perp \to X^*$ is inclusion, and (2.1) becomes (2.2).

COROLLARY 2.3. *Let X be a Banach space and let T: $X \to Y$ be linear, with a closed nullity N, and suppose P is a closed cone in X. Then $TP = TX$ iff (2.2) holds.*

For clearly $TP = TX$ is equivalent to $P + N = X$, so Corollary 2.3 follows from 2.2.

3. Interpolation by positive definite functions. Let G be a locally compact abelian group with dual Γ. For $f \in L_1^R(G)$, $\hat{f}(-\gamma) = \overline{\hat{f}(\gamma)}$, so $\hat{f}$ is hermitian symmetric, and if E is a closed symmetric subset of Γ, $L_1^R(G)\hat{}\,|E \subset C_{0h}(E)$, the hermitian symmetric C_0 functions on E.

We want to consider the question ([2]) of when

$$L_1^{R_+}(G)\hat{}\,|E = C_{0h}(E). \tag{3.1}$$

([1]) There are instances where even less is needed; for example $TP = Y$ if $P^\circ \cap \operatorname{kernel} T \neq \emptyset$ since if x_0 lies in this set and $x_0 + \delta B_X \subset P^\circ$ then for any $x \in X$ we have $x' = x_0 + \frac{\delta}{2\|x\|}x \in P^\circ$ and $Tx = T\left(\frac{2\|x\|}{\delta}x'\right) \in TP$. (This occurs in §3 if we replace $L_1(G)$ by $C(G)$.)

([2]) At least when Γ has no elements of order 2 we could equally well take E a set with $E \cap (-E) = \emptyset$ and consider interpolation of all C_0 functions on E, but the formulation of results in the present terms is more convenient.

Of course that implies $0 \notin E$ since $\hat{f}(0) = \int f\,dx \geqslant 0$ for $f \in L_1^{R+}(G)$, and also that $L_1^R(G)\hat{\ }|E = C_{0h}(E)$. The last is easily seen to be equivalent to $L_1(G)\hat{\ }|E = C_0(E)$, or that E is a Helson set. So restricting our attention to a symmetric Helson set E not containing 0 we can apply 2.1 to our map T: $f \to \hat{f}|E$ of $L_1^R(G)$ onto $C_{0h}(E)$ and the cone $P = L_1^{R+}(G)$, once we note that the dual of the real Banach space $C_{0h}(E)$ is the space of hermitian symmetric measures on E, i.e., those $\mu \in M(E)$ for which $\mu(-F) = \overline{\mu(F)}$, $F \subset E$, or alternatively those with $\hat{\mu}$ real on G. We shall denote the set of such measures by $M_h(E)$.

Here T^* is precisely the map $\mu \to \hat{\mu}$ from $M_h(E)$ into $L_\infty^R(G)$ since $\int_\Gamma \hat{f}\,d\mu = \int_G f\hat{\mu}\,dx$, and (2.1) thus asserts we have a constant $c > 0$ with

$$\|\hat{\mu}\|_\infty \leqslant c\sup\left\{\int f\hat{\mu}\,dx\colon f \geqslant 0 \text{ in } L_1(G),\ \int f\,dx \leqslant 1\right\},\qquad \mu \in M_h(E),$$

or

$$-\inf\hat{\mu}(G) \leqslant c\sup\hat{\mu}(G),\qquad \mu \in M_h(E). \tag{3.2}$$

Since we can freely translate $\hat{\mu}$ about (i.e., multiply μ by a character of Γ), (3.2) is equivalent to the non-existence of a sequence $\{\mu_n\}$ in $M_h(E)$ with

$$\sup\hat{\mu}_n(G) \leqslant 2^{-n},\quad -1 = \hat{\mu}_n(0),$$

or

$$\inf\hat{\mu}_n(G) \geqslant -2^{-n},\qquad 1 = \hat{\mu}_n(0). \tag{3.3}$$

and for the same reason we can take $|\hat{\mu}_n| \leqslant 2$.

Since E Helson implies $\|\mu\| \leqslant k\|\hat{\mu}\|_\infty$ for $\mu \in M(E)$, we have a w^* cluster point μ of $\{\mu_n\}$ in $M(E)$, and, provided E is compact, the w^* topology on $M(E)$ coincides with the topology of pointwise convergence of Fourier–Stieltjes transforms, so that $\hat{\mu} \geqslant 0$, $\hat{\mu}(0) = 1$, and $\mu \in M_h(E)$. Now if we choose $\varphi \in C(\Gamma)$ non-negative, non-zero, and with small support very close to 0 then $\Psi = \mu^*\varphi^*\varphi^*$ will be an integrable element of $C(\Gamma)$ which vanishes near 0 (since the support of μ lies in the compact set E and $0 \notin E$), while $\hat{\Psi}(0) = \hat{\mu}(0)|\hat{\varphi}(0)|^2 = \hat{\mu}(0)\cdot|\int\varphi\,d\gamma|^2 \neq 0$ shows $\Psi \neq 0$ and $\hat{\Psi} = \hat{\mu}|\hat{\varphi}|^2 \geqslant 0$ shows Ψ is positive definite. But then $|\Psi(x)| \leqslant \Psi(0) = 0$, $x \in G$, a contradiction showing (3.3) fails, and thus (3.2) and (3.1) hold *if E is compact.*

When E is non-compact we can only apply this argument by considering $\{\mu_n\}$ as a sequence of measures on Γ^a, the Bohr compactification of Γ; then we have a cluster point μ in $M_h(E^-)$, where E^- is the closure of E in Γ^a, with $\hat{\mu} \geqslant 0$, $\hat{\mu}(0) = 1$, and can obtain the same contradiction only under some additional hypothesis like $0 \notin E^-$, for example. In the particular case of $G = T^1$, $\Gamma = \boldsymbol{Z}$, E would be a Sidon set, and it would suffice to know $E \cap n\boldsymbol{Z} = \emptyset$ for some n: for then if m is Haar measure on $(n\boldsymbol{Z})^\perp$, $\hat{m}$ is an almost periodic function 0 on E and 1 on $n\boldsymbol{Z}$. (For the general case of G compact, the analogous hypothesis is that E misses a subgroup of finite index in Γ.)

