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On characterizations of interpolable and minimal stationary processes

by

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Abstract. In this paper some characterizations of interpolable and minimal stationary processes over locally compact Abelian (LCA) groups are established by using an isomorphism theorem between the time, spectral and Hellinger spectral domains of stationary processes. In the note [15] we studied the interpolation problem on LCA groups and announced several results without proofs. We present them here in a complete form. Our results constitute a natural extension to the case of an LCA group of Kolmogorov's, Yaglom's and Salehi's results on interpolation for the simple stationary stochastic processes.

1. INTRODUCTION

Classical least squares linear prediction theory is concerned with a stationary stochastic process, that is, a family X_s ($s \in Z$ — the discrete Abelian group of integers or $s \in R$—the Abelian group of reals with natural topology) of complex-valued random variables on a probability space, with zero means and finite covariances (X_s, X_t) depending only on $s-t$. Then the family of random variables forms a Hilbert space and, consequently, Hilbert space methods play a key role there.

Two important cases are considered; extrapolation and interpolation. One accomplishment of the theory in both cases is an analytical characterization of those processes which are errorlessly predictable; deterministic processes in extrapolation and interpolable processes in interpolation. Prediction theory of stationary sequences (processes over Z) has been studied by Kolmogorov in his fundamental paper [5]. Basing himself on the isomorphism between the time and the spectral domain of the univariate stationary processes (cf. [5], Th. 2.7) he has obtained analytical characterizations of deterministic and minimal processes. Yaglom [17] has obtained a characterization of interpolable processes in an analogous way. Next, these results have been extended to the multivariate case by Wiener and Masani [16], Rozanov [9] and many others.

In the present paper some analytical characterizations related to interpolation are given in the more general setting of q-variate stationary processes over LCA groups. The study of stationary processes over LCA groups was initiated by Kampé de Fériet [4] in 1948. We refer the reader to Yaglom [18] for an account of the theory of stationary processes over topological groups.

In Section 3 we obtain the spectral representations of a q-variate stationary process and of a correlation function. We present an isomorphism theorem between the time, spectral and Hellinger-spectral domain of a process after having given the preliminary results in Section 2. The significance of Hellinger integrals has been pointed out by Salehi [11] in relation to multivariate processes. In Section 4 a characterization of the "space of errors" of interpolation in terms of Hellinger integrals is established. This result is suggested by recent papers of Salehi [12]–[14]. Finally in Section 5 the interpolable and minimal processes are characterized in terms of the spectral measure of the process. Using these results we have obtained an extension of Bruckner's results [1] concerning univariate processes over a discrete Abelian group given in Section 6.

The results of this paper are taken from my Ph. D. thesis at the Technical University in Wrocław, September 1972. I wish to thank my thesis advisor, Professor S. Gładysz for his advice, many valuable conversations and constant encouragement. I also wish to thank Professor C. Ryll-Nardzewski for his lively interest and encouragement.

2. PRELIMINARIES

The space $L_{2,F}$. Let $\mathscr{B}$ be a σ-algebra of subsets of a space Ω and let $\Phi = [\varphi_{ij}]$ where $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$ be a matrix-valued function on Ω. A function Φ is $\mathscr{B}$-measurable if each function φ_{ij} is $\mathscr{B}$-measurable. If m is a nonnegative real-valued measure on $\mathscr{B}$, then by $L_{1,m}$ we denote the class of all Φ such that each φ_{ij} is integrable with respect to (abbreviated to "w.r.t.") m. For $\Phi \in L_{1,m}$ we put $\int_\Omega \Phi\, dm = \left[\int_\Omega \varphi_{ij}\, dm\right]$.

If $F = [F_{ij}]$ is a $q\times q$ nonnegative Hermitian-valued measure on $(\Omega, \mathscr{B})$, then each F_{ii} is a nonnegative real-valued measure and each F_{ij} for $i \neq j$ is a complex-valued measure on $(\Omega, \mathscr{B})$. Consequently, each F_{ij} is absolutely continuous (a.c.) w.r.t. the measure $\operatorname{tr} F$ (tr = trace). This follows from the inequality $0 \leqslant F \leqslant (\operatorname{tr} F) I$, which is satisfied for any nonnegative Hermitian matrix.

LEMMA 2.1. *Let F be as before and let m, n be σ-finite nonnegative real-valued measures on $\mathscr{B}$ w.r.t. which F is a.c. If Φ, Ψ are $p\times q$ matrix-valued functions on Ω, then*

(a) *the integral $\int_\Omega \Phi(dF/dm)\Psi^* dm$ exists if only if the integral $\int_\Omega \Phi(dF/dn)\Psi^* dn$ exists,*

(b) *if these integrals exist, they are equal.*

Proof. Let $w = m+n$. If the integral $\int_\Omega \Phi(dF/dm)\Psi^* dm$ exists, then

$$\int_\Omega \Phi(dF/dm)\Psi^* dm = \int_\Omega \Phi(dF/dm)\Psi^*(dm/dw)\, dw = \int_\Omega \Phi(dF/dw)\Psi^* dw.$$

