On characterizations of interpolable and minimal stationary processes

by

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Abstract. In this paper some characterizations of interpolable and minimal stationary processes over locally compact Abelian (LCA) groups are established by using an isomorphism theorem between the time, spectral and Hellinger spectral domains of stationary processes. In the note [15] we studied the interpolation problem on LCA groups and announced several results without proofs. We present them here in a complete form. Our results constitute a natural extension to the case of an LCA group of Kolmogorov's, Yaglom's and Salehi's results on interpolation for the simple stationary stochastic processes.

1. INTRODUCTION

Classical least squares linear prediction theory is concerned with a stationary stochastic process, that is, a family $X_t (t \in \mathbb{Z} -$ the discrete Abelian group of integers or $t \in \mathbb{R} -$ the Abelian group of reals with natural topology) of complex-valued random variables on a probability space, with zero means and finite covariances $(X_t, X_s)$ depending only on $t-s$. Then the family of random variables forms a Hilbert space and, consequently, Hilbert space methods play a key role there.

Two important cases are considered; extrapolation and interpolation. One accomplishment of the theory in both cases is an analytical characterization of those processes which are errorlessly predictable; deterministic processes in extrapolation and interpolable processes in interpolation. Prediction theory of stationary sequences (processes over $\mathbb{Z}$) has been studied by Kolmogorov in his fundamental paper [5]. Basing himself on the isomorphism between the time and the spectral domain of the univariate stationary processes (cf. [5], Th. 2.7) he has obtained analytical characterizations of deterministic and minimal processes. Yaglom [17] has obtained a characterization of interpolable processes in an analogous way. Next, these results have been extended to the multivariate case by Wiener and Masani [16], Rozanov [9] and many others.

In the present paper some analytical characterizations related to interpolation are given in the more general setting of q-variate stationary
processes over LCA groups. The study of stationary processes over LCA
groups was initiated by Kampé de Fériet [4] in 1948. We refer the reader
to Yaglom [18] for an account of the theory of stationary processes over
topological groups.

In Section 3 we obtain the spectral representations of a q-variate
stationary process and of a correlation function. We present an isomorphism
domain between the time, spectral and Hellinger-spectral processes. In Section 4 a characterization of the "space of errors" of interpolation in terms of Hellinger integrals is established. This result is suggested by recent papers of Salehi
[12]–[14]. Finally in Section 5 the interpolable and minimal processes are
categorized in terms of the spectral measure of the process. Using these
results we have obtained an extension of Bruckner's results [1] concerning univariate processes over a discrete Abelian group given in
Section 6.

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2. PRELIMINARIES

The space $L_{q}$. Let $\mathcal{B}$ be a σ-algebra of subsets of a space $\Omega$ and
let $\Phi = \{\Phi_{j}\}$ where $1 \leq i \leq p, 1 \leq j \leq q$ be a matrix-valued function on $\Omega$. A function $\Phi$ is $\mathcal{B}$-measurable if each function $\Phi_{j}$ is $\mathcal{B}$-measurable. If $\mathcal{B}$ is a nonnegative real-valued measure on $\mathcal{B}$, then by $L_{q}$ we denote the class of all $\Phi$ such that each $\Phi_{j}$ is integrable with respect to (abbreviated to "w.r.t.") $\mathcal{B}$. For $\Phi \in L_{q}$ we put $\int \Phi \, d\mathcal{B} = \int \Phi_{j} \, d\mathcal{B}_{j}$.

If $F = \{F_{j}\}$ is a $p \times q$ nonnegative Hermitian-valued measure on $(\Omega, \mathcal{B})$, then each $F_{j}$ is a nonnegative real-valued measure and each $F_{ij}$ for $i \neq j$ is a complex-valued measure on $(\Omega, \mathcal{B})$. Consequently, each $F_{ij}$ is absolutely continuous (a.e.) w.r.t. the measure $\operatorname{tr} F = \operatorname{tr} \Phi$ (tr. trace). This follows from the inequality $0 \leq F \leq (\operatorname{tr} F) I$, which is satisfied for any nonnegative Hermitian matrix.

**Lemma 2.1.** Let $F$ be as above and let $m, n$ be σ-finite nonnegative real-valued measures on $\mathcal{B}$ w.r.t. which $F$ is a.e. If $\Phi, \Psi$ are $p \times q$ matrix-valued functions on $\Phi$, then

\[
\int \Phi(\mathcal{B}) \, d\Psi \, d\mathcal{B} = \int \Phi(\mathcal{B}) \, d\mathcal{B} \, d\Psi
\]

(b) if these integrals exist, they are equal.

Proof. Let $w = m + n$. If the integral $\int \Phi(\mathcal{B}) \, d\Psi \, d\mathcal{B}$ exists, then

\[
\int \Phi(\mathcal{B}) \, d\mathcal{B} \, d\Psi = \int \Phi(\mathcal{B}) \, d\mathcal{B} \, d\Psi \, d\mathcal{B} = \int \Phi(\mathcal{B}) \, d\mathcal{B} \, d\Psi
\]

Hence the integral $\int \Phi(\mathcal{B}) \, d\Psi \, d\mathcal{B}$ exists if and only if the integral $\int \Phi(\mathcal{B}) \, d\mathcal{B} \, d\Psi$ exists. A similar argument may be used to show that the integral $\int \Phi(\mathcal{B}) \, d\mathcal{B} \, d\Psi$ exists if and only if the integral $\int \Phi(\mathcal{B}) \, d\mathcal{B} \, d\Psi$ exists. Hence (a) and (b) are proved.

Thus the following definition makes sense. Let $\Phi, \Psi$ and $F$ be as before. We say that $\Phi, \Psi$ are integrable w.r.t. $F$ if $\int \Phi(\mathcal{B}) \, d\mathcal{B} \, d\Psi$ exists. Hence (a) and (b) are proved.

Theorem 2.2 (cf. [7], p. 295 and p. 296). (a) $L_{q}$ is a Hilbert space under the inner product

\[
\langle \Phi, \Psi \rangle = \int \Phi(\mathcal{B}) \, d\Psi
\]

(b) $\Phi \in L_{q}$ if and only if there exists a Cauchy sequence in $L_{q}$ of simple functions $\Phi_{n}$ such that $\Phi_{n} \rightarrow \Phi$ everywhere (tr $F$).