Again in our particular case of a compact G and E a Sidon set, we can take the elements of $M_h(E)$ as functions, rather than measures, and we can rephrase (3.3) in an interesting fashion. Indeed, if we set $p_n(\gamma) = (1+2^{-n})^{-1}(2^{-n}\delta_0\{\gamma\}+\mu_n\{\gamma\})$ then p_n is an integrable positive definite function (since $\hat{p}_n = (1+2^{-n})^{-1}(2^{-n}+\hat{\mu}_n) \geqslant 0$) supported by $E_0 = \{0\}\cup E$, and, in terms of the p_n the non-existence of a sequence $\{\mu_n\}$ satisfying (3.3) becomes the non-existence of a sequence of integrable positive definite functions supported by E_0 satisfying

$$p_n(0) \leqslant 2^{-n},\qquad 1 = \int p_n\,d\gamma \qquad (= \hat{\mu}_n(0)) \tag{3.4}$$

and thus equivalent to the denial of an inequality

(3.5) $\int p\,d\gamma \leqslant \text{const}\cdot p(0)$, *for p integrable positive definite, supported by E_0.*

Thus (3.5) is equivalent to (3.1) for G compact; but in fact this is equivalent to the integrability of all positive definite functions supported by E_0. For a given sequence satisfying (3.4) we have $p_n(0) = \|p\|_\infty \leqslant 2^{-n}$ and thus $p = \sum_1^\infty p_n$ defines a positive definite function supported by E_0, which cannot be integrable: for $p_n = \hat{f}_n$, where $0 \leqslant f_n = \hat{p}_n \in C(G)$, and $\|f_n\|_1 = \int f\,dx = p_n(0) \leqslant 2^{-n}$, so that $f = \sum_1^\infty f_n \in L_1(G)$, and $\hat{f} = \sum_1^\infty \hat{f}_n$ by dominated convergence (since $0 \leqslant \sum_1^N |f_n| \leqslant f$), so that $\hat{f} = \sum_1^\infty p_n = p$. Consequently if p were integrable f would be essentially bounded, indeed $f = \hat{p}$ a.e.; but $f = \sum_1^\infty f_n$ is lower-semi-continuous so that $f(0) = \infty > m > \hat{p}(0)$ implies $f > m$ on a neighborhood of 0, whence $f > \hat{p}$ on a neighborhood of 0 and $f = \hat{p}$ a.e. fails. So if every positive definite function supported by E_0 is integrable then (3.1) follows. (It should be noted that this integrability implies directly that E_0 is a Sidon set: for if g is a bounded E_0-function then $(\|g\|_\infty+g)\hat{\ }$ is a positive definite function supported by E_0, so that the integrability of $(\|g\|_\infty+g)\hat{\ }$ is assured, and that of $\hat{g}$ follows (cf. [8], § 5.73).)

Now suppose our Sidon set E_0 supports a non-integrable positive definite function $p = \hat{\mu}$, where $\mu \geqslant 0$ is a measure on G. Let $\{u_\delta\}$ be an approximate identity for $L_1(G)$ consisting of non-negative trigonometric polynomials, with $\int u_\delta\,dx = 1$. If $\{\|\mu * u_\delta\|_\infty\}$ were bounded then $\{\mu * u_\delta\}$ would have a w^* cluster point f in $L_\infty(G)$, and $f = \lim\hat{\mu}\ \hat{u}_\delta = \hat{\mu} = p$, whence p is integrable since f is a bounded E_0-function. Thus we must have $u_n = u_{\delta_n}$ for which $n \leqslant \|\mu * u_n\|_\infty = \mu * u_n(-x_n) = \mu * u_n * \delta_{x_n}(0)$, so the transform $p_n = \frac{1}{n} x_n \cdot \hat{\mu} \hat{u}_n$ of the non-negative trigonometric polynomial $\frac{1}{n}(\mu * u_n * \delta_{x_n})$ is an integrable positive definite function supported by E_0 with $\hat{p}_n(0) = \int p_n d\gamma = \frac{1}{n}(\mu * u_n * \delta_{x_n}(0)) \geqslant 1$ and $p_n(0) = \frac{1}{n}\hat{\mu}(0)\hat{u}_n(0) = \frac{1}{n}\hat{\mu}(0) = \frac{1}{n} p(0)$, so (3.5) fails.

We have proved all but the final parts of the following

THEOREM 3.1. *Suppose $0 \notin E = -E$, a closed Helson set in the dual Γ of a locally compact abelian group G. If E is compact, or 0 is not in the closure of E in the Bohr compactification Γ^a of Γ, then*

$$L_1^{R+}(G)\hat{}\,|E = C_{0h}(E). \tag{3.1}$$

In general (3.1) *is equivalent to*

$$-\inf \hat{\mu}(G) \leqslant c \sup \hat{\mu}(G), \quad \mu \in M_h(E), \tag{3.2}$$

and if G is compact (so E is Sidon) then (3.1) *is equivalent to each of the following*:

(3.5) *$\int p\, d\gamma \leqslant c\, p(0)$ for p integrable positive definite, supported by $E_0 = \{0\} \cup E$.*

(3.6) *All positive definite functions on Γ supported by E_0 are integrable.*

(3.7) *The positive definite functions on Γ interpolate the bounded hermitian symmetric functions on E.*

(3.8) *Some positive definite function on Γ supported by $\Gamma \setminus E$ has positive mean square.*

With the exception of (3.8), the equivalence of these conditions (in sometimes slightly different form) when G is compact is given in [3].

(3.7) amounts to the assertion that $M^{R+}(G)\hat{}\,|E = C_h(E)$. That this implies (3.1) for E Sidon is most easily seen by noting that by (3.7) we have a $\nu \geqslant 0$ on G for which $\hat{\nu} \equiv -1$ on E, so $\nu * L_1^{R+}(G) \subset L_1^{R+}(G)$ implies

$$-L_1^{R+}(G)\hat{}\,|E = \big(\nu * L_1^{R+}(G)\big)\hat{}\,|E \subset L_1^{R+}(G)\hat{}\,|E,$$

and hence

$$L_1^R = L_1^{R+} - L_1^{R+} \quad \text{implies} \quad L_1^R(G)\hat{}\,|E \subset L_1^{R+}(G)\hat{}\,|E,$$

and therefore $C_{0h}(E) \subset L_1^{R+}(G)\hat{}\,|E$ since E is Sidon.

Conversely if (3.1) holds, $g \in C_h(E)$, and $\{u_\delta\}$ is a real approximate identity in $L_1(G)$, then from the final assertion of 2.1 we have a constant k and $f_\delta \in L_1^{R+}(G)$ with $\hat{f}_\delta = \hat{u}_\delta g$ on E and $\|f_\delta\|_1 \leqslant k\|\hat{u}_\delta g\|_\infty \leqslant k\|g\|_\infty$. For a w^* cluster point μ in $M^{R+}(G)$ of the net of measures corresponding to $\{f_\delta\}$ we of course have $\hat{\mu} = g$ on E (and $\|\mu\| \leqslant k\|g\|_\infty$), yielding (3.7).

