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Linear operators on $L^{1/\alpha}(0, \infty)$ and Lorentz spaces: The Krasnosel'skii–Zabrieko characteristic sets

by

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Abstract. The concept of the L-characteristic set for a linear operator as introduced by M. A. Krasnosel'skii and P. P. Zabrieko is discussed in the context of the Lebesgue and Lorentz spaces of functions on $(0, \infty)$. Necessary and sufficient conditions for a set to be a L-characteristic is given for the case $L^{1/\alpha}(0, \infty)$. A necessary condition for a "characteristic" set is given in a general setting, and sufficient conditions for the set are given in the case of Lorentz spaces.

Introduction. In the book [5], Krasnosel'skii, Zabrieko and others used the concept of L-characteristics to describe the actions of linear and non-linear operators on spaces of summable functions. Let $L^{1/\alpha}(M)$, $0 \leqslant \alpha < +\infty$, be the $1/\alpha$-summable Lebesgue measurable functions on the bounded measurable set M in Euclidean space. The L-characteristic of an operator T is simply the set of points (α, β), $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta < +\infty$, for which the operator maps $L^{1/\alpha}(M_1)$ continuously into $L^{1/\beta}(M_2)$. On page 42 of [5], the problem of characterizing the L-characteristic sets was posed, and a partial solution was given. In [7], the author solved the problem for the spaces $L^{1/\alpha}[0,1]$. In this paper, the problem is considered for functions defined on $(0, \infty)$ in the context of the $L^{1/\alpha}(0, \infty)$ spaces and the Lorentz spaces $L(1/\alpha, r)$.

§ 1. A necessary condition. Let $(X_\alpha, \|\cdot\|_\alpha)$, $\alpha \in I_\alpha = [\alpha_0, \alpha_1]$, be a family of Banach spaces continuously imbedded in some topological vector space $\mathscr{X}$. Suppose that there is a set $\Sigma \subseteq X_{\alpha_0}$ which is dense in each X_α, $\alpha_0 < \alpha \leqslant \alpha_1$. Further, assume that $\varphi_x(\alpha) = \|x\|_\alpha$ is a continuous function of α for each fixed $x \in \Sigma$.

Let (Y_β, ϱ_β), $\beta \in I_\beta$ where $I_\beta = [\beta_0, \beta_1]$ or $I_\beta = [\beta_0, \infty)$, be a family of complete linear metric spaces continuously imbedded in a topological vector space $\mathscr{Y}$. Let θ represent the origin in $\mathscr{Y}$. Suppose that $\psi_y(\beta) = \varrho_\beta(y, \theta)$ is a lower semi-continuous function of β for each fixed y. Further, suppose

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that $\varrho_\beta(ty, \theta) = g(t, \beta)\varrho_\beta(y, \theta)$ where t is a positive scalar, and $g(t, \beta)$ is continuous as a function of t and β.

Let T be a linear operator defined on Σ with values in $\mathscr{Y}$. The (X_α, Y_β)-characteristic of T is the set of all points $(\alpha, \beta) \in I_\alpha \times I_\beta$ for which T can be extended to a continuous linear operator from X_α to Y_β. We have the following theorem about (X_α, Y_β)-characteristics.

THEOREM 1. *If X_α, $\alpha \in I_\alpha$, Y_β, $\beta \in I_\beta$, and T are as described above, then the (X_α, Y_β)-characteristic of T is an F_σ-set of the region $I_\alpha \times I_\beta$.*

Proof. The proof can be carried out as the proof in § 2 of [7] with little change.

We note that the theorem applies when X_α and Y_β are the usual Lebesgue spaces $L^{1/\alpha}(M_1, m_1)$ and $L^{1/\beta}(M_2, m_2)$ for totally σ-finite non-atomic measure spaces (M_i, m_i) $i = 1, 2$ and $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta < +\infty$. The Lorentz spaces as will be considered in Section 3 also satisfy the requirements of the theorem. Other more general examples could be found by taking the spaces X_α and Y_β to be composed of (1) continuous regular normal scales of Banach spaces in the sense of Krein and Petunin [6]; (2) interpolation spaces constructed by the complex method of Calderón [2]; and (3) interpolation spaces constructed by the K-method of Lions–Peetre (see [1], pp. 165–191).