Stochastic integral. Let $H$ be a Hilbert space over the field of complex
numbers with inner product $(\cdot, \cdot)$, let $H_{q} = \{1 \leq q < \infty\}$ be the Cartesian product of q copies of $H$. If $X, Y$ are in $H_{q}$ and $A, B$ are $q \times q$ matrices with complex entries, then $AX + BY$ is in $H_{q}$. For $X = (x_{1}, x_{2}, \ldots, x_{q})$ and $Y = (y_{1}, y_{2}, \ldots, y_{q})$ denote by

\[
(x, y) = [x_{1}, y_{1}, x_{2}, y_{2}, \ldots, x_{q}, y_{q}]
\]

the Gramian of $X$ and $Y$. In $H_{q}$, $X$ and $Y$ are orthogonal if $(X, Y) = 0$, i.e., if each $x_{k}$ is orthogonal to each $y_{j}$ for $i, j = 1, 2, \ldots, q$. The space $H_{q}$ with the inner product $(X, Y) = \operatorname{tr} (X, Y)$ becomes a Hilbert space (cf. [18], Section 5).

We shall call $S$ an orthogonally-scattered vector-valued measure o.v.m. on $(\Omega, \mathcal{B})$ if $S$ is a countably additive function on $\mathcal{B}$ such that

\[
1^\circ \quad S(E) \in H \quad \text{for each } E \in \mathcal{B},
\]

\[
2^\circ \quad \langle S(E), S(E') \rangle = 0 \quad \text{whenever } E \cap E' = \emptyset.
\]
If $S$ is an o.v.m. and $F(E) = (S(E), S(E))$ for each $E \in \mathcal{B}$, then $F$ is a matrix-valued measure. This is a consequence of the formula

$$F\left(\sum E_k\right) = (S(\sum E_k), S(\sum E_k)) = \sum (S(E_k), S(E_k)) = \sum F(E_k).$$

Moreover, $F$ is obviously a nonnegative Hermitian measure. We may define an integral w.r.t. an o.v.m. $S$ (stochastic integral, cf. [7], p. 297) in such a way that it has the following properties:

$$\int \Phi \, dF = \int \Phi \, dS = \int \Phi \, dS,$$

and

$$\int \Phi \, dS = \int \Psi \, dS.$$  \hspace{1cm} (2.3)

$$\int \Phi \, dS = \int \Psi \, dS = \int \Phi \, dF.$$  \hspace{1cm} (2.4)

where $\Phi, \Psi \in L_{p,p}$ and $A, B$ are $p \times p$ matrices.

Let $\mathcal{G}$ denote the class of all stochastic integrals $\int \Phi \, dS$ with $\Phi \in L_{p,p}$; then

**Lemma 2.5** (cf. [7], p. 297). The correspondence $V : \Phi \rightarrow \int \Phi \, dS$ is an isomorphism from $L_{p,p}$ onto $\mathcal{G}$ such that (2.3) and (2.4) hold.

**Hellinger integral.** The following lemma belongs to elementary matrix theory.

**Lemma 2.6** (cf. [6], p. 406). The four equations $AXA = A, XAX = X, (AX)^* = AX$ and $(XA)^* = XA$ have a unique solution for any matrix $A$.

The unique solution of this equations is called the generalized inverse of $A$ and written $A^- = A^{-1}$ and for scalars $k = 1/k$ if $k \neq 0$ and $k = 0$ if $k = 0$.

If $m$ is a $\mathcal{B}$-finite nonnegative real-valued measure on $\mathcal{B}$, then $dF(m)$ is $\mathcal{B}$-measurable matrix-valued function. The proof of the following lemma is analogous to that of Lemma 2.1.

**Lemma 2.7** (cf. [11]). Let $M$ and $N$ be $p \times q$ matrix-valued measures on $\mathcal{B}$ and let $m$ and $n$ be $\mathcal{B}$-finite nonnegative real-valued measures on $\mathcal{B}$ w.r.t. which $M, N$ and $F$ are a.e. Then

(a) the integral $\int (dM(m))(dF(m)^{-1})(dN(m)^{-1}) \, dm$ exists if and only if the integral $\int (dM(m))(dF(m)^{-1})(dN(m)^{-1}) \, dm$ exists,

(b) if these integrals exist, they are equal.

Thus, the following definition makes sense. Let $M, N, F$ and $m$ be as in the previous lemma. Then we say that $M, N$ is Hellinger integrable w.r.t. $F$ if

$$\int (dM(m))(dF(m)^{-1})(dN(m)^{-1}) \, dm \in L_{1,m}.$$  

We write

$$\int (M,N)_F = \int (dM(m))(dF(m)^{-1})(dN(m)^{-1}) \, dm.$$  \hspace{1cm} (3.1)

By $H_{p,p}$ we denote the class of all $p \times q$ matrix-valued measures $M$ on $\mathcal{B}$ for which Hellinger integral $(M,N)_F$ exists.

It is known (cf. [11]) that $H_{p,p}$ is a Hilbert space under the inner product $(M,N)_F = \text{tr}(M^*N_F)$ and that $M \in H_{p,p}$ if and only if there exists a $\mathcal{B}$-measurable, matrix-valued function $\Phi \in L_{p,p}$ such that, for each $E \in \mathcal{B}$, $M(E) = \int_E \Phi \, dF$.

### 3. Spectral Representations and the Isomorphism Theorem

Let $G$ be any LCA group with multiplication. The set of all characters of $G$, i.e., continuous homomorphism of $G$ into the group $\mathbb{T} = \{\exp(2\pi i x) : 0 \leq x < 1\}$ forms a group $\Gamma$, the dual group of $G$ (cf. [18], p. 7). In view of the duality between $G$ and $\Gamma$ (the Pontryagin duality theorem [15], p. 25) we will denote the characters by $(g, \gamma), g \in G$ and $\gamma \in \Gamma$. From the definition it follows immediately that

$$\langle g, \gamma \rangle = \langle g, 1 \rangle = 1,$$

$$\langle g^{-1}, \gamma \rangle = \langle g, \gamma^{-1} \rangle = \overline{\gamma(1)}.$$  \hspace{1cm} (3.2)

$\Gamma$ with the compact-open topology is also an LCA group. The Borel field of the LCA group is the minimal $\sigma$-field generated by the closed subsets. Throughout this paper the latter $\Gamma$ will denote the dual group of $G$ and $\mathcal{B}$ the Borel field of the dual group. On every LCA group there exists a nonnegative measure, finite on compact sets and positive on nonempty open sets, the so-called Haar measure of the group, which is translation-invariant. We denote by $dg$ and $d\gamma$ the Haar measures on $G$ and $\Gamma$.

**Definition 3.3** (cf. [15]). A $q$-variate stationary process over any LCA group $G$ is a function $(X_g)_{g \in G}$ such that

(i) $X_g \in H^q$ for all $g \in G$,

(ii) the $q \times q$ Gram matrix $(X_g, X_h) = (X_{gh^{-1}}, X_h) = K(g^{-1})$ depends only on $gh^{-1}$ for all $g, h \in G$,

(iii) the correlation function $K(g)$ is continuous on $G$.