As we have just seen we have a $\nu \geqslant 0$ with $\hat{\nu} \equiv -1$ on E if (3.1) holds, so $(\delta_0 + \nu)\hat{}$ is a positive definite function on Γ supported by $\Gamma \setminus E$ whose mean square is non-zero since $\delta_0 + \nu$ has a non-zero discrete part. But the full equivalence of (3.8) and (3.1) arises from a bit of argument which applies somewhat more generally. Suppose (still for G compact) we ask for which symmetric E $(0 \notin E)$ we have

$$L_1^{R+}(G)\hat{}\,|E = L_1^R(G)\hat{}\,|E. \tag{3.9}$$

(For E Sidon this is our original question about (3.1).) Noting that the nullity N of $f \to \hat{f}|E$ in $L_1^R(G)$ consists of the real elements of kE, the kernel of E, it is easy to see that $N^\perp$ in the real dual $L_\infty^R(G)$ is $(kE)_R^\perp$, the set of real valued elements of $(kE)^\perp$ in $L_\infty(G)$, thus the real elements in the w^* closed span of E. We can apply 2.3 here to assert that (3.9) is equivalent to the existence of a $c > 0$ for which

$$-\operatorname{ess\,inf} \varphi \leqslant c \operatorname{ess\,sup} \varphi, \quad \varphi \in (kE)_R^\perp. \tag{3.10}$$

But in fact it will suffice to merely note that (3.9) implies (3.10) (thus using only the easier half of 2.1) and of course that implies (3.10) for the trigonometric polynomials in $(kE)_R^\perp$, i.e.,

(3.11) $-\inf \varphi(G) \leqslant c \sup \varphi(G)$, *$\varphi$ a real trigonometric polynomial in* $\operatorname{span} E$.

Now for a non-negative element $\Psi = r + \varphi$ of the real trigonometric polynomials in span E_0, where $r \in R$ and $\varphi \in \operatorname{span} E$, we have $r \geqslant \sup(-\varphi) \geqslant -\frac{1}{c}\inf(-\varphi) = \frac{1}{c}\sup\varphi$, and thus since $r = \int \Psi\, dx$, for another c,

(3.12) $\Psi(0) \leqslant c \int_G \Psi\, dx$ *for $\Psi \geqslant 0$ a trigonometric polynomial in* $(kE_0)_R^\perp$.

With m Haar measure on G, (3.12) says $cm - \delta_0$ provides a non-negative linear functional on $(kE_0)_R^\perp$, and so we have a non-negative extension by Hahn–Banach to $C^R(G)$. So we have a $\nu \geqslant 0$ on G with $cm - \delta_0 - \nu \perp (kE_0)_R^\perp$, hence orthogonal to E_0: in particular $\hat{\nu} = \hat{\delta}_0 = -1$ on E, and (3.8) follows. On the other hand $\hat{\nu} = -1$ on E implies (3.1) as we know, and in showing that we really showed (3.9). So we have shown (3.9), (3.10), (3.12) are equivalent, and imply (3.8).

Now suppose (3.8) holds and note that the positive definite function whose existence is asserted there corresponds to a $\lambda \geqslant 0$ with discrete part $\lambda_d \neq 0$, and with $\hat{\lambda} \equiv 0$ on E. Thus $\hat{\lambda}_c = -\hat{\lambda}_d$ on E, so for $a > 0$ large, $b = \hat{\lambda}_d(0) + \hat{\lambda}_c(0) + a > 0$ and

$$(am + \lambda_c)\hat{} = \big[\big(\hat{\lambda}_d(0) + \hat{\lambda}_c(0) + a\big)m - \lambda_d\big]\hat{} = (bm - \lambda_d)\hat{} \quad \text{on } E_0. \tag{3.13}$$

Now since $am+\lambda_c \geqslant 0$, $bm-\lambda_d$ is non-negative on $(kE_0)^{\perp}_R$ by (3.13), and if x_0 is in the support of λ_d, with $c = \lambda_d\{x_0\} > 0$, then $bm - c\delta_{x_0} = bm-\lambda_d+(\lambda_d - c\delta_{x_0}) \geqslant bm-\lambda_d$ shows $bm-c\delta_{x_0}$ is non-negative on the invariant space $(kE_0)^{\perp}_R$, so $bm-c\delta_0$ is also, and (3.12) and therefore (3.9) follow. We have thus proved (3.8), (3.9), (3.10) and (3.12) are all equivalent, and have completed our proof of 3.1.

COROLLARY 3.2 (of the proof). *For any subset E of Γ with $0 \notin E = -E$, if G is compact the following are equivalent*:

(3.8) *Some positive definite function on Γ supported by $\Gamma \setminus E$ has positive mean square.*

(3.9) $$L_1^{R+}(G)\hat{}\,|E = L_1^R(G)\hat{}\,|E.$$

(3.10) $-\operatorname{ess\,inf}\varphi \leqslant c\operatorname{ess\,sup}\varphi$, *$\varphi$ real valued in the w^* closed span of E.*

(3.12) $\Psi(0) \leqslant c\int_G \Psi\,dx$ *for all non-negative elements of the span of $E_0 = \{0\}\cup E$.*

The preceding argument shows we can obtain the equivalence in 3.1 for G compact using only the easier half of 2.1: in effect that shows (3.1) implies (3.2) (or, what is the same, (3.10)), and (3.10) then produces a $\nu \geqslant 0$ with $\hat{\nu} \equiv -1$ on E, which shows directly that (3.2) implies (3.1). (The fact that G is compact is essential; otherwise ν is a measure on the Bohr compactification of G.) This use of ν depends of course on the fact that $L_1^{R+}(G)$ is translation invariant, and we now want to point out applications of 2.1 where invariance is missing.

Suppose $\mu \geqslant 0$ is in $M_0(G)$, i.e., $\mu \in C_0(\Gamma)$. Then a well known argument due to Rajchman shows the set M_μ of finite ν absolutely continuous with respect to μ lies in $M_0(G)$, and thus $M_\mu\hat{}\,|E \subset C_0(E)$. Now if the closed support F of μ in G is also the closed support of $\chi_F\cdot m$ and has the property that $(\chi_F\cdot L_1(G))\hat{}$ interpolates $C_0(E)$, the same is true of $M_\mu\hat{}$ and conversely: for $(\chi_F\cdot L_1(G))\hat{}\,|E = C_0(E)$ and $M_\mu\hat{}\,|E = C_0(E)$ are equivalent, respectively, to inequalities

$$\|\lambda\| \leqslant c\|\hat{\lambda}\|_{L_\infty(\chi_F\cdot m)}, \qquad \lambda \in M(E),$$

and

$$\|\lambda\| \leqslant c'\|\hat{\lambda}\|_{L_\infty(\mu)}, \qquad \lambda \in M(E),$$

and in each case the right hand term features precisely $\sup|\hat{\lambda}(F)|$. The same takes place when we consider interpolation using positive cones: $(\chi_F\cdot L_1^{R+}(G))\hat{}\,|E = C_{0h}(E)$ and $(M_\mu^{R+})\hat{}\,|E = C_{0h}(E)$ both come to the further inequality

$$-\inf\hat{\lambda}(F) \leqslant c\sup\hat{\lambda}(F), \qquad \lambda \in M_h(E)$$