§ 2. The $L^{1/\alpha}(0, \infty)$ spaces. Let T be a linear operator mapping the simple functions into the space of Lebesgue measurable functions on $(0, \infty)$. The L-characteristic of T, $L(T)$, is the set of all (α, β), $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta < +\infty$, such that T can be extended to a linear operator from L_α to L_β. Two important properties of the L-characteristic are that (i) $L(T)$ is convex (Riesz convexity theorem), and (ii) if T_1 and T_2 are positive linear operators, then $L(T_1+T_2) = L(T_1) \cap L(T_2)$. A characterization of $L(T)$ is given by the following theorem.

THEOREM 2. *Let Ω be a point set in the strip $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta < +\infty$. Then Ω is the L-characteristic of some linear operator T if and only if Ω is convex and F_σ.*

The necessity of the condition has already been established.

The proof of sufficiency is based on the proof of Theorem 1 in [7]. The proof in [7] was connected to the particular spaces in question through the use of Lemma 4 of that paper. The lemma was sufficient for the purposes needed there since the L-characteristics considered had the monotone property, which implied that the non-vertical boundary of the set in question was a non-decreasing convex curve. However, in our present case, the non-vertical boundary of the convex set will consist of two pieces; a convex curve, and a concave curve. Thus, we need a more general lemma.

Let Γ_1 and Γ_2 be two distinct parallel non-vertical lines of non-zero slope in the strip $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta < +\infty$. Suppose that Γ_1 is above Γ_2. Let A be the region in the strip on and above Γ_1, and B be the region on and below Γ_2. With this notation, we give the following lemma.

LEMMA 1. *Let $M > 0$ be given. There are bounded positive kernels $K_1(s, t)$ and $K_2(s, t)$ with supports of finite measure in $(0, \infty) \times (0, \infty)$ such that the resulting integral operators K_1 and K_2 satisfy:*

(i) $\|K_1\|_{\alpha,\beta} \leqslant 1$ *for* $(\alpha, \beta) \in A$, $\|K_1\|_{\alpha,\beta} \geqslant M$ *for* $(\alpha, \beta) \in B$,

(ii) $\|K_2\|_{\alpha,\beta} \leqslant 1$ *for* $(\alpha, \beta) \in B$, $\|K_2\|_{\alpha,\beta} \geqslant M$ *for* $(\alpha, \beta) \in A$

where

$$\|K_i\|_{\alpha,\beta} = \begin{cases} \sup\limits_{x>0} \dfrac{\|K_i x\|_\beta}{\|x\|_\alpha}, & 0 \leqslant \beta \leqslant 1, \\[2ex] \sup\limits_{x>0} \dfrac{\|K_i x\|_\beta^\beta}{\|x\|_\alpha}, & 1 < \beta < +\infty. \end{cases}$$

Proof. There are actually four cases to be considered; namely, whether the lines have positive or negative slope for each of (i) and (ii). Since the estimates are similar to those obtained in Lemma 3 of [7]; we shall give the kernels for each of the cases and a brief word about the proof.

The basic kernel needed for all cases is given by

$$K(s, t) = [t^{a_1}(1+|\log t|)^{2a_1} s(1+|\log s|)^2 + t^{a_2}(1+|\log t|)^{2a_2}]^{-1}$$

where $\beta = (a_2 - a_1)\alpha + a_1$ represents one of the lines under consideration. For lines with positive slope, we take the equation to describe Γ_1 and multiply $K(s, t)$ by the characteristic functions $\chi_{(\frac{1}{N},1)}(s)\chi_{(\frac{1}{N},1)}(t)$ for case (i), or take the equation to describe Γ_2 and multiply by $\chi_{(1,N)}(s)\chi_{(1,N)}(t)$ for case (ii). In the case of negative slope, we take the equation to describe Γ_1 and multiply $K(s, t)$ by $\chi_{(1,N)}(s)\chi_{(\frac{1}{N},1)}(t)$ for (i), or take the equation to describe Γ_2 and multiply by $\chi_{(\frac{1}{N},1)}(s)\chi_{(1,N)}(t)$ for (ii). The choice of N will depend on the given bound M.