Let $\mathcal{R}$ denote the time domain of the stationary process $(X_g)_{g \in G}$, i.e., the closed subspace of $H^q$ spanned over the elements $X_g, g \in G$ with $q \times q$ matrix-valued measures.
matrix coefficients. If \((x_0^i, x_1^i, \ldots, x_q^i)\) are the components of \(X_q^i\), then each \(x_k^i \in \mathcal{H}\) and by (ii)

\[(x_0^i, x_1^i) = k \gamma^{-1}\]

depends only on \(gh^{-1}\). Thus the \(q\)-variate process \((X_q^i)_{h \in \mathcal{H}}\) is associated with \(q\) simple processes \((x_k^i)_{h \in \mathcal{H}}\) which are stationary in the wide sense (cf. [3]). Each simple process defines in the space \(\mathfrak{M}\), the time domain of process \((x_k^i)_{h \in \mathcal{H}}\), a unitary representation of the group \(G\). Namely, the suitable unitary operators \(U_h^i\) are defined by the formulas

\[U_h^i x_k^i = x_k^{ih} \quad \text{for} \quad g, h \in G, \quad i = 1, 2, \ldots, q,
\]

and for the remaining points of the space \(\mathfrak{M}\) the operators \(U_h^i\) are defined by a natural extension. It is known (cf. [8], Lemma 2.1 or [16], p. 135) that we may take \(U_h^i = U_h^s\), so that there exists a unitary operator \(U_h^s = (U_h^1, U_h^2, \ldots, U_h^q)\) on \(\mathfrak{M}\) such that

\[X_q^i = U_h^s X_q^s = (U_h^s x_k^s)_{h \in \mathcal{H}}, \quad \text{for} \quad g \in G.
\]

Following Jajte [3] we have by the generalized theorem of Stone for the operators \(U_h^i\) the spectral representation

\[U_h^i = \int h \gamma^s I P(d\gamma),
\]

where \(P(\gamma)\) is a regular, normed and orthogonal spectral family of projectors in \(\mathfrak{M}\) defined on \(\mathfrak{S}\). If we put

\[S(\gamma) = P^0(\gamma) X_q^s,
\]

where \(P^0(\gamma) = (P_1^0, P_2^0, \ldots, P_q^0)\) is a spectral family of projectors in \(\mathfrak{M}\), then \(S(\gamma)\) is an o.v.m. on \(\mathfrak{A}\). According to (3.4) and (3.5) we have the spectral representation for the stationary process \((X_q^s)_{h \in \mathcal{H}}\)

\[X_q^i = U_h^s x_k^s = \int h \gamma I S(d\gamma), \quad \text{for} \quad g \in G.
\]

where \(I\) denotes the \(q \times q\) unit matrix.

The matrix-valued function \(F\) on \(\mathfrak{A}\), \(F(\gamma) = (S(\gamma), S(\gamma))\), is called a spectral measure of the process \((X_q^i)_{h \in \mathcal{H}}\). Clearly, \(F\) is a nonnegative Hermitian-valued measure. Henceforth, the letter \(F\) will denote the spectral measure; the spaces \(L^2\) and \(H^2\) related to \(F\)—as in Section 2—are called the spectral domains and the Hellinger-spectral domain of the process \((X_q^i)_{h \in \mathcal{H}}\), respectively.

We note that according to (3.6), (2.4) and (3.1) we obtain the spectral representation for the correlation function

\[K(g) = (X_q^i, X_q^j) = \int \langle g, \gamma \rangle IS(d\gamma), \quad \int \langle \gamma, \gamma \rangle IS(d\gamma) = \int \langle g, \gamma \rangle I dF(\gamma, \gamma) I^* = \int \langle g, \gamma \rangle I dF.
\]

The main result of this section is the following isomorphism theorem.

**Theorem 3.7.** If \((X_q^i)_{h \in \mathcal{H}}\) is a \(q\)-variate stationary process over any LCA group \(G\), with the spectral measure \(I\), then Hilbert spaces \(L^2\), \(L^2_\beta\), and \(H^2_\beta\) are isomorphic, where

(a) the mapping \(V_1: X_q^i \to \langle g, \gamma \rangle I\), I denoting the unit matrix, is an isomorphism between \(\mathfrak{M}\) and \(L^2_\beta\); 

(b) the mapping \(V_2: \Phi \to M_\beta\) for any matrix-valued function \(\Phi \in L^\beta_\beta\) with values on the set of measures \(M_\beta\) on \(\mathfrak{A}\) given by \(M_\beta(\mathfrak{A}) = \prod_\gamma \Phi(d\gamma)\), is an isomorphism between \(L^2\) and \(H^2_\beta\).

**Proof.** (a) Let \(X_q^i = X_q^s\) in \(\mathfrak{M}\) then according to (2.4) and (3.2)

\[\int I dF = (X_q^i, X_q^j) = (X_q^s, X_q^s) = \int \langle g, \gamma \rangle I dF, \quad \int \langle \gamma, \gamma \rangle I dF = \int \langle g, \gamma \rangle I dF,
\]

so that \(\langle g, \gamma \rangle I = \langle h, \gamma \rangle I\) in \(L^2_\beta\), and the mapping \(V_1: \langle X_q^i, g \cdot h^{-1} \rangle \to L^2_\beta\), is well defined and may be extended in a natural way to a mapping of \(\mathfrak{M}\) into \(L^2_\beta\). If we prove that \(\mathfrak{M} = \mathfrak{S}\), then by Lemma 2.5 and the obvious equality \(V_1 = V_2\) the proposition (a) will be proved. Since \(\langle g, \gamma \rangle I\) \(L^2_\beta\), then from the representation (3.6) it follows that \(\mathfrak{S} = \mathfrak{S}\).

Conversely, let \(\Phi \in L^2\) and let \(F = \int \Phi dS\). By Theorem 2.2, there exists in \(L^2\) a Cauchy sequence of simple functions \(\Phi_n \to \Phi\) everywhere. If we put \(Y_n = \int \Phi_n dS\), then clearly a sequence of \(Y_n\) is convergent in \(\mathfrak{S}\). Since \(\Phi_n\) is a simple function,

\[Y_n = \int \Phi_n dS = \sum_{A \in \mathfrak{A}} \int \Phi(A) S(A) = \sum_{A \in \mathfrak{A}} \int \Phi(A) \mathcal{P}^0(\gamma) X_q^s,
\]

where \(A^0\) are \(q \times q\) matrices, \(\mathcal{P}^0(\gamma) \in \mathfrak{A}\), and \(\mathcal{P}^0(\gamma)\) is a spectral family of projectors in \(\mathfrak{M}\). Of course, \(\mathcal{P}^0(\gamma) X_q^s \in \mathfrak{S}\), thus \(Y_n \in \mathfrak{S}\) and consequently \(Y \in \mathfrak{S}\). It follows that \(\mathfrak{S} \subseteq \mathfrak{M}\).