(perhaps with distinct constants). In particular

THEOREM 3.3. *Suppose $0 \notin E = -E$, a closed Helson set in Γ, and $\mu \geqslant 0$ is in $M_0(G)$ and has global support. Then*

$$(M_\mu^{R+})\hat{}\,|E = (L_1^{R+}(\mu)\cdot\mu)\hat{}\,|E = C_{0h}(E)$$

iff

$$L_\mu^{R+}(G)\hat{}\,|E = C_{0h}(E).$$

In [3] the familiar Sidon sets are shown to be Fatou–Zygmund (i.e., satisfy (3.1)). Noting (as in [3]) that such sets are closed under the taking of finite unions by Drury's argument, one can trace this result of [3] via (3.2) to the fact that, in showing these familiar sets Sidon one produces transforms of Riesz products giving "arbitrary signs" on the set, and these Riesz products are *non-negative*. Indeed in order to show E satisfies (3.1) from (3.2) it suffices to see that there is a constant $\beta > 0$ for which, for every function φ on E assuming values ± 1, there is a probability measure ν on G for which $\operatorname{sgn}\hat{\nu} = \varphi$ and $|\hat{\nu}| \geqslant \beta$ on E: for then given $\mu \in M_h(E)$ we can set $\varphi(\gamma) = \operatorname{sgn}\mu\{\gamma\}$, so that for the corresponding ν we have

$$\beta\|\mu\| = \beta\sum|\mu\{\gamma\}| \leqslant \sum\mu\{\gamma\}\hat{\nu}(\gamma) = \int_G \hat{\mu}\,d\nu \leqslant \sup\hat{\mu}(G).$$

We can do approximately the same thing for E a non-compact Helson set, and obtain the following sufficient condition for (3.1).

LEMMA 3.4. *Suppose $0 \notin E = -E$, a Helson set, and*

(3.14) $$\sup\left\{\frac{\|\mu\|}{\|\hat{\mu}\|_\infty} :\ \mu \in M_h(E_0)\right\} < 3.$$

Then (3.1) *holds for E.*

Let $\alpha \in R$ lie between the quantities in (3.14).

If (3.1) fails we have μ_n in $M_h(E)$ with $\hat{\mu}_n \geqslant -\frac{1}{n}$, $\hat{\mu}_n(0) = 1$, or alternatively, adding $\frac{1}{n}\delta_0$ to μ_n and renormalizing, $\mu_n \in M_h(E_0)$ with $\hat{\mu}_n \geqslant 0$, $\hat{\mu}_n(0) = 1$, $0 < \mu_n\{0\} \leqslant \frac{1}{n}$. Fix n and let $\theta\mu_n = |\mu_n|$, where θ has real values and $|\theta| \equiv 1$; by Lusin we can choose a compact symmetric subset F of E_0 with $\theta|E$ continuous, $|\mu_n|(E_0 \setminus F) < \frac{1}{n}$, and since 0 is isolated in E_0 we can take $0 \in F$.

The fact that α exceeds the left side of (3.14) means that for $\varphi \in C_{0h}(F) = C_h(F)$ of unit norm we can find a $\nu \in M^R(G)$ with $\hat{\nu} = \varphi$ on F, $\|\nu\| < \alpha$, as usual. (3) With $\varphi(0) = 1$ and $\varphi = -\theta$ elsewhere on F we have $\hat{\nu}\cdot(\mu_n)_{F\setminus\{0\}} = |\mu_n|_{F\setminus\{0\}}$ (the subscript denoting restriction of the measure), and $\hat{\nu}\{0\} = 1$.

Let $\nu = \nu_+ - \nu_-$ be the Jordan decomposition of ν. Since $\hat{\nu}(0) = \hat{\nu}_+(0) - \hat{\nu}_-(0) = 1$ while $\hat{\nu}_+(0) + \nu_-(0) = \|\nu\| < \alpha$,

$$\hat{\nu}_-(0) = \|\nu_-\| = \tfrac{1}{2}(\|\nu\| - 1) \leqslant \tfrac{1}{2}(\alpha - 1) = 1 - \beta$$

where β (independent of n of course) is positive since $\alpha < 3$. For $\gamma \in F$ $\hat{\nu}(\gamma) = \pm 1$, and if $\hat{\nu}(\gamma) = 1$ then

$$\hat{\nu}_+(\gamma) = \hat{\nu}(\gamma) + \hat{\nu}_-(\gamma) \geqslant 1 - (1-\beta) = \beta$$

while if $\hat{\nu}(\gamma) = -1$

$$\hat{\nu}_+(\gamma) = -1 + \hat{\nu}_-(\gamma) \leqslant -1 + (1-\beta) = -\beta$$

so that $\hat{\nu}_+$ has the same sign as $\hat{\nu}$ and $|\hat{\nu}_+| \geqslant \beta$ on F. Since $\nu_+ \geqslant 0$ and $\hat{\mu}_n \geqslant 0$, we have

$$\begin{aligned} 0 &\leqslant \int \hat{\mu}_n d\nu_+ = \int \hat{\nu}_+ d\mu_n \leqslant \|\nu\| \mu_n\{0\} + \int_E \hat{\nu}_+ d\mu_n \\ &\leqslant \frac{\alpha}{n} + \int_{E\setminus F} \|\nu\| d|\mu_n| + \int_{F\setminus\{0\}} \hat{\nu}_+ d\mu_n \\ &\leqslant \frac{\alpha}{n} + \frac{\alpha}{n} - \beta|\mu_n|(F\setminus\{0\}) \leqslant \frac{2\alpha}{n} - \beta|\mu_n|(E_0) + \frac{\beta}{n} + \frac{\beta}{n} \\ &\leqslant \frac{2}{n}(\alpha+\beta) - \beta \end{aligned}$$

since $\|\mu_n\| \geqslant \hat{\mu}_n(0) = 1$. Thus $\beta \leqslant \dfrac{2}{n}(\alpha+\beta)$ for all n, and $\beta \leqslant 0$, our contradiction.