We briefly sketch the proofs. For points of the form $(0, \beta)$ or $(1, \beta)$ in the region where $\|\cdot\|_{\alpha,\beta} \leqslant 1$ is desired, one majorizes the kernel by omitting one of the terms in the denominator. Then $\left|\int_0^\infty K(s, t)x(s)\,ds\right|$ is majorized by the norm of x multiplied by a function of t whose norm is one.

For $0 < \alpha < 1$ in the region where $\|\cdot\|_{\alpha,\beta} \leqslant 1$ is desired, we apply the Kantorovitch criterion; if the function $\varphi(t) = \|K(\cdot, t)\|_{1-\alpha} \in L_\beta$, then the integral operator is bounded from L_α to L_β with $\|Kx\|_\beta \leqslant \|\varphi\|_\beta \|x\|_\alpha$. An estimate for $\varphi(t)$ is obtained by breaking up the integration in $\|K(\cdot, t)\|_{1-\alpha}$ over the intervals $[0, S]$ and $[S, \infty)$ where $S = S(t)$ is determined so that the summands in the denominator of the kernel $K(s, t)$ are equal at $s = S(t)$. This results in $\varphi(t)$ being majorized by a function of t whose norm is uniformly bounded for (α, β) in the desired region.

To show that the norm can be uniformly large in the other region, we estimate the action of the operators on the functions $x_{n,\alpha}(s) = n^{-\alpha}\chi_{(0,n)}(s)$, (or $= n^{\alpha}\chi_{\left(0,\frac{1}{N}\right)}(s)$) $0 < \alpha < 1$ $(n = 2, 3, \ldots)$. For fixed α and all n, $\|x_{n,\alpha}\|_\alpha = 1$. However, we can obtain a lower bound that will blow up uniformly on any region that is a fixed distance from the region where $\|\cdot\|_{\alpha,\beta} \leqslant 1$. The choice of the N for the kernels is made at this point.

Lemma 3 of [7] provides the prototype for the above discussion.

The proof of Theorem 2 now proceeds as in [7]. In so doing, the concave part of the boundary and the convex part of the boundary are treated separately.

§ 3. The Lorentz spaces $L(1/\alpha, r)$. In this section, we give sufficient conditions for a "characteristic" set in the case when the spaces are Lorentz spaces. Let $x(s)$ be a Lebesgue measurable function on $(0, \infty)$. We let $x^*(s)$ denote the non-increasing rearrangement of $|x(s)|$; i.e. the non-increasing function which is equimeasurable with $|x(s)|$. The Lorentz space $L(1/\alpha, r)$, $0 < \alpha < 1$, $1 \leqslant r \leqslant +\infty$, is the collection of all Lebesgue measurable functions $x(s)$ for which the quantity $\|\cdot\|^*_{\alpha,r}$ is finite where

$$(3.1) \qquad \|x\|^*_{\alpha,r} = \begin{cases} \Big\{\int\limits_0^\infty s^{\alpha r-1} x^*(s)^r\, ds\Big\}^{1/r}, & 1 \leqslant r < +\infty, \\ \sup\limits_{0<s<\infty} s^\alpha x^*(s), & r = +\infty. \end{cases}$$

The quantity $\|\cdot\|^*_{\alpha,r}$ does not define a norm on $L(1/\alpha, r)$. However, if we replace $x^*(s)$ by the function $x^{**}(s) = \frac{1}{s}\int\limits_0^s x^*(t)\,dt$ in equation (3.1), then we do obtain a norm which we will denote by $\|x\|_{\alpha,r}$.