Part (b) is a special case of the isomorphism theorem (cf. [11], Th. 1) between the space \(L^2_\beta\) and \(H^2_\beta\) on any space \(\Omega\).
4. THE SPACE \( R_c \)

Let \((X_p)_{p \in G}\) be a \(q\)-variate stationary process over LCA group \(G\) and let \(G\) be any proper and nonempty compact subset of \(G\). If \(R_{G,c}\) denotes the closed linear subspace of \(L^1(G)\) spanned by \(X_p, g \in G\), then we denote

\[
R_c = R_c \oplus R_{G,c}.
\]

The space \(R_c\) plays an essential role in an interpolation problem, which will be considered in the next section. The purpose of this section is a construction of isomorphism \(T_\phi\) on \(R_c\) into \(L_{A,G}\) and a characterization of the range of \(T_\phi\) (Th. 4.9). First we prove several lemmas.

Let \(\mathcal{D}(G)\) denote the set of all \(p \times q\)-matrix-valued functions \(\Phi\) on \(G\) which are representable in the form

\[
\Phi(g) = \int G \langle g, \gamma \rangle dM, \quad g \in G,
\]

where \(M\) is a \(p \times q\) matrix-valued measure on \(G\), i.e., each \(M_{ij}\) is a regular, complex-valued measure on \(G\). We note that Bochner's theorem (cf. [10], p. 19) in combination with the Jordan decomposition theorem (cf. [2], p. 309) implies that \(\mathcal{D}(G)\) is exactly the set of all matrix-valued functions where entries are finite linear combinations of continuous positive-definite functions on \(G\).

For all \(p \times q\) matrix-valued functions \(\Phi \in L_{A,G}\), the \(p \times q\) matrix-valued function \(\hat{\Phi}\) defined on dual group \(\hat{G}\) by

\[
\hat{\Phi}(\gamma) = \int G \langle g, \gamma \rangle \Phi(g) dg
\]

is called the Fourier transform of \(\Phi\).

**Lemma 4.1.**

(a) If \(\Phi \in L_{A,G} \cap \mathcal{D}(G)\), then \(\hat{\Phi} \in L_{A,\hat{G}}\).

(b) If the Haar measure of \(G\) is fixed, the Haar measure of \(\hat{G}\) can be normalized so that the inversion formula

\[
\Phi(g) = \int \hat{G} \langle g, \gamma \rangle \hat{\Phi}(\gamma) d\gamma, \quad g \in G
\]

is valid for every \(\Phi \in L_{A,G} \cap \mathcal{D}(G)\).

**Proof.** We note that in virtue of the definition of a matrix-valued integral w.r.t. a scalar measure it remains to prove that

\[
\hat{\Phi}(g) = \int \langle g, \gamma \rangle \hat{\Phi}(\gamma) d\gamma, \quad g \in G
\]

where the Haar measure \(d\gamma\) is suitably normalized, \(1 \leq i \leq p, 1 \leq j \leq q\).

\(g \in G\) and \(\gamma \in \Gamma\). Since \(\Phi \in \mathcal{D}(G)\), then \(\Phi(g) = \int \langle g, \gamma \rangle dM\). If \(M\) is a \(\sigma\)-finite nonnegative real-valued measure on \(A\) w.r.t. which \(M\) is a.e., then

\[
\Phi(g) = \int \langle g, \gamma \rangle I dM = \int \langle g, \gamma \rangle I (dM/dm) dm
\]

\[
= \int \langle g, \gamma \rangle I (dM_{ij}/dm) dm = \int \langle g, \gamma \rangle dM_{ij},
\]

and so \(\hat{\Phi}(g) = \int \langle g, \gamma \rangle dM_{ij}\), where \(M_{ij}\) is a bounded regular complex-valued measure on \(A\). Moreover \(\hat{\Phi} \in L_{A,\hat{G}}\) according to the assumption \(\Phi \in L_{A,G}\) and hence by the inversion formula for complex-valued functions (cf. [10], p. 22) we obtain \((*)\) and \((***)\).

From now on, it will always be tacitly assumed that the Haar measures \(d\gamma\) and \(d\gamma\) are so adjusted that the inversion formula holds.

**Lemma 4.2.**

(a) If \(X \in \mathcal{M}\), then \((X, X) \in \mathcal{D}(G)\).

(b) If \(X \in \mathcal{M}\), then \((X, X) \in L_{A,G}\).

**Proof.** Let \(X\) be in \(\mathcal{M}\) and let \(\Phi \in L_{A,G}\) such that \(\langle V, X \rangle = \Phi\), where \(V\) is an isomorphism from \(\mathcal{M}\) onto \(L_{A,G}\) as in Theorem 3.7. We have

\[
(X, X) = \langle V, X, V, X \rangle = \langle \Phi, \gamma, \gamma \rangle I = \int \langle g, \gamma \rangle \Phi dF.
\]

If \(m\) is a \(\sigma\)-finite nonnegative real-valued measure on \(A\) w.r.t. which \(F\) is a.e., then for each \(E \in \mathcal{M}\) we have

\[
M_m(E) = \int E (d\Phi F) = \int E (dF dM) dm.
\]

Consequently, \(dM_{ij}/dm) = \Phi (dF/\Phi) dm\) and therefore

\[
\int \langle g, \gamma \rangle \Phi dF = \int \langle g, \gamma \rangle \Phi (dF/\Phi) dm = \int \langle g, \gamma \rangle dM_{ij}.
\]

It remains to prove that the entries of \(M_m\) are regular measures on \(A\).

From Section 2 it is clear that we may put \(m = \tau F\), where \(F\) is the spectral measure of the process \((X_{ij})_{ij}\). Each measure \(F_{ij}\) for \(i = 1, \ldots, q\) as the spectral measure of a simple stationary process is regular and nonnegative. This is a consequence of Bochner's theorem (cf. [3]). Thus \(m = \sum_{i=1}^{q} F_{ij}\) is also a nonnegative regular measure.

Let \(\Phi_m = \Phi (dF/\Phi) dm\) and let \(M_{ij} = [M_{ij}]\) where \(1 \leq i \leq p, 1 \leq j \leq q\). Then for each \(E \in \mathcal{M}\), \(m(E) = \int \Phi_m dm\). Since \(m\) is regular, then \(M_m\) are complex regular measures on \(A\). Thus part (a) is proved.
For $X \in \mathbb{R}$ the matrix-valued function $(X, X)$ is by the definition 3.3 (iii) a continuous function on $G$. Moreover, if $X \in \mathcal{R}_0$, then $(X, X) = 0$ for $g \not\in C$. Hence for $X \in \mathcal{R}_0(X, X)$ is a continuous function with a compact support and so $(X, X) \in L_{a_0}$.