The preceding results give some sufficient conditions for the Fatou–Zygmund interpolation (3.1). As we have already noted when G is compact, if Λ is a subgroup of finite index in Γ and $E \cap \Lambda = \emptyset$ then E satisfies (3.1); thus *if a positive definite function p is supported by our Sidon set E_0 and $p\chi_\Lambda$ is integrable, p is integrable*: for if $g = (p\cdot\chi_\Lambda)\hat{}$, $g \in C(G)$ and

$$p' = p + (\|g\|_\infty - g)\hat{} = p + \|g\|_\infty \delta_0 - p\chi_\Lambda$$

is positive definite by the first equality and supported by $(E\setminus\Lambda)_0$ by the second, while (3.1) holds for $E\setminus\Lambda$ so that p' is integrable by Theorem 3.1. Thus by our Lemma 3.4, *if $E\cap\Lambda$ satisfies* (3.14) *for some Λ of finite index* (*or is a finite union of sets which do*) *then E satisfies* (3.1). (Alternatively

(3) The left side of (3.14) gives the interpolation constant for $L_1^R(G) \to L_1^R(G)\hat{}\,|E = C_{0h}(E)$, and thus for the interpolation of bounded continuous hermitian functions on E by $M^R(G)\hat{}$.

some condition like (3.8) might apply to Λ and its subset $\Lambda\cap E$ which would yield the integrability of $p\chi_\Lambda$ by 3.1.)

Finally, in connection with (3.6) for G compact it should be noted that integrability of all positive definite functions supported by E_0 is of course far from the integrability of all Fourier–Stieltjes transforms supported by E_0. In fact the last condition holds only when E_0 is finite. A simple argument to show this is the following, pointed out to me by John Fournier: if E_0 is infinite, we can choose $\varphi \in l_2(E_0)\setminus l_1(E_0)$, and (extended to be zero off E_0) $\varphi = \hat{g}$, for $g \in L_2(G) \subset L_1(G)$ by Plancherel, so that φ (extended) is a non-integrable Fourier–Stieltjes transform supported by E_0.

4. Interpolation by other cones. In the present section we point out the possibility of interpolation by other cones in $L_1(G)$, and also in some other algebras, in a series of examples.

4.1. When G is compact and E is a Fatou–Zygmund set, the existence of a $\nu \geqslant 0$ on G with $\hat{\nu} \equiv -1$ on E shows that any invariant cone P in $L_1(G)$ interpolates as well as its real span, as we have seen; thus, for example, for

$$P_0 = \{f \in L_1(G)\colon\ \varepsilon \operatorname{Re} f \geqslant |\operatorname{Im} f|\},$$

we have $P_0\hat{}\,|E_0 = C_0(E)$. On the other hand it is trivial to design a similar cone which is not invariant and which has the same property (but not so trivially evident) as in 3.3. Let U be an open (say) dense subset of G but of small measure, and consider

$$P = \chi_U \cdot P_0.$$

This is a smaller cone and far from invariant, but we can easily see $P\hat{}\,|E = C_0(E)$. For as we observed before Theorem 3.3, $(P-P)\hat{} = (\chi_U L_1(G))\hat{}$ interpolates $C_0(E)$, and applying 2.1 to this cone in $\chi_U L_1(G)$, $P\hat{}\,|E = C_0(E)$ iff we have a k for which, for each $\mu \in M(E)$,

$$\|\hat{\mu}\|_\infty \leqslant k \sup\{\operatorname{Re}\textstyle\int \hat{\mu} f\,dx\colon\ f \in P,\ \|f\|_1 \leqslant 1\} \tag{4.1}$$

since each real linear functional is the real part of a complex linear one (so that $f \to \operatorname{Re}\int \hat{f}\,d\mu = \operatorname{Re}\int f\hat{\mu}\,dx$ yields the (real) adjoint to $f \to \hat{f}|E$) and $\|\hat{\mu}\|_\infty = \sup\{\operatorname{Re}\int \hat{\mu} f\,dx\colon\ f \in L_1,\ \|f\|_1 \leqslant 1\}$. Now the right side of (4.1) is $k \sup_{x\in U} \Psi_\mu(x) = k \sup_{x \in U} |\hat{\mu}(x)|\varphi(-\arg\hat{\mu}(x))$ where φ is an even periodic function, $= 1$ on $[0, \tan^{-1}\varepsilon]$, $= \cos(\theta - \tan^{-1}\varepsilon)$ on $[\tan^{-1}\varepsilon, \frac{1}{2}\pi + \tan^{-1}\varepsilon]$ and 0 on $[\frac{1}{2}\pi + \tan^{-1}\varepsilon, \pi]$ as is easily seen ($\Psi_\mu(x)$ is the maximum of the projection of $e^{i\theta}\hat{\mu}(x)$ onto R for $|\theta| < \tan^{-1}\varepsilon$, or 0, if that is negative). Since U is dense the supremum coincides with that over all of G, and from the form of φ the supremum exceeds $\sup \operatorname{Re}\hat{\mu}(x)$ and also $\sin(\tan^{-1}\varepsilon)\sup|\operatorname{Im}\hat{\mu}(x)|$. Now if E is a Fatou–Zygmund set then by (3.2) for $\nu \in M_h(E)$,

$$\sup\hat{\nu}(x) = \sup\operatorname{Re}\hat{\nu}(x) \geqslant \frac{1}{c}\|\hat{\nu}\|$$

while for any $\mu \in M(E)$ we have $\mu = \lambda + i\nu$, where $\lambda = \frac{1}{2}(\mu+\mu^*)$ and $\nu = \frac{1}{2i}(\mu-\mu^*)$ lie in $M_h(E)$; and $\hat{\lambda} = \operatorname{Re}\hat{\mu}$, $\hat{\nu} = \operatorname{Im}\hat{\mu}$ so

$$\sup\operatorname{Re}\hat{\mu}(x) = \sup\hat{\lambda}(x) \geqslant \frac{1}{c}\|\hat{\lambda}\|,$$

$$\sup\operatorname{Im}\hat{\mu}(x) = \sup\hat{\nu}(x) \geqslant \frac{1}{c}\|\hat{\nu}\|$$

whence

$$\|\hat{\mu}\| \leqslant \|\hat{\lambda}\| + \|\hat{\nu}\| \leqslant c\sup\operatorname{Re}\hat{\mu} + c\sup\operatorname{Im}\hat{\mu}$$

$$\leqslant \left(c + \frac{c}{\sin(\tan^{-1}\varepsilon)}\right)\sup|\hat{\mu}(x)|\varphi(\arg\hat{\mu}(x))$$

yielding (4.1). (We could equally well let

$$P = \left\{\nu \ll \mu:\ \varepsilon \operatorname{Re}\frac{d\nu}{d\mu} \geqslant \left|\operatorname{Im}\frac{d\nu}{d\mu}\right|\right\}$$

where μ is a measure as in Theorem 3.3, again assuming E Fatou–Zygmund.)