The quantity $\|\cdot\|^*_{\alpha,r}$ is easier to work with in our computations, but $\|\cdot\|_{\alpha,r}$ is needed to define the norms of our operators. These two quantities are related by the inequalities:

$$(3.2) \qquad \|x\|_{\alpha,r} \leqslant \alpha^{1/r}(1-\alpha)^{(1/r)-1}\|x\|^*_{\alpha,r} \leqslant (1-\alpha)^{-1}\|x\|_{\alpha,r}.$$

This relation can be found in A. P. Calderón's paper [3]. A complete discussion of the Lorentz spaces and their properties can be found in the work of R. A. Hunt [4].

Let X_α, $0 < \alpha < 1$, be $L(1/\alpha, r)$, and Y_β, $0 < \beta < 1$, be $L(1/\beta, q)$, where $1 \leqslant r, q \leqslant +\infty$. Further, set $X_0 = Y_0 = L_0 = L^\infty(0, \infty)$ and $X_1 = Y_1 = L_1 = L^1(0, \infty)$. Let T be a linear operator mapping the simple functions into Lebesgue measurable functions on $(0, \infty)$. The $L_{r,q}$-characteristic of T, $L_{r,q}(T)$, is the set $\{(\alpha, \beta)\colon T\colon X_\alpha \to Y_\beta \text{ continuously}\}$. We shall employ the notation $\|T\|_{\alpha,r\to\beta,q}$ to indicate the norm of the operator T as a mapping from $L(1/\alpha, r)$ to $L(1/\beta, q)$. This norm is defined in the usual way.

For the $L_{r,q}$-characteristics, we have the following theorem.

THEOREM 3. *If Ω is an arbitrary convex, F_σ-set of the square $0 \leqslant \alpha$, $\beta \leqslant 1$, then Ω is the $L_{r,q}$-characteristic for some linear operator T.*

As in the case of $L^{1/\alpha}(0, \infty)$, we need only prove a lemma like Lemma 1. However, in this case, our norm estimates degenerate near the boundary of the square. We first give the needed integral operators to more effectively point out the bounds on the norm estimates, and then give the geometrical lemma relating the norms of these operators to the square.

Again, we shall consider parallel lines Γ_1 and Γ_2 with strictly positive or negative slope and passing through the square $0 \leqslant \alpha$, $\beta \leqslant 1$. The basic kernel for the Lorentz space argument will be

$$(3.3) \qquad K(s, t) = (1+|\log t|)^{-2}(b+|\log s|)^{-2}t^{-a_1}[s+t^{a_2-a_1}]^{-1},$$

where $\beta = (a_2-a_1)\alpha + a_1$ will be the equation of one of the lines above.

In order to illustrate the argument, we shall state one of the cases as a lemma and sketch its proof. There are two major differences from the procedure in the last section. Firstly, the norms require the use of the decreasing rearrangement, and secondly, the bounds on the norms will degenerate for (α, β) close to the boundary.

LEMMA 2. *The integral operator given by the kernel* (3.3) *multiplied by $\chi_{(0,1)}(s)\chi_{(0,1)}(t)$ with $b = 4$, $a_2 > a_1$, $a_2 > 0$, $a_1 < 1$ has the $L_{r,q}$-characteristic $\{(\alpha, \beta)\colon 0 \leqslant \alpha,\ \beta \leqslant 1,\ \beta \geqslant (a_2-a_1)\alpha + a_1\}$.*

Proof. Suppose that $1 \leqslant r \leqslant +\infty$, $0 < \alpha < 1$, and r' is such that $\frac{1}{r} + \frac{1}{r'} = 1$ (with the usual convention when $r = \infty$ or 1). Let $f_t(s) = \left(4+\log\frac{1}{s}\right)^{-2}[s+t^{a_2-a_1}]^{-1}$. Observe that $f_t(s)$ is decreasing for $s > S$ where S is defined by $2t^{a_2-a_1} = S(2+\log 1/S)$. Let $g_t(s)$ be $f_t(s)$ on $(S, 1)$ and be equal to $f_t(S)$ on $(0, S)$. For any $x(s) \geqslant 0$, by replacing $f_t(s)$ by $g_t(s)$ in the integral defining $Kx(t)$, majorizing the resulting integral by the integral obtained by replacing $x(s)$ by $x^*(s)$, introducing $s^{1-\alpha}s^{\alpha-1}$ in the integrand, and applying Hölder's inequality, we obtain