Now for each $X \in \mathcal{R}_0$ we let

$$P_X(\gamma) = \int \langle g, \gamma \rangle (X, X) \, dg, \quad \gamma \in \Gamma.$$  \hfill (4.4)

$P_X(\gamma)$ is the Fourier transform of $(X, X)$. Its properties are given in the following lemma.

**Lemma 4.5.**
(a) $P_X(\gamma) \in L_{a_0}$,
(b) If for each $E \in \mathcal{B}$ we define $N_{P_X}(E) = \int P_X(\gamma) \, d\gamma$, then $N_{P_X} = M_\Phi$, where $M_\Phi = V_\Phi \mathcal{Y}(X)$ (see Theorem 3.7),
(c) $N_{P_X} \in H_{a_0}$.

Proof. (a) If $X \in \mathcal{R}_0$, then from Lemma 4.2 $(X, X) \in L_{a_0} \cap \mathcal{B}(G)$. Hence by Lemma 4.1 (a) $P_X \in L_{a_0}$.

(b) Simultaneously, by Lemma 4.1 (b), $(X, X) = \int \langle g, \gamma \rangle P_X(\gamma) \, dg$.

Since the definition of $N_{P_X}$ implies $dN_{P_X}(d\gamma) = P_X$,

$$(X, X) = \int \langle g, \gamma \rangle (dN_{P_X}(d\gamma)) = \int \langle g, \gamma \rangle I dN_{P_X}.$$  

On the other hand, by (4.3) we have $(X, X) = \int \langle g, \gamma \rangle dM_\Phi$. Hence for each $g \in \mathcal{G}$ we get

$$\int \langle g, \gamma \rangle I dN_{P_X} = \int \langle g, \gamma \rangle I dM_\Phi.$$  \hfill (5.1)

It follows according to the uniqueness theorem for the inverse Fourier-Stieltjes transform of measure (cf. [10], p. 17) that $N_{P_X} = M_\Phi$.

(c) From Theorem 3.7 (b) $M_\Phi \in H_{a_0}$; thus from (b) $N_{P_X} \in H_{a_0}$. \hfill \blacksquare

**Definition 4.6.** Let $C$ be any proper and nonempty compact subset of $G$. Then

(a) $\mathcal{R}_C$ will denote the set of $q \times q$ matrix-valued functions $Q(g)$ on $G$ such that

(*)

$Q(g) \in L_{a_0} \cap \mathcal{B}(G),$

(**)

$\text{supp } Q(g) \subset C.$

(b) $\mathcal{R}_\Delta$ will denote the set of Fourier transforms of all functions from $\mathcal{R}_C$. The properties of the set $\mathcal{R}_\Delta$ are given in the following two lemmas.

**Lemma 4.7.**
(a) If $X \in \mathcal{R}_\Delta$, then $P_X \in \mathcal{R}_\Delta$, where $P_X$ is as in (4.4).

(b) If $G$ is a discrete Abelian group, then $\mathcal{R}_\Delta$ is exactly the set of all trig-polynomials $W(\gamma) = \sum_{k=1}^n a_k \langle \gamma, \gamma \rangle$ with matrix coefficients.

Proof. (a) If $X \in \mathcal{R}_\Delta$ then by Lemma 4.2 $(X, X) \in L_{a_0} \cap \mathcal{B}(G)$; moreover, the support of $(X, X)$ is in $C$. Thus $(X, X) \in \mathcal{R}_\Delta$ and consequently by (4.4) $P_X(\gamma) \in \mathcal{R}_\Delta$.

(b) follows readily from the fact that in discrete topology each compact subset is finite. \hfill \blacksquare

**Lemma 4.8.** If $P(\gamma) \in \mathcal{R}_\Delta$, then

(a) $P(\gamma) \in L_{a_0}$,

(b) $N_P$ is a matrix-valued measure on $G$ if we put for each $E \in \mathcal{B}$

$$N_P(E) = \int E P(\gamma) \, d\gamma,$$

(c) $Q(g) = \int \langle g, \gamma \rangle P(\gamma) \, d\gamma$.

Proof. Since $P(\gamma)$ is by definition a Fourier transform of the matrix-valued function $Q(g)$ which is in $L_{a_0} \cap \mathcal{B}(G)$, (a) and (c) follow immediately from Lemma 4.1. According to (a) the definition of a measure $N_P$ in (b) makes sense. \hfill \blacksquare

Now we define the operator $T_C$ on $\mathcal{R}_\Delta$ into $H_{a_0}$. For each $X \in \mathcal{R}_C$

$$T_C X = N_{P_X},$$

where $N_{P_X}$ is as in Lemma 4.5 (b). A construction of $T_C$ is presented in the following diagram:
We are now ready to give a characterization for the range of the operator $T_C$.

**Theorem 4.9.** (a) $T_C$ is a single-valued linear operator on $\mathfrak{H}_C$, i.e., if $X, Y \in \mathfrak{H}_C$ and $A, B$ are $q \times q$ matrices, then

$$T_C(AX + BY) = AT_C X + BT_C Y.$$

(b) $T_C$ is an isometry on $\mathfrak{H}_C$ into $H_{A,\mathfrak{P}}$, i.e., for $X, Y \in \mathfrak{H}_C$

$$(X, Y) = (T_C X, T_C Y).$$

(c) The range of $T_C$ is a closed subspace of the Hilbert space $H_{A,\mathfrak{P}}$.

(A) The range of $T_C$ consists of all matrix-valued measures $N_X$ for which the Hellinger integral $\int dN_X dN'_{X'}$, $d\mathfrak{P}$ exists, where $P \in \mathfrak{P}_C$ and $N_{X'}$ is related to $P$ as in Lemma 4.8 (b).

**Proof.** (a) Let $X, Y \in \mathfrak{H}_C$ and $A, B$ be $q \times q$ matrices; then $Z = AX + BY \in \mathfrak{H}_C$ and for each $E \in \mathfrak{S}$ we obtain

$$N_Z(E) = \int_E Z_Y d\mathfrak{P} = \int_E (AX + BY, Y) d\mathfrak{P} = \int_E (A, Y) (X, Y) d\mathfrak{P} = AN_X(E) + BN_{X'}(E).$$

Consequently, $T_C Z = AT_C X + BT_C Y$.