4.2. If $E \subset T^1$ is a Helson set then a well known theorem of Wik [5, 6, 9] shows $l_1(\boldsymbol{Z}_+)^\wedge|E = C(E)$ and $\|\mu\| \leqslant c\sup|\hat{\mu}(\boldsymbol{Z}_+)|$, $\mu \in M(E)$; in fact

(4.2) $$\|\mu\| \leqslant c \lim_{n\to+\infty} |\hat{\mu}(n)|.$$

If we take $1 \notin E = E^{-1}$ then of course $l_1^R(\boldsymbol{Z}_+)^\wedge|E = C_h(E)$, and now we want to note that

$$l_1^{R+}(Z_+)^\wedge|E = C_h(E)$$

holds here too.

Were this to fail we would have $\mu_n \in M_h(E)$ with $\sup\hat{\mu}_n(Z_+) = 1$, $\inf\hat{\mu}_n(Z_+) \geqslant -\frac{1}{n}$, and since we can replace μ_n by $e^{ik}\mu_n$, $k \in Z_+$, we can assume instead that $\hat{\mu}_n(0) \geqslant \frac{1}{2}$, $1 \geqslant \hat{\mu}_n \geqslant -\frac{1}{n}$ on $\boldsymbol{Z}_+$. Then $\{\mu_n\}$ has a w^* cluster point in $M_h(E)$ (since $\|\mu_n\| \leqslant c$) and $\hat{\mu} \geqslant 0$ on $\boldsymbol{Z}_+$, $\hat{\mu}(0) \geqslant \frac{1}{2}$ (so $\|\mu\| \geqslant \frac{1}{2}$). By (4.2) we have $k_j \nearrow +\infty$ for which $\hat{\mu}(k_j) \geqslant \frac{1}{2c}$, and now $\{e^{ik_j}\mu\}$ has a w^* cluster point $\nu \in M_h(E)$ with $\hat{\nu}(0) \geqslant \frac{1}{2c}$ and $\hat{\nu} \geqslant 0$ on *all* of Z. But exactly as in § 3, just after (3.3), we can use ν to construct a non-zero positive definite function on T^1 which vanishes at the identity, the desired contradiction.

We can use the same argument for any half-group: if $\Psi: \Gamma \to R$ is a non-trivial homomorphism then $l_1(\Psi^{-1}([0,\infty)))^\wedge|E = C(E)$ when $l_1(\Gamma)^\wedge|E = C(E)$ as a consequence of a result of Bernard [1, 6] which yields this extension of Wik's result. So the preceding argument applies to show $l_1^{R+}(\Psi^{-1}([0,\infty)))^\wedge|E = C_h(E)$ once we have the analogue of (4.2), which here reads

$$\|\mu\| \leqslant c\sup\{|\hat{\mu}(\gamma)|:\ \Psi(\gamma) \geqslant n\},\qquad \mu \in M(E),$$

for all n, and is a consequence of the following

LEMMA 4.2.1. *Let $S \subset \Gamma$ and suppose $l_1(S)^\wedge|E = C(E)$. Then there is a constant c for which, for each $\gamma \in \Gamma$,*

(4.3) $$\|\mu\| \leqslant c\sup|\hat{\mu}(\gamma+S)|,\qquad \mu \in M(E).$$

For each γ the map T_γ from $l_1(\gamma+S)$ to $C(E)$ defined by $f \to \hat{f}|E$ has as its adjoint T_γ^*: $M(E) \to l_\infty(\gamma+S)$ the map $\mu \to \hat{\mu}|E$, as does the induced $1-1$ map $\overline{T}_\gamma$: $l_1(\gamma+S)/\operatorname{kern}T_\gamma \to C(E)$. Of course (4.3) asserts that T_γ^* has a bounded inverse of norm $\leqslant c$, so that the same must follow for $\overline{T}_\gamma$; and since we are assuming interpolation for $\gamma = 0$ we thus have a c_0 for which (4.3) holds for $\gamma = 0$. So for $\varphi \in C(E)$, $\gamma \in \Gamma$ and $\varepsilon > 0$ we can choose $f \in l_1(S)$ with $\hat{f}|E = \overline{\gamma}\varphi$, $\|f\|_1 \leqslant (c_0+\varepsilon)\|\overline{\gamma}\varphi\|_\infty = (c_0+\varepsilon)\|\varphi\|_\infty$; with $R_\gamma f(\gamma') = f(\gamma'-\gamma)$ we have $(R_\gamma f)^\wedge = \gamma\hat{f} = \varphi$ on E, $\|R_\gamma f\|_1 = \|f\|_1 \leqslant (c_0+\varepsilon)\|\varphi\|_\infty$ and $R_\gamma f \in l_1(\gamma+S)$. Since $\varepsilon > 0$ is arbitrary, this says $\overline{T}_\gamma$ is an onto map with bounded inverse of norm $\leqslant c_0$, so $\|(\overline{T}_\gamma)^{*-1}\| = \|T_\gamma^{*-1}\| \leqslant c_0$, and (4.3) holds with $c = c_0$.

The lemma is no doubt well known, and can easily be formulated for Γ not necessarily discrete. We should probably also note that it yields an analogue of Helson's result that a non-zero measure carried by a Helson set cannot have a C_0 transform: *Suppose $S \subset \Gamma$ is a subsemigroup with the property that for each compact $K \subset S$ there is a γ in S with $K \cap (\gamma+S) = \emptyset$* (which occurs, for example, if there is a $\gamma_0 \in S$ with $-\gamma_0 \notin S$ and $\Gamma = \bigcup_0^\infty(-n\gamma_0+S)$). *Then if $l_1(S)^\wedge|E = C(E)$, $\mu \in M(E)$, $\mu \neq 0$ imply $\hat{\mu} \notin C_0(S)$.* Indeed if $|\hat{\mu}(S\setminus K)| < \varepsilon$ for some K then for our γ, $\gamma+S \subset S$ and is disjoint from K, so $\|\mu\| \leqslant c\sup|\hat{\mu}(\gamma+S)| \leqslant c\varepsilon$.

Now suppose S is a subsemigroup of our discrete Γ, $0 \in S$, $l_1(S)^\wedge|E = C(E)$, $0 \in E = -E$, and there is a $\gamma_0 \in S$ with $\Gamma = \bigcup_{n=1}^\infty(S-n\gamma_0)$. Then $l_1^{R+}(S)^\wedge|E = C_h(E)$.