$$Kx(t) \leqslant \left(1+\log\frac{1}{t}\right)^{-2} t^{-a_1}\chi_{(0,1)}(t)\|x\|^*_{\alpha,r}\left\{\int\limits_0^1 \left(s^{1-\alpha}g_t(s)\right)^{r'}\frac{ds}{s}\right\}^{1/r'} \quad (1 < r < \infty).$$

Estimating the last integral over $(0, S)$ and $(S, 1)$, and using the relation between S and t, one can obtain

$$(3.4)\qquad Kx(t) \leqslant C(r)(1-\alpha)^{-1/r'}\|x\|^{*}_{\alpha,r}\left(1+\log\frac{1}{t}\right)^{-2} t^{-(a_2-a_1)\alpha-a_1}\chi_{(0,1)}(t)$$

where $C(r)$ is independent of t and α. Similarly, $Kx(t)$ can be estimated so that (3.4) holds for $r = \infty$, $r' = 1$. For $r = 1$, $r' = \infty$, the term $C(r)(1-\alpha)^{-1/r'}$ can be omitted.

If $1 > \beta \geqslant (a_2-a_1)\alpha + a_1$, $\beta > 0$, we have by (3.4) and (3.2)

$$(3.5)\qquad \|Kx\|_{\beta,q} \leqslant C(r, q)(1-\beta)^{-1+1/q}\alpha^{-1/r}(1-\alpha)^{-1}\|x\|_{\alpha,r}.$$

If $\beta = 1$, $0 < \alpha < 1$, then $1-(a_2-a_1)\alpha - a_1 \geqslant 0$, and we obtain

$$(3.6)\qquad \|Kx\|_1 \leqslant C(r, q)\,\alpha^{-\frac{1}{r}}(1-\alpha)^{-1}\|x\|_{\alpha,r}.$$

We shall only state the available bounds for the other cases. For $\alpha = 0$,

$$(3.7)\qquad \|Kx\|_{\beta,q} \leqslant C(q)(1-\beta)^{-1/q'}\|x\|_0$$

holds for $\beta \geqslant a_1$, and if $a_1 = 0$, then $\|Kx\|_0 \leqslant \|x\|_0$ holds. If $\alpha = 0$ and $\beta = 1$, then $\|Kx\|_1 \leqslant \|x\|_0$.

If $\beta = 0$ and $0 < \alpha < -a_1/(a_2-a_1)$, then we can obtain

$$(3.8)\qquad \|Kx\|_0 \leqslant C(r)\,\alpha^{-1/r}(1-\alpha)^{-1}\|x\|_{\alpha,r}.$$

If $\alpha = 1$, then

$$(3.9)\qquad \|Kx\|_{\beta,q} \leqslant C(q)(1-\beta)^{-1/q'}\|x\|_1$$

for $1 > \beta \geqslant a_2$, and $\|Kx\|_1 \leqslant \|x\|_1$. (Note that, in general, $a_2 \leqslant 1$ need not hold.)

The above estimates show that the desired region is in the $L_{r,q}$-characteristic. To show that no other points belong, we estimate the operator on $x_{n,\alpha}(s) = n^{\alpha}\chi_{(0,1/n)}(s)$. For $0 < \alpha < 1$, $0 < \beta < 1$, we estimate the value of $Kx_{n,\alpha}(t)$ by integrating over $(1/2n, 1/n)$ to obtain

$$Kx_{n,\alpha}(t) \geqslant Cn^{\frac{(a_2-a_1)\alpha+a_1}{a_2-a_1}}\left[1+\frac{\log n}{a_2-a_1}\right]^{-2}(4+\log 2n)^{-2},$$

for $n^{-\frac{1}{a_2-a_1}} \leqslant t \leqslant 2n^{-\frac{1}{a_2-a_1}}$ where C is independent of α, t and n. This implies

$$(3.10)\qquad \|K\|_{\alpha,r\to\beta,q} \geqslant C(1-\beta)^{1/q}(1-\alpha)^{1/r'}n^{\frac{(a_2-a_1)\alpha+a_1-\beta}{a_2-a_1}}\left[1+\frac{\log n}{a_2-a_1}\right]^{-2}(4+\log 2n)^{-2}$$

where C is independent of n, α, and β.