(b) Let $X, Y \in \mathfrak{H}_C$. According to Theorem 3.7 there exist matrix-valued functions $\Psi, V$ in $L_{A,\mathfrak{P}}$ and matrix-valued measures $M_{\Psi}, M_V$ in $H_{A,\mathfrak{P}}$ such that

$$(X, Y) = (\Psi, V) = (M_\Psi, M_V).$$

But from Lemma 4.5 (b) $M_{\Psi} = N_{X'}$ and $M_V = N_{X'}$, hence

$$(M_\Psi, M_V) = (N_{X'}, N_{X'}).$$

and consequently

$$(X, Y) = (T_C X, T_C Y).$$

(c) Since $\mathfrak{H}_C$ is a closed subspace of $\mathfrak{H}$ and since by (b) $T_C$ is an isometry on $\mathfrak{H}_C$ into $H_{A,\mathfrak{P}}$, the range of $T_C$ is a closed subspace of $H_{A,\mathfrak{P}}$.

(d) Let $X \in \mathfrak{H}_C$ and $T_C X = N_{X'}$. From Lemma 4.5 (c) $N_{X'}$ is in $H_{A,\mathfrak{P}}$ and consequently the Hellinger integral $\int dN_{X'} dN'_{X''}, d\mathfrak{P}$ exists. Moreover, from Lemma 4.7 (a) $P_{X'}$ is in $\mathfrak{P}_C$.

Conversely, let $N$ be a matrix-valued measure on $\mathfrak{S}$ given for each $E \in \mathfrak{S}$ by $N(E) = \int P_X d\mathfrak{P}$, where $P_X$ is in $\mathfrak{P}_C$ and the Hellinger integral $\int dN_X dN'_{X'}, d\mathfrak{P}$ exists. Since $N_X$ is in $H_{A,\mathfrak{P}}$, then there exists (cf. [11], Th. 1) a $\mathfrak{S}$-measurable matrix-valued function $\Phi$ in $L_{A,\mathfrak{P}}$ such that for each $E \in \mathfrak{S}$ $N_X(E) = \int (X, Y) \Phi d\mathfrak{P} = \int (X, Y) \Phi(d\mathfrak{P}) d\mathfrak{P}$, where $m$ is an arbitrary $\mathfrak{S}$-finite nonnegative real-valued measure on $\mathfrak{S}$ w.r.t. which $\Phi$ is a.c. By the Theorem 3.7 we may choose $Y \in \mathfrak{M}$ such that $V Y = \Phi$. Thus according to Theorem 3.7 (a) and Lemma 4.8 we obtain

$$N_Y(E) = \int (X, Y) \Phi d\mathfrak{P} = \int (X, Y) \Phi(d\mathfrak{P}) d\mathfrak{P} = \int (X, Y) \Phi d\mathfrak{P} = \int (X, Y) \Phi d\mathfrak{P} = \int (X, Y) \Phi d\mathfrak{P}.$$

Hence by Definition 4.6 the support of $(X, Y')$ is contained in $C$. Since $X \in \mathfrak{M}$ and $(X, Y') = 0$ for $g \in C$ we conclude that $Y \in \mathfrak{M}_C$. But also from equation $(Y, X') = Q(g)$ we obtain for each $E \in \mathfrak{S}$

$$N_{X'}(E) = \int (X, Y') = \int (X, Y') Q(g) d\mathfrak{P} = \int (X, Y') Q(g) d\mathfrak{P} = \int (X, Y') = N_{X'}(E).$$

Hence it follows that we find $Y \in \mathfrak{M}_C$ such that $N_Y = T_C Y$. □

5. INTERPOLABLE AND MINIMAL PROCESSES

Let $(X_t)_{t \in \Delta}$ be a $\Psi$-variate stationary process over an LCA group $G$ and let $C$ be a proper and nonempty compact subset of $G$. Suppose that only $X_g$ for $g \in C - \{0\}$ are known. The prediction problem for a compact subset $C$ will be called interpolation (for classical processes cf. [9], p. 134 and p. 186).

We say that $\mathfrak{M}_{g, C}$ is the "space of observation!" of the process $(X_t)_{t \in \Delta}$ and consider each $X \in \mathfrak{M}_{g, C}$ to be a prediction of the process $(X_t)_{t \in \Delta}$ based on observations of the outside of a compact subset $C$. The error of this prediction may be expressed with the aid of the norm in $\mathfrak{M}$. That is to say, we are looking for a predictor $X_g$ satisfying $X_g \in \mathfrak{M}_{g, C}$ and

$$||X_g - X||_g = \min_{Y \in \mathfrak{M}_{g, C}} ||X_g - Y||_g.$$

It follows that $X_g$ is the projection of $X_g$ onto $\mathfrak{M}_{g, C}$. We note that the closed subspace of $\mathfrak{M}$ spanned by all $X_g - X_g$, for $g \in C$ is exactly the space $\mathfrak{M}_{g, C}$, which was defined in the previous section. Hence, the space $\mathfrak{M}_{g, C}$ plays the role of the "space of errors!" of interpolation.

**Definition 5.1.** (cf. [12] and [15]). We say that

(a) $C$ is interpolable w.r.t. $(X_t)_{t \in \Delta}$ if $\mathfrak{M}_{g, C} = 0$. 

(b) \((X_p)_p G\) is interpolable if each proper and nonempty compact subset of \(G\) is interpolable w.r.t. \((X_p)_p G\).

e) \((X_p)_p G\) is minimal if, for each \(h \in G\), \((b)\) is not interpolable w.r.t. \((X_p)_p G\).

Now we give an analytical characterization of interpolable and minimal processes. We start from an analytical characterization of interpolable subsets.

**Theorem 5.2.** A compact subset \(C\) of an LCA group \(G\) is interpolable w.r.t. a \(q\)-variately stationary process \((X_p)_p G\) with the spectral measure \(\mathbb{P}\) if and only if for every matrix-valued function \(P \in \mathbb{P}\), the Helinger integral \(\int dN_p dN'_p / d\mathbb{P}\) is zero or does not exist.

**Proof.** Necessity. Let \(T_G\) be an isomorphism on \(\mathbb{R}_0\) onto \(H_{p_k}\) defined in the previous section. If \(G\) is interpolable w.r.t. \((X_p)_p G\) then by definition \(\mathbb{R}_0 = 0\). Hence by Theorem 4.9 (b) the range of \(T_G\) is a null-point in \(H_{p_k}\) and so by Theorem 4.9 (d) for each matrix-valued function \(F\) in \(\mathbb{P}\) the measure \(N_p\) is a null-point in \(H_{p_k}\) or \(N_p \notin H_{p_k}\).

In other words, for every measure \(N_p\) the Helinger integral \(\int dN_p dN'_p / d\mathbb{P}\) is zero or does not exist.