The assertion is equivalent to

$$\sup|\hat{\mu}(S)| \leqslant c\sup\hat{\mu}(S), \qquad \mu \in M_h(E),$$

and as before we have to obtain a contradiction from a sequence $\{\mu_n\}$ with $\sup\hat{\mu}_n(S) = 1$, $\inf\hat{\mu}_n(S) \geqslant -\frac{1}{n}$. Since S is a semigroup, we again have another sequence $\{\mu_n\}$ with $1 \geqslant \hat{\mu}_n \geqslant -\frac{1}{n}$ on S, $\hat{\mu}_n(0) \geqslant \frac{1}{2}$; moreover, by our lemma,

$$\|\mu\| \leqslant c\sup|\hat{\mu}(\gamma+S)| \tag{4.4}$$

for all $\mu \in M(E)$ and $\gamma \in S$. So once more we get a w^* cluster point with $1 \geqslant \hat{\mu} \geqslant 0$ on S, $\hat{\mu}(0) \geqslant \frac{1}{2}$, and by (4.4) for each n we have a $\gamma_n \in n\gamma_0 + S$ with $\hat{\mu}(\gamma_n) \geqslant \frac{1}{2c}$. But now any w^* cluster point ν of $\{\gamma_n\mu\}$ in $M_h(E)$ satisfies $\hat{\nu}(0) \geqslant \frac{1}{2}c$ while $\hat{\nu} \geqslant 0$ on all of Γ: for $S - n\gamma_0 \subset S + (-\gamma_n + S) \subset S - \gamma_n$ while $\gamma_0 \in S$ implies the sets $S - n\gamma_0$ increase with n, so each $\gamma \in \Gamma$ lies in $S - \gamma_n$ for $n \geqslant n_\gamma$, and thus $(\gamma_n\mu)\hat{\ }(\gamma) = \mu(\gamma+\gamma_n) \geqslant 0$ for $n \geqslant n_\gamma$. Of course, ν provides our contradiction as before.

4.3. As a variant of the first example of 4.2 we can note that for $f \in l_1(Z_+)$ we have $\hat{f}(z) = \sum_0^\infty f(n)z^n$, $|z| \leqslant 1$, an extension of the Fourier transform analytic on $|z| < 1$ (which is, of course, the Gelfand representative of f), and the elements $g = \hat{f}$ of $l_1^R(Z_+)\hat{\ }$ are hermitian symmetric in the sense that

$$\tilde{g}(z) = \overline{g(\bar{z})} = g(z).$$

Thus we can take $1 \notin E = \bar{E} \subset D$ and ask when $l_1^{R+}(Z_+)\hat{\ }|E = C_h(E) = \{g \in C(E)\colon\ g = \tilde{g}\}$. Of course, $l_1^R(Z_+)\hat{\ }|E = C_h(E)$ is equivalent to $l_1(Z_+)\hat{\ }|E = C(E)$ as usual, and the latter is equivalent to $E \cap T^1$ being Helson and $E \cap D^{\circ}$ being finite.

Indeed if $l_1(Z_+)\hat{\ }$ interpolates, $E \cap T^1$ is certainly Helson, while the finiteness of $E \cap D^{\circ}$ is a result of S.A. Vinogradov ([5], p. 145). (The argument runs as follows: if $E \cap D^{\circ}$ were infinite we could choose a sequence $\{z_n\}$ therein converging to $z_0 \in D$ (necessarily in T^1) and an $f_n \in l_1(Z_+)$ which has $\hat{f}_n(z_j) = (-1)^j$ for $j \leqslant n$, $= 0$ elsewhere on E, with $\|f_n\|_1 \leqslant c$, the constant of interpolation; then any w^* cluster point f of $\{f_n\}$ would have $\hat{f}(z_j) = \sum_0^\infty f(n)z_j^n$ a cluster point of $\{f_n(z_j)\}$ (since $|z_j| < 1$), hence $= (-1)^j$, for each j, so that $\hat{f}$ could not be continuous at z_0.) Conversely if $F = E \cap T^1$ is Helson and $E \cap D^{\circ}$ finite then the fact that F is hull kernel closed in the maximal ideal space D of $l_1(Z_+)$ (the hull of the kernel

of F being precisely the maximal ideal space of the quotient algebra $l_1(Z_+)\hat{\ }|E = C(E)$) shows that for any $z_1 \in E \cap D^{\circ}$ we have an $f_1 \in l_1(Z_+)$ with $\hat{f}_1 = 0$ on $F = E \cap T^1$, $\hat{f}_1(z_1) = 1$. Thus given $\varphi \in C(E)$ we can choose $g \in l_1(Z_+)$ with $\hat{g} = \varphi$ on F and a multiple tf_1 of f_1 so that $t\hat{f}_1(z_1) + \hat{g}(z_1) = \varphi(z_1)$, whence $tf_1 + g = \varphi$ on $F_1 = \{z_1\} \cup F$. Evidently, $l_1(Z_+)\hat{\ }|F_1 = C(F_1)$ and we can replace F by F_1 and continue.

Now for E conjugate symmetric $l_1^{R+}(Z_+)\hat{\ }|E = C_h(E)$ iff $E \cap T^1$ is Helson and $E \cap D^{\circ}$ is finite and misses the non-negative real axis, i.e. *iff $l_1(Z_+)\hat{\ }|E = C(E)$ and $E \cap [0,1] = \emptyset$*. Of course, the elements of $l_1^{R+}(Z_+)$ are non-negative (and non-decreasing) on the segment $[0,1]$, so that $E \cap [0,1] = \emptyset$ if our interpolation holds, and "only if" is clear. On the other hand since

$$\mu(\hat{f}) = \int \hat{f}\,d\mu = \sum_0^\infty f(n)\mu(z^n)$$

for $\mu \in M(E)$, $\mu \in M_h(E)$ amounts to $\mu(z^n)$ being real for $n \geqslant 0$, and (2.1) becomes

$$-\inf_{n\geqslant 0}\mu(z^n) \leqslant c\sup_{n\geqslant 0}\mu(z^n), \qquad \mu \in M_h(E), \tag{4.5}$$

which is equivalent to the desired interpolation since $l_1^R(Z_+)\hat{\ }|E = C_h(E)$. Again (4.5) fails only if we have a sequence $\{\mu_k\}$ in $M_h(E)$ with $1 \geqslant \mu_k(z^n) \geqslant -\frac{1}{k}$ and (replacing μ_k by $z^m\mu_k$ for some $m > 0$ if necessary) $\mu_k(1) \geqslant \frac{1}{2}$. So if (4.5) fails we have a μ in $M_h(E)$ with $\mu(1) \geqslant \frac{1}{2}$, $1 \geqslant \mu(z^n) \geqslant 0$ for $n \geqslant 0$.