Hence the integral $\int_\Omega \Phi(dF/dw)\Psi^* dw$ exists if and only if the integral $\int_\Omega \Phi(dF/dm)\Psi^* dm$ exists. A similar argument may be used to show that the integral $\int_\Omega \Phi(dF/dn)\Psi^* dn$ exists if and only if the integral $\int_\Omega \Phi(dF/dw)\Psi^* dw$ exists. Hence (a) and (b) are proved. ■

Thus the following definition makes sense. Let Φ, Ψ and F be as before. We say that Φ, Ψ are *integrable* w.r.t. F if $\Phi(dF/dm)\Psi^* \in L_{1,m}$, where m is an arbitrary σ-finite nonnegative real-valued measure w.r.t. which F is a.c. We note that such a measure m always exists, $m = \operatorname{tr} F$. We write

$$(\Phi, \Psi)_F = \int_\Omega \Phi\, dF \Psi^* = \int_\Omega \Phi(dF/dm)\Psi^* dm.$$

By $L_{2,F}$ we denote the class of all $\mathscr{B}$-measurable $p\times q$ matrix-valued functions Φ on Ω for which the integral $\int_\Omega \Phi\, dF \Psi^*$ exists.

THEOREM 2.2 (cf. [7], p. 295 and p. 296). (a) *$L_{2,F}$ is a Hilbert space under the inner product*

$$\big((\Phi, \Psi)\big) = \operatorname{tr}(\Phi, \Psi)_F.$$

(b) *$\Phi \in L_{2,F}$ if and only if there exists a Cauchy sequence in $L_{2,F}$ of simple functions Φ_n such that $\Phi_n(\omega) \to \Phi(\omega)$ everywhere $(\operatorname{tr} F)$.*

Stochastic integral. Let H be a Hilbert space over the field of complex numbers with inner product $(\cdot,\cdot)$, let H^q, $1 \leqslant q < \infty$, be the Cartesian product of q copies of H. If X, Y are in H^q and A, B are $q\times q$ matrices with complex entries, then $AX+BY$ is in H^q. For $X = (x_1, x_2, \ldots, x_q)$ and $Y = (y_1, y_2, \ldots, y_q)$ denote by

$$(X, Y) = [(x_i, y_j)]$$

the Gramian of X and Y. In H^q, X and Y are orthogonal if $(X, Y) = 0$, i.e., if each x_i is orthogonal to each y_j for $i, j = 1, 2, \ldots, q$. The space H^q with the inner product $\big((X, Y)\big) = \operatorname{tr}(X, Y)$ becomes a Hilbert space (cf. [16], Section 5).

We shall call S an *orthogonally-scattered vector-valued measure* o.v.m. on $(\Omega, \mathscr{B})$ if S is a countably additive function on $\mathscr{B}$ such that

1° $S(E) \in H^q$ for each $E \in \mathscr{B}$,

2° $\big(S(E_1), S(E_2)\big) = 0$ whenever $E_1 \cap E_2 = \emptyset$.

If S is an o.v.m. and $F(E) = (S(E), S(E))$ for each $E \in \mathscr{B}$, then F is a matrix-valued measure. This is a consequence of the formula

$$F\Big(\sum E_k\Big) = \Big(S\Big(\sum E_k\Big), S\Big(\sum E_k\Big)\Big) = \sum \big(S(E_k), S(E_k)\big) = \sum F(E_k).$$

Moreover, F is obviously a nonnegative Hermitian measure. We may define an integral w.r.t. an o.v.m. S (*stochastic integral*, cf. [7], p. 297) in such a way that it has the following properties:

$$(2.3)\qquad \int_\Omega (A\Phi + B\Psi)\,dS = A\int_\Omega \Phi\,dS + B\int_\Omega \Psi\,dS,$$

$$(2.4)\qquad \Big(\int_\Omega \Phi\,dS, \int_\Omega \Psi\,dS\Big) = \int_\Omega \Phi\,dF\Psi^*,$$

where $\Phi, \Psi \in L_{2,F}$ and A, B are $p \times p$ matrices.

Let $\mathfrak{S}$ denote the class of all stochastic integrals $\int_\Omega \Phi\,dS$ with $\Phi \in L_{2,F}$; then

LEMMA 2.5 (cf. [7], p. 297). *The correspondence* $V\colon \Phi \to \int_\Omega \Phi\,dS$ *is an isomorphism from* $L_{2,F}$ *onto* $\mathfrak{S}$ *such that* (2.3) *and* (2.4) *hold.*

Hellinger integral. The following lemma belongs to elementary matrix theory.

LEMMA 2.6 (cf. [6], p. 406). *The four equations* $AXA = A$, $XAX = X$, $(AX)^* = AX$ *and* $(XA)^* = XA$ *have a unique solution for any matrix* A.

The unique solution of this equations is called *the generalized inverse* of A and written $X = A^-$. If A is nonsingular then $A^- = A^{-1}$ and for scalars $k^- = 1/k$ if $k \neq 0$ and $k^- = 0$ if $k = 0$.(1)

If m is a σ-finite nonnegative real-valued measure on w.r.t. which F is a.c., then $(dF/dm)^-$ is $\mathscr{B}$-measurable matrix-valued function. The proof of the following lemma is analogous to that of Lemma 2.1.