Sufficiency. Let \(X\) be in \(\mathbb{P}\) then by Lemma 4.5 (c) \(N_p X_{p_k} H_{p_k}\) and by Lemma 4.7 (a) \(F X_{p_k} \mathbb{P}\). Since for each matrix-valued function \(P\) from \(\mathbb{P}\) a suitable measure \(N_p\) is a null-point in \(H_{p_k}\) or \(N_p \notin H_{p_k}\), it follows that \(N_p X_{p_k}\) is a null-point in \(H_{p_k}\). Consequently, by Theorem 4.9 (b)

\[
0 = (N_p X_{p_k})_p = (T_G X_{p_k} T_G X_{p_k}) = (X, X).
\]

Since \(X\) is arbitrary in \(\mathbb{P}\), it follows that \(\mathbb{R}_0 = 0\) and by definition the set \(G\) is interpolable w.r.t. \((X_p)_p G\).

**Definition 5.3.** (a) \(\Omega\) will denote the set of \(q \times q\) matrix-valued functions \(Q(g)\) on \(G\) such that

\[
Q(g) \in L_{2,0} \cap \mathbb{P}(G),
\]

\((*)\) the support of \(Q(g)\) is contained in any proper compact subset of \(G\).

(b) \(\mathbb{P}\) will denote the set of Fourier transforms of all functions from \(\Omega\).

We note that according to Definition 4.6 \(\mathbb{P} = \bigcup \mathbb{P}\), where \(G\) is any proper compact subset of \(G\). Hence the following generalization of Salehi’s results (cf. [15], Th. 2, [16], Th. 3 and [14], Th. 3.6) concerning \(q\)-variately stationary processes over the group \(Z\), \(Z^n\) and \(R\) follows directly from Theorem 5.2.

**Corollary 5.4.** A \(q\)-variately stationary process \((X_p)_p G\) over an LCA group \(G\), with spectral measure \(\mathbb{P}\) is interpolable if and only if for any matrix-valued function \(P \in \mathbb{P}\) the Helinger integral \(\int dN_p dN'_p / d\mathbb{P}\) is zero or does not exist.

Now, for discrete Abelian groups we obtain the following group analogue of Yaglom’s result (cf. [17]) concerning simple discrete parameter processes, i.e., with \(q = 1\) and \(G = Z\).

**Theorem 5.5.** Let \(G\) be a discrete Abelian group and \((X_p)_p G\) a \(q\)-variately stationary process over \(G\), where the spectral measure \(F\) is a.e. w.r.t. the Haar measure \(dy\) on dual group \(G^*\). Then \((X_p)_p G\) is interpolable if and only if for any trig-polynomials \(W(\gamma)\) with matrix coefficients, the integral \(\int W(\gamma)(dF \gamma)^* W(\gamma)\gamma^* dy\) is zero or does not exist.

**Proof.** Since \(G\) is discrete, then according to the Pontryagin theorem (cf. [10], p. 9) the dual group of \(G\) is compact and consequently the Haar measure \(dy\) is finite on \(G^*\). Since by assumption \(F\) is a.e. w.r.t. \(dy\), the Radon-Nikodym derivative \((dF \gamma)^*\) exists. By Lemma 4.7 (b) \(\mathbb{P}\) is exactly the set of all trig-polynomials \(W(\gamma)\) with matrix coefficients, on dual group \(G^*\). Let \(X\) for \(E \in \mathbb{P}\) \(N_p(E) = \int W(\gamma)\gamma^* d\gamma\), then \((dN_p)^* d\gamma\). Thus we get the formula

\[
\int dN_p dN'_p / d\mathbb{P} = \int \left(\int dN_p dN'_p / d\mathbb{P}\right)(dF \gamma)^* (dF \gamma)^* dy
\]

\[= \int W(\gamma)(dF \gamma)^* W(\gamma)\gamma^* dy.
\]

Now our assertion is a consequence of Corollary 5.4.

We note that by Definition 3.3 (ii) minimal processes exist only on discrete groups. Let \(G\) be a discrete Abelian group. Let \(Y_g\) denote the orthogonal projection of \(X_g\) on \(\mathbb{R}_0\) and let \(J\) denote the projection matrix on the subspace \(S^0\) of \(q\)-tuples of complex numbers onto the range of \((Y_g, Y_g)\) in the privileged basis of \(S^0\). The next theorem is an extension to the case of an LCA group of Salehi’s results on processes over \(Z\) and \(Z^n\) (cf. [13], Th. 3 and [14], Th. 3.7).

**Theorem 5.6.** Let \(G\) be a discrete Abelian group and \((X_p)_p G\) a \(q\)-variately stationary process over \(G\), with spectral measure \(\mathbb{P}\). Then \((X_p)_p G\) is minimal if and only if the Helinger integral \(\int dN_p dN'_p / d\mathbb{P}\) is zero, where, for each \(E \in \mathbb{P}\) \(N_p(G) = \int d\gamma\).

**Proof.** Since the process \((X_p)_p G\) is stationary, \(\mathbb{R}_0 = 0\) if and only if for each \(g \in G\) \(\mathbb{R}_0 = 0\). Hence it remains to prove that \(Y_g\) is not a null-point in \(\mathbb{R}_0\). Let, for each \(E \in \mathbb{P}\) \(N_p(E) = \int (Y_g, Y_g) d\gamma = (Y_g, Y_g)\) \(\int d\gamma\).

Let \(F_{Y_g}\) be as in (4.4) then according to the definition of \(Y_g\) we get \(F_{Y_g}(\gamma) = \int (\gamma, \gamma)(Y_g, Y_g) d\gamma = (Y_g, Y_g)\). It follows that \(T_G Y_g = N_{Y_g}\).

Consequently, by Theorem 4.9 (b) \((Y_g, Y_g) = (N_{Y_g}, N_{Y_g})\) \(\mathbb{P}\). From Lemma
2.6. we obtain

$$(Y_1, Y_2)^{-} = (Y_1, Y_2)^{-} (Y_1, Y_2)^{-} (Y_1, Y_2)^{-}$$

thus

$$(Y_1, Y_2)^{-} = (Y_1, Y_2)^{-} (N_{Y_1}, N_{Y_2}) (Y_1, Y_2)^{-} = (Y_1, Y_2)^{-} (Y_1, Y_2)^{-} (N_{Y_2}, N_{Y_2}) (Y_1, Y_2)^{-}$$

where for each \( E \in \mathcal{B} \) we have

$$\int_E \int \text{d}y I$$

and \( I \) denotes the unity matrix.

Since \((Y_1, Y_2)^{-} (Y_1, Y_2) = I\), we get

$$(Y_1, Y_2)^{-} = \int_N (N_1, N_2) \int (N_1, N_2) = \int \int \text{d}N_1 \text{d}N_2 \text{d}F.$$ 

From Lemma 2.6 it follows also that \((Y_1, Y_2) \neq 0\) if and only if \((Y_1, Y_2)^{-} \neq 0\). Hence it follows that \(Y_2\) is a null-point in \(R_{Y_1}\) if and only if \(\int \int \text{d}N_1 \text{d}N_2 \text{d}F \neq 0\).