Similarly, for $\beta = 1$ and $(a_2-a_1)\alpha + a_1 > 1$, we obtain

$$(3.11)\qquad \|K\|_{\alpha,r\to 1} \geqslant C(1-\alpha)^{1/r'}n^{\frac{(a_2-a_1)\alpha+a_1-1}{a_2-a_1}}\left[1+\frac{\log n}{a_2-a_1}\right]^{-2}(4+\log 2n)^{-2}.$$

For $\alpha = 1$,

$$(3.12)\qquad \|K\|_{1\to\beta,q} \geqslant C(1-\beta)^{1/q}n^{\frac{a_2-\beta}{a_2-a_1}}\left[1+\frac{\log n}{a_2-a_1}\right]^{-2}(4+\log 2n)^{-2}.$$

In the event that there are points $(0, \beta)$ not in our set (which is the case when $a_1 \geqslant 0$), we estimate the action of K on $x(s) \equiv 1$ to obtain $K(1, t) \geqslant \frac{1}{2}\left(1+\log\frac{1}{t}\right)^{-2} t^{-a_1}\chi_{(0,1)}(t)$. Thus, $K(1, t)$ does not belong to $L_{\beta,q}$ for any $\beta < a_1$.

The lemma is proven.

Remark 1. Inequality (3.5) shows that we may obtain a uniform bound for the norm on that portion of $L_{r,q}(K)$ contained in the interior of the square provided that we stay away from the boundary of the square by a fixed distance. Inequalities (3.6), (3.7), (3.8) and (3.9) show that we may obtain a uniform bound for the norm on the boundary provided we stay away from the corners. Finally, the norm at those corners which lie in $L_{r,q}(K)$ is bounded by 1.

Remark 2. If we truncate the kernel in Lemma 2 by multiplying by $\chi_{\left(\frac{1}{N},1\right)}(s)\chi_{\left(\frac{1}{N},1\right)}(t)$, then it is easy to see that the $L_{r,q}$-characteristic is all of the square and that the estimates (3.10), (3.11), and (3.12) still hold for this truncation ($n \leqslant \min N, N^{\frac{1}{a_2-a_1}}$). These inequalities allow us to obtain a uniformly large lower bound for the norm of the truncation provided that $\beta \leqslant (a_2-a_1)\alpha + a_1 - \varepsilon$ $(\varepsilon > 0)$, and that we stay away from the lines $\beta = 1$, $\alpha = 1$. On these lines, we can obtain a uniformly large lower bound if we stay away from $(1, 1)$; yet, at $(1, 1)$ itself (if it is not in $L_{r,q}(K)$), we can obtain a large lower bound.

We are now able to present the lemma analogous to Lemma 1. Let Γ_1 and Γ_2 be two parallel lines in the square of positive (> 0) or negative (< 0) slope. Suppose that Γ_1 is above Γ_2 and let A be the region of the square above both lines and B be the region below both lines. Let $\delta > 0$ be a small number. Let E_δ be the subset of the square containing (a) the four corner points, (b) the boundary of the square except that portion within δ of the corners, and (c) the interior of the square except that portion within δ of the boundary. With this notation, we have the following lemma.