LEMMA 2.7 (cf. [11]). *Let* M *and* N *be* $p\times q$ *matrix-valued measures on* $\mathscr{B}$ *and let* m *and* n *be* σ*-finite nonnegative real-valued measures on* $\mathscr{B}$ *w.r.t. which* M, N *and* F *are a.c. Then*

(a) *the integral* $\int_\Omega (dM/dm)(dF/dm)^-(dN/dm)^*\,dm$ *exists if and only if the integral* $\int_\Omega (dM/dn)(dF/dn)^-(dN/dn)^*\,dn$ *exists,*

(b) *if these integrals exist, they are equal.*

Thus, the following definition makes sense. Let M, N, F and m be as in the previous lemma. Then we say that M, N is *Hellinger integrable* w.r.t. F if

$$(dM/dm)(dF/dm)^-(dN/dm)^* \in L_{1,m}.$$

We write

$$(M, N)_F = \int_\Omega dM\,dN^*/dF = \int_\Omega (dM/dm)(dF/dm)^-(dN/dm)^*\,dm.$$

By $H_{2,F}$ we denote the class of all $p\times q$ matrix-valued measures M on $\mathscr{B}$ for which Hellinger integral $(M, N)_F$ exists.

It is known (cf. [11]) that $H_{2,F}$ is a Hilbert space under the inner product $((M, N)) = \operatorname{tr}(M, N)_F$ and that $M \in H_{2,F}$ if and only if there exists a $\mathscr{B}$-measurable matrix-valued function $\Phi \in L_{2,F}$ such that, for each $E \in \mathscr{B}$, $M(E) = \int_E \Phi\,dF$.

(1) cf. also A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Providence 1961, p. 63.

3. SPECTRAL REPRESENTATIONS AND THE ISOMORPHISM THEOREM

Let G be any LCA group with multiplication. The set of all characters of G, i.e., continuous homomorphism of G into the group $T = \{\exp 2\pi i x;\ 0 \leqslant x < 1\}$ forms a group Γ, the dual group of G (cf. [10], p. 7). In view of the duality between G and Γ (the Pontryagin duality theorem [10], p. 28) we will denote the characters by $\langle g, \gamma\rangle$, $g \in G$ and $\gamma \in \Gamma$. From the definition it follows immediately that

$$(3.1)\qquad \langle e, \gamma\rangle = \langle g, 1\rangle = 1,$$

$$(3.2)\qquad \langle g^{-1}, \gamma\rangle = \langle g, \gamma^{-1}\rangle = \overline{\langle g, \gamma\rangle}.$$

Γ with the compact-open topology is also an LCA group. The Borel field of the LCA group is the minimal σ-field generated by the closed subsets. Throughout this paper the letter Γ will denote the dual group of G and $\mathscr{B}$ the Borel field of the dual group Γ. On every LCA group there exists a nonnegative measure, finite on compact sets and positive on nonempty open sets, the so-called Haar measure of the group, which is translation-invariant. We denote by dg and $d\gamma$ the Haar measures on G and Γ.

DEFINITION 3.3 (cf. [8]). *A q-variate stationary process over any LCA group G is a function $(X_g)_{g\in G}$ such that*

(i) $X_g \in H^q$ *for all* $g \in G$,

(ii) *the* $q\times q$ *Gram matrix* $(X_g, X_h) = (X_{gh^{-1}}, X_e) = K(gh^{-1})$ *depends only on* gh^{-1} *for all* $g, h \in G$,

(iii) *the correlation function* $K(g)$ *is continuous on* G.

Let $\mathfrak{M}$ denote *the time domain* of the stationary process $(X_g)_{g\in G}$, i.e., the closed subspace of H^q spanned over the elements X_g, $g \in G$ with $q\times q$

matrix coefficients. If $(x_g^1, x_g^2, \ldots, x_g^q)$ are the components of X_g, then each $x_g^i \in H$ and by (ii)

$$(x_g^i, x_h^i) = k(gh^{-1})$$

depends only on gh^{-1}. Thus the q-variate process $(X_g)_{g\in G}$ is associated with q simple processes $(x_g^i)_{g\in G}$ which are stationary in the wide sense (cf. [3]). Each simple process defines in the space $\mathfrak{M}^i$, the time domain of process $(x_g^i)_{g\in G}$, a unitary representation of the group G. Namely, the suitable unitary operators U_h^i are defined by the formula

$$U_h^i x_g^i = x_{gh}^i \quad \text{for } g, h \in G,\ i = 1, 2, \ldots, q,$$

and for the remaining points of the space $\mathfrak{M}^i$ the operators U_h^i are defined by a natural extension. It is known (cf. [8], Lemma 2.1 or [16], p. 135) that we may take $U_g^i = U_g^j$, so that there exists a unitary operator $U_g^{(q)} = (U_g, U_g, \ldots, U_g)$ on $\mathfrak{M}$ such that

$$X_g = U_g^{(q)} X_e = [U_g x_e^i]_{i=1}^q \quad \text{for } g \in G. \tag{3.4}$$

Following Jajte [3] we have by the generalized theorem of Stone for the operators U_h the spectral representation

$$U_h = \int_\Gamma \overline{\langle h, \gamma\rangle} P(d\gamma), \tag{3.5}$$

where $P(\cdot)$ is a regular, normed and orthogonal spectral family of projectors in $\mathfrak{M}^i$ defined on $\mathscr{B}$. If we put

$$S(\cdot) = P^{(q)}(\cdot) X_e,$$

where $P^{(q)} = (P, P, \ldots, P)$ is a spectral family of projectors in $\mathfrak{M}$, then S is an o.v.m. on $\mathscr{B}$. According to (3.4) and (3.5) we have the spectral representation for the stationary process $(X_g)_{g\in G}$

$$X_g = U_g^{(q)} X_e = \int_\Gamma \overline{\langle g, \gamma\rangle} I S(d\gamma), \quad g \in G \tag{3.6}$$

where I denotes the $q \times q$ unit matrix.