Finally we obtain a natural extension to the case of an LCA group of a result due to Kolmogorov (cf. [5]) concerning univariate processes, which was generalized by Rozanov (cf. [9], p. 138) to the multivariate case.

Theorem 5.7. Let \(G\) be a discrete Abelian group and \((X_g)_{g \in G}\) be a \(q\)-variate stationary process over \(G\), where the spectral measure \(F\) is a.c. w.r.t. the Haar measure on \(G\). Then \((X_g)_{g \in G}\) is minimal if and only if the integral \(\int \text{tr}(DF) \text{d}F\) exists.

Proof. From Definition 5.1 (c) for each \(g \in G\) the set \(\{g\}\) is not interpolable w.r.t. \((X_g)_{g \in G}\) if and only if \(\int \text{tr}(DF) \text{d}F\) exists.

A q-variate stationary process \((X_g)_{g \in G}\) is called I-regular if and only if \(\int \text{tr}(DF) \text{d}F\) is minimal.

Therefore, \((X_g)_{g \in G}\) is I-regular if \(\int \text{tr}(DF) \text{d}F\) is minimal.

6. REGULARITY AND SINGULARITY

The interpolation problem for univariate stationary processes over any discrete Abelian group was studied by Bruckner in [1]. In this section we shall demonstrate how the results of the previous section extend his results to the case of \(q\)-variate stationary processes over any LCA group.

Theorem 6.1. If the family \(I\) is closed under translations for all \(g \in G\) (i.e., \(A I = I\) and \(g \in G\) implies \(Ag = (hg) \in I\)), then every \(q\)-variate stationary process \((X_g)_{g \in G}\) is called \(I\)-regular if \(\int \text{tr}(DF) \text{d}F\) is minimal and \(N_{\text{tr}} \neq 0\) over any LCA group \(G\).

Thus any \(q\)-variate stationary process \((X_g)_{g \in G}\) over any LCA group \(G\) is a sum of two \(q\)-variate stationary processes \((X_g^1)_{g \in G}\) and \((X_g^2)_{g \in G}\) (i.e., \(g \in G\), \(X_g = X_g^1 + X_g^2\)) such that the following statements are true:

(i) for the spaces \(R^1\) and \(R^2\) spanned by \((X_g^1)_{g \in G}\) and \((X_g^2)_{g \in G}\) respectively, we have

\[ \mathbb{R} = \mathbb{R}^1 \oplus \mathbb{R}^2 \]

(ii) \((X_g^1)_{g \in G}\) is I-regular.

(iii) \((X_g^2)_{g \in G}\) is I-regular.
The proof of this result is essentially the same as in the classical case (cf. [9], p. 75) and we omit it.

Let \( G \) denote the family of complements of all compact subsets in \( G \). If \( G \) is a discrete group then \( I_{\Phi} \) coincides with the family \( \text{Int}_G \) of complements of all finite subsets. Denote by \( I_{\Phi} \) the family of complements of all singletons of \( G \).

**Lemma 6.3** Let \((X_{\alpha})_{\alpha \in G}\) be a \( \Phi \)-variate stationary process over any LCA group \( G \); then

(a) \((X_{\alpha})_{\alpha \in G}\) is \( I_{\Phi} \)-singular if and only if it is interpolable,

(b) \((X_{\alpha})_{\alpha \in G}\) is not \( I_{\Phi} \)-singular if and only if it is minimal.

**Proof.** (a) If \((X_{\alpha})_{\alpha \in G}\) is \( I_{\Phi} \)-singular then for each compact subset \( C \) of a group \( G \) we have \( \mathcal{M} = \mathcal{M}_G \). Hence \( \mathcal{M} = 0 \) and consequently by Definition 5.1 (b) \((X_{\alpha})_{\alpha \in G}\) is interpolable. Conversely, if for each compact subset \( C \) \( \mathcal{M} = 0 \) then \( \mathcal{M} = \mathcal{M}_G \) and \( \bigcap_{\alpha \in G} \mathcal{M} = \mathcal{M} \). Hence \((X_{\alpha})_{\alpha \in G}\) is \( I_{\Phi} \)-singular.

Of course, part (b) may be proved in the same way.

**Corollary 5.4.** Theorem 5.6 and Lemma 6.2 immediately yield the following

**Corollary 6.3.** Let \((X_{\alpha})_{\alpha \in G}\) be a \( \Phi \)-variate stationary process over any LCA group \( G \), with the spectral measure \( P \); then

(a) \((X_{\alpha})_{\alpha \in G}\) is \( I_{\Phi} \)-singular if and only if for any matrix-valued function \( P \in \mathcal{B} \) the Hellinger integral \( \int_{\mathcal{B}} dP \mathbb{E}_{X_{\alpha}} \mathcal{M}_\alpha \mathcal{M}_\alpha^* d\mathcal{M} = 0 \) is zero and does not exist.

(b) \((X_{\alpha})_{\alpha \in G}\) over a discrete group is not \( I_{\Phi} \)-singular if and only if the Hellinger integral \( \int_{\mathcal{B}} dP \mathbb{E}_{X_{\alpha}} \mathcal{M}_\alpha \mathcal{M}_\alpha^* d\mathcal{M} \neq 0 \).

Similarly, Theorems 5.5 and 5.7 with Lemma 6.2 yield the following multivariate analogue of the results obtained by Bruckner (cf. [1], Th. 4.1 and Th. 5.2).

**Corollary 6.4.** Let \( G \) be a discrete Abelian group and \((X_{\alpha})_{\alpha \in G}\) a \( \Phi \)-variate stationary process, where the spectral measure \( P \) is a.c. w.r.t. the Haar measure \( d\gamma \) on dual group \( \hat{G} \); then

(a) \((X_{\alpha})_{\alpha \in G}\) is \( I_{\Phi} \)-singular if and only if for any trig-polynomial \( W(\gamma) \) with matrix coefficients the integral \( \int W(\gamma)d\gamma(\mathbb{E}_{X_{\alpha}} W(\gamma))d\gamma \) is zero or does not exist.

(b) \((X_{\alpha})_{\alpha \in G}\) is not \( I_{\Phi} \)-singular if and only if the integral \( \int \text{tr}(d\gamma(\mathbb{E}_{X_{\alpha}} W(\gamma))d\gamma) \) exists.

Added in proof. In the case of discrete LCA groups similar problems, concerning interpolation of multivariate stationary processes, were considered also by H. Salehi and J. K. Schofield, Journal of Multivariate Analysis 2 (1972), pp. 307-331.

References