Now $\overline{\lim}_{n\to+\infty}\mu(z^n) = \overline{\lim}_{n\to+\infty}\mu_{T^1}(z^n)$, and vanishes only if $\mu_{T^1} = 0$, by Lemma 4.2.1 applied to $S = Z_+$. If $\overline{\lim}_{n\to+\infty}\mu(z^n) \neq 0$, then we have an $\varepsilon > 0$ and $n_j \nearrow \infty$ for which $\mu(z^{n_j}) > \varepsilon$ so that any w^* cluster point ν of $\{z^{n_j}\mu\}$ has $\nu(z^0) \geqslant \varepsilon$ and $\nu(z^k) \geqslant 0$ for all $k \in \boldsymbol{Z}$; since ν is carried by $E \cap T^1$, we now have $\nu * \varphi * \varphi^*$ a non-zero positive definite function on T^1 vanishing at 1 if $\varphi \in C(T^1)$ is chosen appropriately, yielding the desired contradiction.

On the other hand if $\overline{\lim}_{n\to+\infty}\mu(z^n) = 0$, so that $\mu_{T^1} = 0$ and μ is carried by the finite set $E \cap D^{\circ}$, set $r = \max\{|z|\colon z \in E \cap D^{\circ}\}$ (so $0 < r < 1$) and let $\varrho(z) = \frac{1}{r}z$; then $\varrho^*\mu(z^k) = \mu(r^{-k}z^k) = r^{-k}\mu(z^k) \geqslant 0$ for $k \geqslant 0$ while $\varrho^*\mu(z^0) = \varrho^*\mu(1) \geqslant \frac{1}{2}$. But $\varrho^*\mu$ is carried by the finite subset $F = \frac{1}{r}(E \cap D^{\circ})$ of D, and $C(F \cap T^1)$ is certainly interpolated by $l_1(Z_+)\hat{\ }$, so that $\varrho^*\mu \neq 0$ implies $(\varrho^*\mu)\hat{\ } \notin C_0(Z_+)$ as before by Lemma 4.2.1. Thus we again have an $\varepsilon > 0$ and $n_j \nearrow \infty$ with $\varrho^*\mu(z^{n_j}) \geqslant \varepsilon$. Now any w^* cluster point ν of $\{z^{n_j}\varrho^*\mu\}$ in $M_h(F)$ is non-zero, supported by $F\cap T^1$ (which doesn't contain 1), and has $\hat{\nu}\geqslant 0$ on all of $\mathbf{Z}$, so we arrive at the same contradiction as before, completing our proof.

4.4. As a final variant we consider interpolation by Beurling algebras [7]. Suppose $w\geqslant 1$ is an even continuous function on G with $w(x+y)\leqslant w(x)w(y)$; then $L_w(G)=L_1(w\,dx)$ is contained in $L_1(G)$, is an algebra under convolution, and for a closed $E\subset\Gamma$, $L_w(G)\hat{}\,|E\subset C_0(E)$. By Bernard's theorem [1, 6] if $G_+=\Psi^{-1}([0,\infty))$, where $\Psi\colon G\to R$ is a non-trivial homomorphism, then $L_w(G_+)\hat{}\,|E=C_0(E)$ follows from $L_w(G)\hat{}\,|E=C_0(E)$ (for $\operatorname{Re}L_w(G_+)\hat{}=\operatorname{Re}L_w(G)\hat{}$ since

$$\operatorname{Re}\hat{f}=\operatorname{Re}(f\cdot\chi_{G_+})\hat{}+\operatorname{Re}(f\cdot\chi_{G\setminus G_+})\hat{}=\operatorname{Re}(f\cdot\chi_{G_+})\hat{}+\operatorname{Re}\big((f\cdot\chi_{G\setminus G_+})^*\big)\hat{}$$

and $(f\cdot\chi_{G\setminus G_+})^*\in L_w(G_+)$). Thus we can consider the questions of when an interpolation set E (with $0\notin E=-E$) has $L_w^{R+}(G)\hat{}\,|E=C_{0h}(E)$ or $L_w^{R+}(G_+)\hat{}\,|E=C_{0_h}(E)$, and again both occur if E is compact or, in the Bohr compactification of Γ, $0\notin E^-$.

For convenience we'll consider the second interpolation with E compact. Where 3.3 is concerned with the map $fw\to(fw)\hat{}\,|E$, here we must consider $f\to\hat{f}|E$, whose adjoint takes $\mu\in M(E)$ into the element $\dfrac{\hat{\mu}}{w}$ of $\chi_{G_+}L_\infty(w\,dx)$ since

$$\int_E f\,d\mu=\int_{G_+}\hat{\mu}f\,dx=\int_{G_+}\frac{\hat{\mu}}{w}fw\,dx,\qquad f\in L_w(G_+),$$

and thus $L_w(G_+)\hat{}\,|E=C(E)$ iff

$$\|\mu\|\leqslant c\sup\left\{\left|\frac{\hat{\mu}(x)}{w(x)}\right|:\ x\in G_+\right\},\tag{4.7}$$

and $L_w^{R+}(G_+)\hat{}\,|E=C_h(E)$ if we also have

$$\|\mu\|\leqslant k\sup\left\{\frac{\hat{\mu}(x)}{w(x)}:\ x\in G_+\right\}.\tag{4.8}$$

Once more if the last fails we have μ_n in $M_h(E)$ with $1=\sup\limits_{G_+}\dfrac{\hat{\mu}_n}{w}$, $\dfrac{\hat{\mu}_n}{w}\geqslant -\dfrac{1}{n}$ on G_+, so that (4.7) implies $\|\mu_n\|\leqslant c$; again we have $x_n\in G_+$ with $\hat{\mu}_n(x_n)\geqslant\frac{1}{2}w(x_n)$, so that the sequence $\{x_n\mu_n\}$ in $M_h(E)$ has a w^* cluster point ν with $1\geqslant\dfrac{\hat{\nu}}{w}\geqslant 0$ on G_+ and $\hat{\nu}(0)\geqslant\frac{1}{2}$. Now

$$L_1(G_+)\hat{}\,|E\supset L_w(G_+)\hat{}\,|E=C(E),$$

so by Lemma 4.2.1

$$\|\mu\|\leqslant c\sup|\hat{\mu}(g+G_+)|,\qquad \mu\in M(E),$$

for each $g\in G$, and thus we have g_n with $0\leqslant\Psi(g_n)\to\infty$ and $\|\nu\|\leqslant c|\hat{\nu}(g_n)|=c\hat{\nu}(g_n)$. But $\{g_n\nu\}$ now has a w^* cluster point λ in $M_h(E)$ with $\hat{\lambda}(0)\geqslant c^{-1}\|\nu\|>0$ and $\hat{\lambda}\geqslant 0$ on all of G since $g+g_n\in G_+$ for $n\geqslant n_g$, so that $\hat{\nu}(g+g_n)=(g_n\nu)\hat{}\,(g)\geqslant 0$ for $n\geqslant n_g$. Our usual contradiction now follows.

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