The matrix-valued function F on $\mathscr{B}$, $F(\cdot) = \big(S(\cdot), S(\cdot)\big)$ is called *a spectral measure* of the process $(X_g)_{g\in G}$. Clearly, F is a nonnegative Hermitian-valued measure. Henceforth, the letter F will denote the spectral measure; the spaces $L_{2,F}$ and $H_{2,F}$ related to F—as in Section 2—are called *the spectral domain* and *the Hellinger-spectral domain* of the process $(X_g)_{g\in G}$, respectively.

We note that according to (3.6), (2.4) and (3.1) we obtain the spectral representation for the correlation function

$$K(g) = (X_g, X_e) = \Big(\int_\Gamma \overline{\langle g, \gamma\rangle} I S(d\gamma),\ \int_\Gamma \overline{\langle e, \gamma\rangle} I S(d\gamma)\Big)$$
$$= \int_\Gamma \overline{\langle g, \gamma\rangle} I\, dF \langle e, \gamma\rangle I^* = \int_\Gamma \overline{\langle g, \gamma\rangle} I\, dF.$$

The main result of this section is the following isomorphism theorem.

THEOREM 3.7. *If $(X_g)_{g\in G}$ is a q-variate stationary process over any LCA group G, with the spectral measure F, then Hilbert spaces $\mathfrak{M}$, $L_{2,F}$ and $H_{2,F}$ are isomorphic, where*

(a) *the mapping V_1: $X_g \to \overline{\langle g, \gamma\rangle} I$, I denoting the unit matrix, is an isomorphism between $\mathfrak{M}$ and $L_{2,F}$,*

(b) *the mapping V_2: $\Phi \to M_\Phi$, for any matrix-valued function $\Phi \in L_{2,F}$ with values on the set of measures M_Φ on $\mathscr{B}$ given by $M_\Phi(E) = \int_E \Phi dF$, is an isomorphism between $L_{2,F}$ and $H_{2,F}$.*

Proof. (a) Let $X_g = X_h$ in H^q; then according to (2.4) and (3.2)

$$\int_\Gamma I\, dF = (X_g, X_g) = (X_g, X_h) = \Big(\int_\Gamma \overline{\langle g, \gamma\rangle} I\, dS,\ \int_\Gamma \overline{\langle h, \gamma\rangle} I\, dS\Big)$$
$$= \int_\Gamma \overline{\langle g, \gamma\rangle} I\, dF \langle h, \gamma\rangle I^* = \int_\Gamma \langle hg^{-1}, \gamma\rangle I\, dF,$$

so that $\overline{\langle g, \gamma\rangle} I = \overline{\langle h, \gamma\rangle} I$ in $L_{2,F}$ and the mapping V_1: $\{X_g, g \in G\} \to L_{2,F}$, is well defined and may be extended in a natural way to a mapping of $\mathfrak{M}$ into $L_{2,F}$. If we prove that $\mathfrak{M} = \mathfrak{S}$, then by Lemma 2.5 and the obvious equality $V_1 = V$ the proposition (a) will be proved. Since $\overline{\langle g, \gamma\rangle} I \in L_{2,F}$, then from the representation (3.6) it follows that $\mathfrak{M} \subset \mathfrak{S}$.

Conversely, let $\Phi \in L_{2,F}$ and let $Y = \int_\Gamma \Phi\, dS$. By Theorem 2.2, there exists in $L_{2,F}$ a Cauchy sequence of simple functions $\Phi_n \to \Phi$ everywhere. If we put $Y_n = \int_\Gamma \Phi_n dS$, then clearly a sequence of Y_n is convergent in δ. Since Φ_n is a simple function,

$$Y_n = \int_\Gamma \Phi_n dS = \sum_{i=1}^k A_i^n S(E_i^n) = \sum_{i=1}^k A_i^n P^{(q)}(E_i^n) X_e,$$

where A_i^n are $q \times q$ matrices, $E_i^n \in \mathscr{B}$ and $P^{(q)}$ is a spectral family of projectors in $\mathfrak{M}$. Of course, $P^{(q)}(E_i^n) X_e \in \mathfrak{M}$, thus $Y_n \in \mathfrak{M}$ and consequently $Y \in \mathfrak{M}$. It follows that $\mathfrak{S} \subset \mathfrak{M}$.

Part (b) is a special case of the isomorphism theorem (cf. [11], Th. 1) between the space $L_{2,F}$ and $H_{2,F}$ on any space Ω. ■

4. THE SPACE $\mathfrak{N}_C$

Let $(X_g)_{g\in G}$ be a q-variate stationary process over LCA group G and let C be any proper and nonempty compact subset of G. If $\mathfrak{M}_{G-C}$ denotes the closed linear subspace of H^q spanned by X_g, $g\in G-C$, then we denote

$$\mathfrak{N}_C = \mathfrak{M}\ominus\mathfrak{M}_{G-C}.$$

The space $\mathfrak{N}_C$ plays an essential role in an interpolation problem, which will be considered in the next section. The purpose of this section is a construction of isomorphism T_C on $\mathfrak{N}_C$ into $H_{2,F}$ and a characterization of the range of T_C (Th. 4.9). First we prove several lemmas.