LEMMA 3. *Let M and δ be given positive numbers. There are bounded positive kernels $K_1(s,t)$ and $K_2(s,t)$ with supports of finite measure in $(0,\infty)\times(0,\infty)$ such that the resulting integral operators K_1 and K_2 satisfy:*

(i) $\|K_1\|_{a,r\to\beta,q}\leqslant 1$ *for* $(a,\beta)\in A\cap E_\delta$, $\|K_1\|_{a,r\to\beta,q}\geqslant M$ *for* $(a,\beta)\in B\cap E_\delta$;

(ii) $\|K_2\|_{a,r\to\beta,q}\leqslant 1$ *for* $(a,\beta)\in B\cap E_\delta$, $\|K_2\|_{a,r\to\beta,q}\geqslant M$ *for* $(a,\beta)\in A\cap E_\delta$.

Proof. Lemma 2 provides the operator in the case when the lines have positive slope and (i) is desired, if $\beta=(a_2-a_1)a+a_1$ represents Γ_1 and $\beta=(a_2-a_1)a+a_1-\varepsilon$ represents Γ_2. By the first remark following Lemma 2, we can divide by a constant so that the first condition in (i) is satisfied, and by the second remark, we can truncate the kernel in such a way that the second condition in (i) is satisfied.

The remaining cases can be obtained from the basic kernel (3.3) in a similar way. For example, the case of negative slope and (i) is obtained by letting Γ_1 be $\beta=(a_2-a_1)a+a_1$, $b=1$, and multiplying (3.3) by the characteristic functions $\chi_{(1,N)}(s)\chi_{(1/N,1)}(t)$. The procedure for establishing this is essentially the same as in Lemma 2, and the resulting inequalities will lead to the same type of remarks as those after Lemma 2.

The proof of Theorem 3 now proceeds as in [7] with the additional consideration of a concave upper boundary. The construction is also complicated by the additional condition in Lemma 3; namely, that the estimates only hold on E_δ. However, the construction utilizes only countably many operators, so that one may associate a δ_k ($\delta_k\to 0$) with each operator. Then for any fixed (a,β) in the square, the desired estimates will hold for all but a finite number of the operators. We remark that the rotating parallel lines method in the proof of Proposition 1 of [7] can be used for the vertical and horizontal portions of the boundary in the general case.

§4. Remarks. The linear operators of Lemma 1 are compact whenever they are continuous. Therefore, Theorem 2 may be stated for the "compact characteristic", $L_c(T)\equiv\{(a,\beta)\colon T\colon L_a\to L_\beta \text{ compactly}\}$.

In [7], the theorem required a monotonicity property on the set Ω. This property was equivalent to the fact that $L_a\subset L_\beta$ if $a\leqslant\beta$. If we were to consider mappings from $L_a(M)$ to $L_\beta(N)$ where M and N were chosen from $[0,1]$, $(0,\infty)$, then, depending on the choice, a monotonicity property on Ω would be required. For example, if $M=(0,\infty)$ and $N=[0,1]$, then the set Ω would necessarily have the property: $(a_0,\beta_0)\in\Omega$ implies $(a_0,\beta)\in\Omega$ for $\beta\geqslant\beta_0$.

In the case of $L_{r,q}$-characteristics, we do not necessarily know that Ω must be convex. However, by a theorem of Hunt [4], we can make the

following statement. If $r\leqslant q$, then the $L_{r,q}$-characteristics is convex, except perhaps on horizontal and vertical segments (the exceptional cases in Hunt's Theorem).

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References

[1] P. L. Butzer and H. Berens, *Semigroups of operators and approximation*, New York 1967.
[2] A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), pp. 113–190.
[3] —, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26 (1966), pp. 273–299.
[4] R. A. Hunt, *On $L(p,q)$ spaces*, L'Enseignement Math. (2) 12 (1966), pp. 249–276.
[5] M. A. Krasnosel'skii, P. P. Zabrieko, E. I. Pustyl'nik, and P. E. Sobolevskii, *Integral operators in spaces of summable functions*, Izdat. Nauka, Moscow 1966.
[6] S. G. Krein and I. Y. Petunin, *A scale of Banach spaces*, Uspeki Mat. Nauk 21 (1966), pp. 89–168.
[7] S. D. Riemenschneider, *The L-characteristics of linear operators on* $L^{1/a}([0,1])$, J. Functional Analysis, 8 (1971), pp. 405–421.

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