Let $\mathscr{D}(G)$ denote the set of all $p\times q$ matrix-valued functions Φ on G which are representable in the form

$$\Phi(g) = \int_\Gamma \langle g,\gamma\rangle dM, \quad g\in G,$$

where M is a $p\times q$ matrix-valued measure on $\mathscr{B}$, i.e., each M_{ij} is a regular, complex-valued measure on $\mathscr{B}$. We note that Bochner's theorem (cf. [10], p. 19) in combination with the Jordan decomposition theorem (cf. [2], p. 309) implies that $\mathscr{D}(G)$ is exactly the set of all matrix-valued functions where entries are finite linear combinations of continuous positive-definite functions on G.

For all $p\times q$ matrix-valued functions $\Phi\in L_{1,dg}$ the $p\times q$ matrix-valued function $\hat{\Phi}$ defined on dual group Γ by

$$\hat{\Phi}(\gamma) = \int_G \overline{\langle g,\gamma\rangle}\,\Phi(g)\,dg$$

is called *the Fourier transform* of Φ.

LEMMA 4.1. (a) *If* $\Phi\in L_{1,dg}\cap\mathscr{D}(G)$, *then* $\hat{\Phi}\in L_{1,d\gamma}$.

(b) *If the Haar measure of* G *is fixed, the Haar measure of* Γ *can be normalized so that the inversion formula*

$$\Phi(g) = \int_\Gamma \langle g,\gamma\rangle\hat{\Phi}(\gamma)\,d\gamma, \quad g\in G$$

is valid for every $\Phi\in L_{1,dg}\cap\mathscr{D}(G)$.

Proof. We note that in virtue of the definition of a matrix-valued integral w.r.t. a scalar measure it remains to prove that

$$(*)\qquad \hat{\varphi}_{ij}(\gamma)\in L_{1,d\gamma},$$

$$(**)\qquad \varphi_{ij}(g) = \int_\Gamma \langle g,\gamma\rangle\hat{\varphi}_{ij}(\gamma)\,d\gamma,$$

where the Haar measure $d\gamma$ is suitably normalized, $1\leqslant i\leqslant p$, $1\leqslant j\leqslant q$, $g\in G$ and $\gamma\in\Gamma$. Since $\Phi\in\mathscr{D}(G)$, then $\Phi(g) = \int_\Gamma\langle g,\gamma\rangle I\,dM$. If m is a σ-finite nonnegative real-valued measure on $\mathscr{B}$ w.r.t. which M is a.c., then

$$\Phi(g) = \int_\Gamma\langle g,\gamma\rangle I\,dM = \int_\Gamma\langle g,\gamma\rangle I(dM/dm)\,dm$$
$$= \Big[\int_\Gamma\langle g,\gamma\rangle I(dM_{ij}/dm)\,dm\Big] = \Big[\int_\Gamma\langle g,\gamma\rangle dM_{ij}\Big],$$

and so $\varphi_{ij}(g) = \int_\Gamma\langle g,\gamma\rangle dM_{ij}$, where M_{ij} is a bounded regular complex-valued measure on $\mathscr{B}$. Moreover $\varphi_{ij}\in L_{1,dg}$ according to the assumption $\Phi\in L_{1,dg}$ and hence by the inversion formula for complex-valued functions (cf. [10], p. 22) we obtain (*) and (**). ■

From now on, it will always be tacitly assumed that the Haar measures dg and $d\gamma$ are so adjusted that the inversion formula holds.

LEMMA 4.2. (a) *If* $X\in\mathfrak{M}$, *then* $(X, X_g)\in\mathscr{D}(G)$.

(b) *If* $X\in\mathfrak{N}_C$, *then* $(X, X_g)\in L_{1,dg}$.

Proof. Let X be in $\mathfrak{M}$ and let Φ be in $L_{2,F}$ such that $V_1X = \Phi$, where V_1 is an isomorphism from $\mathfrak{M}$ onto $L_{2,F}$ as in Theorem 3.7. We have

$$(X, X_g) = (V_1X, V_1X_g)_F = (\Phi, \langle g,\gamma\rangle I)_F = \int_\Gamma\langle g,\gamma\rangle\Phi\,dF.$$

If m is a σ-finite nonnegative real-valued measure on $\mathscr{B}$ w.r.t. which F is a.c., then for each $E\in\mathscr{B}$ we have

$$M_\Phi(E) = \int_E\Phi\,dF = \int_E\Phi(dF/dm)\,dm.$$

Consequently, $(dM_\Phi/dm) = \Phi(dF/dm)$ and therefore

$$(4.3)\qquad (X, X_g) = \int_\Gamma\langle g,\gamma\rangle\Phi\,dF = \int_\Gamma\langle g,\gamma\rangle\Phi(dF/dm)\,dm$$
$$= \int_\Gamma\langle g,\gamma\rangle(dM_\Phi/dm)\,dm = \int_\Gamma\langle g,\gamma\rangle dM_\Phi.$$

It remains to prove that the entries of M_Φ are regular measures on $\mathscr{B}$. From Section 2 it is clear that we may put $m = \operatorname{tr}F$, where F is the spectral measure of the process $(X_g)_{g\in G}$. Each measure F_{ii} for $i = 1,\ldots,q$ as the spectral measure of a simple stationary process is regular and nonnegative. This is a consequence of Bochner's theorem (cf. [3]). Thus $m = \sum_{i=1}^{q}F_{ii}$ is also a nonnegative regular measure.

Let $[\psi_{ij}] = \Phi(dF/dm)$ and let $M_\Phi = [M_{ij}]$ where $1\leqslant i\leqslant p$, $1\leqslant j\leqslant q$. Then for each $E\in\mathscr{B}$ $M_{ij}(E) = \int_E\psi_{ij}\,dm$. Since m is regular, then M_{ij} are complex regular measures on $\mathscr{B}$. Thus part (a) is proved.

(b) For $X \in \mathfrak{M}$ the matrix-valued function (X, X_g) is by the definition 3.3 (iii) a continuous function on G. Moreover, if $X \in \mathfrak{N}_C$ then $(X, X_g) = 0$ for $g \notin C$. Hence for $X \in \mathfrak{N}_C (X, X_g)$ is a continuous function with a compact support and so $(X, X_g) \in L_{1,dg}$. ∎

Now for each $X \in \mathfrak{N}_C$ we let

$$(4.4) \qquad P_X(\gamma) = \int_G \overline{\langle g, \gamma\rangle}(X, X_g)\,dg, \qquad \gamma \in \Gamma.$$

$P_X(\gamma)$ is the Fourier transform of (X, X_g). Its properties are given in the following lemma.

LEMMA 4.5. (a) *$P_X(\gamma) \in L_{1,d\gamma}$,*

(b) *If for each $E \in \mathscr{B}$ we define $N_{P_X}(E) = \int_E P_X(\gamma)\,d\gamma$, then $N_{P_X} = M_\Phi$, where $M_\Phi = V_2 V_1(X)$ (see Theorem 3.7),*

(c) *$N_{P_X} \in H_{2,F}$.*

Proof. (a) If $X \in \mathfrak{N}_C$, then from Lemma 4.2 $(X, X_g) \in L_{1,dg} \cap \mathscr{D}(G)$. Hence by Lemma 4.1 (a) $P_X \in L_{1,d\gamma}$.

(b) Simultaneously, by Lemma 4.1 (b), $(X, X_g) = \int_\Gamma \langle g, \gamma\rangle P_X(\gamma)\,d\gamma$. Since the definition of N_{P_X} implies $(dN_{P_X}/d\gamma) = P_X$,

$$(X, X_g) = \int_\Gamma \langle g, \gamma\rangle (dN_{P_X}/d\gamma)\,d\gamma = \int_\Gamma \langle g, \gamma\rangle I\,dN_{P_X}.$$

On the other hand, by (4.3) we have $(X, X_g) = \int_\Gamma \langle g, \gamma\rangle dM_\Phi$. Hence for each $g \in G$ we get

$$\int_\Gamma \langle g, \gamma\rangle I\,dN_{P_X} = \int_\Gamma \langle g, \gamma\rangle I\,dM_\Phi.$$

It follows according to the uniqueness theorem for the inverse Fourier–Stieltjes transform of measure (cf. [10], p. 17) that $N_{P_X} = M_\Phi$.

(c) From Theorem 3.7 (b) $M_\Phi \in H_{2,F}$; thus from (b) $N_{P_X} \in H_{2,F}$. ∎

DEFINITION 4.6. *Let C be any proper and nonempty compact subset of G. Then*

(a) *$\mathfrak{Q}_C$ will denote the set of $q \times q$ matrix-valued functions $Q(g)$ on G such that*

$$(*) \qquad Q(g) \in L_{1,dg} \cap \mathscr{D}(G),$$

$$(**) \qquad \operatorname{supp} Q(g) \subset C.$$

(b) *$\mathfrak{P}_C$ will denote the set of Fourier transforms of all functions from $\mathfrak{Q}_C$.*

The properties of the set $\mathfrak{P}_C$ are given in the following two lemmas.

LEMMA 4.7. (a) *If $X \in \mathfrak{N}_C$, then $P_X \in \mathfrak{P}_C$, where P_X is as in (4.4).*

(b) *If G is a discrete Abelian group, then $\mathfrak{P}_C$ is exactly the set of all trig-polynomials $W(\gamma) = \sum_{k=1}^{n} A_{g_k} \overline{\langle g_k, \gamma\rangle}$ with matrix coefficients.*

Proof. (a) If $X \in \mathfrak{N}_C$ then by Lemma 4.2. (X, X_g) is in $L_{1,dg} \cap \mathscr{D}(G)$; moreover, the support of (X, X_g) is in C. Thus $(X, X_g) \in \mathfrak{Q}_C$ and consequently by (4.4) $P_X(\gamma) \in \mathfrak{P}_C$.

(b) follows readily from the fact that in discrete topology each compact subset is finite. ∎

LEMMA 4.8. *If $P(\gamma) \in \mathfrak{P}_C$, then*

(a) *$P(\gamma) \in L_{1,d\gamma}$,*

(b) *N_p is a matrix-valued measure on $\mathscr{B}$ if we put for each $E \in \mathscr{B}$ $N_p(E) = \int_E P(\gamma)\,d\gamma$,*

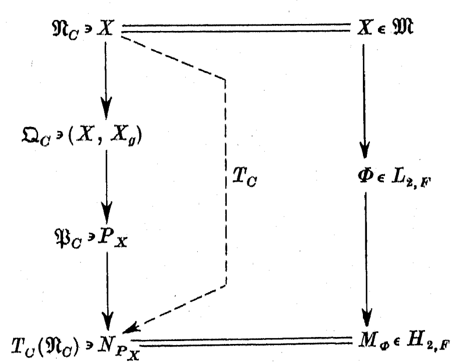
(c) *$Q(g) = \int_\Gamma \langle g, \gamma\rangle P(\gamma)\,d\gamma$.*

Proof. Since $P(\gamma)$ is by definition a Fourier transform of the matrix- valued function $Q(g)$ which is in $L_{1,dg} \cap \mathscr{D}(G)$, (a) and (c) follow immediately from Lemma 4.1. According to (a) the definition of a measure N_P in (b) makes sense. ∎

Now we define the operator T_C on $\mathfrak{N}_C$ into $H_{2,F}$. For each $X \in \mathfrak{N}_C$

$$T_C X = N_{P_X},$$

where N_{P_X} is as in Lemma 4.5 (b). A construction of T_C is presented in the following diagram:



We are now ready to give a characterization for the range of the operator T_C.

THEOREM 4.9. (a) *T_C is a single-valued linear operator on $\mathfrak{N}_C$, i.e., if $X, Y \in \mathfrak{N}_C$ and A, B are $q\times q$ matrices, then*

$$T_C(AX+BY) = AT_CX+BT_CY.$$

(b) *T_C is an isometry on $\mathfrak{N}_C$ into $H_{2,F}$, i.e., for $X, Y \in \mathfrak{N}_C$*

$$(X, Y) = (T_CX, T_CY)_F.$$

(c) *The range of T_C is a closed subspace of the Hilbert space $H_{2,F}$.*

(d) *The range of T_C consists of all matrix-valued measures N_P for which the Hellinger integral $\int_\Gamma dN_P dN_P^*/dF$ exists, where $P \in \mathfrak{P}_C$ and N_P is related to P as in Lemma* 4.8 (b).

Proof. (a) Let $X, Y \in \mathfrak{N}_C$ and A, B be $q\times q$ matrices; then $Z = AX+BY \in \mathfrak{N}_C$ and for each $E \in \mathscr{B}$ we obtain

$$N_Z(E) = \int_E P_Z(\gamma)d\gamma = \int_E\Big[\int_G \overline{\langle g,\gamma\rangle}(AX+BY, X_g)dg\Big]d\gamma$$
$$= \int_E\Big[A\int_G \overline{\langle g,\gamma\rangle}(X, X_g)dg + B\int_G \overline{\langle g,\gamma\rangle}(Y, X_g)dg\Big]d\gamma$$
$$= AN_{P_X}(E)+BN_{P_Y}(E).$$

Consequently, $T_CZ = AT_CX+BT_CY$.

(b) Let $X, Y \in \mathfrak{N}_C$. According to Theorem 3.7 there exist matrix-valued functions Φ, Ψ in $L_{2,F}$ and matrix-valued measures M_Φ, M_Ψ in $H_{2,F}$ such that

$$(X, Y) = (\Phi, \Psi)_F = (M_\Phi, M_\Psi)_F.$$

But from Lemma 4.5 (b) $M_\Phi = N_{P_X}$ and $M_\Psi = N_{P_Y}$, hence

$$(M_\Phi, M_\Psi)_F = (N_{P_X}, N_{P_Y})_F$$

and consequently

$$(X, Y) = (T_CX, T_CY)_F.$$

(c) Since $\mathfrak{N}_C$ is a closed subspace of $\mathfrak{M}$ and since by (b) T_C is an isometry on $\mathfrak{N}_C$ into $H_{2,F}$, the range of T_C is a closed subspace of $H_{2,F}$.

(d) Let $X \in \mathfrak{N}_C$ and $T_CX = N_{P_X}$. From Lemma 4.5 (c) N_{P_X} is in $H_{2,F}$ and consequently the Hellinger integral $\int_\Gamma dN_{P_X}dN_{P_X}^*/dF$ exists. Moreover, from Lemma 4.7 (a) P_X is in $\mathfrak{P}_C$.

Conversely, let N_P be a matrix-valued measure on $\mathscr{B}$ given for each $E \in \mathscr{B}$ by $N_P(E) = \int_E P(\gamma)d\gamma$, where $P(\gamma)$ is in $\mathfrak{P}_C$ and the Hellinger integral $\int_\Gamma dN_P dN_P^*/dF$ exists. Since N_P is in $H_{2,F}$, then there exists (cf. [11], Th. 1) a $\mathscr{B}$-measurable matrix-valued function Φ in $L_{2,F}$ such that for each $E \in \mathscr{B}$ $N_P(E) = \int_E \Phi dF = \int_E \Phi(dF/dm)dm$, where m is an arbitrary σ-finite nonnegative real-valued measure on $\mathscr{B}$ w.r.t. which F is a.c. By the Theorem 3.7 we may choose $Y \in \mathfrak{M}$ such that $V_1Y = \Phi$. Thus according to Theorem 3.7 (a) and Lemma 4.8 we obtain

$$(Y, X_g) = (\Phi, \overline{\langle g,\gamma\rangle}I)_F = \int_\Gamma \langle g,\gamma\rangle\Phi dF = \int_\Gamma \langle g,\gamma\rangle\Phi(dF/dm)dm$$
$$= \int_\Gamma \langle g,\gamma\rangle dN_P = \int_\Gamma \langle g,\gamma\rangle P(\gamma)d\gamma = Q(g).$$

Hence by Definition 4.6. the support of (Y, X_g) is contained in C. Since $Y \in \mathfrak{M}$ and $(Y, X_g) = 0$ for $g \notin C$ we conclude that $Y \in \mathfrak{N}_C$. But also from equation $(Y, X_g) = Q(g)$ we obtain for each $E \in \mathscr{B}$

$$N_P(E) = \int_E P(\gamma)d\gamma = \int_E\Big[\int_G \overline{\langle g,\gamma\rangle}Q(g)dg\Big]d\gamma$$
$$= \int_E\Big[\int_G \overline{\langle g,\gamma\rangle}(Y, X_g)dg\Big]g\gamma = \int_E P_Y(\gamma)d\gamma = N_{P_Y}(E).$$

Hence it follows that we find $Y \in \mathfrak{N}_C$ such that $N_P = T_CY$. ∎

5. INTERPOLABLE AND MINIMAL PROCESSES

Let $(X_g)_{g\in G}$ be a q-variate stationary process over an LCA group G and let C be a proper and nonempty compact subset of G. Suppose that only X_g for $g \in G-C$ are known. The prediction problem for a compact subset C will be called interpolation (for classical processes cf. [9], p. 134 and p. 180).

We say that $\mathfrak{M}_{G-C}$ is the "*space of observation*" of the process $(X_g)_{g\in G}$ and consider each $Y \in \mathfrak{M}_{G-C}$ to be a *prediction* of the process $(X_g)_{g\in G}$ based on observations of the outside of a compact subset C. *The error of this prediction* may be expressed with the aid of the norm in $\mathfrak{M}$. That is to say, we are looking for *a predictor* X_g^0 satisfying $X_g^0 \in \mathfrak{M}_{G-C}$ and

$$\|X_g - X_g^0\|^2 = \min_{Y\in\mathfrak{M}_{G-C}} \|X_g - Y\|^2.$$

It follows that X_g^0 is the projection of X_g onto $\mathfrak{M}_{G-C}$. We note that the closed subspace of $\mathfrak{M}$ spanned by all $X_g - X_g^0$, for $g \in C$ is exactly the space $\mathfrak{N}_C$, which was defined in the previous section. Hence, the space $\mathfrak{N}_C$ plays the role of the "*space of errors*" of interpolation.

DEFINITION 5.1 (cf. [12] and [15]). *We say that*

(a) *C is interpolable w.r.t. $(X_g)_{g\in G}$ if $\mathfrak{N}_C = 0$.*

(b) $(X_g)_{g\in G}$ *is interpolable if each proper and nonempty compact subset of* G *is interpolable w.r.t.* $(X_g)_{g\in G}$.

(c) $(X_g)_{g\in G}$ *is minimal if, for each* $h\in G$, $\{h\}$ *is not interpolable w.r.t.* $(X_g)_{g\in G}$.

Now we give an analytical characterization of interpolable and minimal processes. We start from an analytical characterization of interpolable subsets.

THEOREM 5.2. *A compact subset* C *of an LCA group* G *is interpolable w.r.t. a q-variate stationary process* $(X_g)_{g\in G}$ *with the spectral measure* F *if and only if for any matrix-valued function* $P\in\mathfrak{P}_C$ *the Hellinger integral* $\int_\Gamma dN_P dN_P^*/dF$ *is zero or does not exist.*

Proof. Necessity. Let T_C be an isomorphism on $\mathfrak{N}_C$ onto $H_{2,F}$ defined in the previous section. If C is interpolable w.r.t. $(X_g)_{g\in G}$ then by definition $\mathfrak{N}_C=0$. Hence by Theorem 4.9 (b) the range of T_C is a null-point in $H_{2,F}$ and so by Theorem 4.9 (d) for each matrix-valued function P in $\mathfrak{P}_C$ the measure N_P is a null-point in $H_{2,F}$ or $N_P\notin H_{2,F}$. In other words, for each measure N_P the Hellinger integral $\int_\Gamma dN_P dN_P^*/dF$ is zero or does not exist.

Sufficiency. Let X be in $\mathfrak{N}_C$ and $T_C X=N_{P_X}$, by Lemma 4.5 (c) $N_{P_X}\in H_{2,F}$ and by Lemma 4.7 (a) $P_X\in\mathfrak{P}_C$. Since for each matrix-valued function P from $\mathfrak{P}_C$ a suitable measure N_P is a null-point in $H_{2,F}$ or $N_P\notin H_{2,F}$, it follows that N_{P_X} is a null-point in $H_{2,F}$. Consequently, by Theorem 4.9 (b)

$$0=(N_{P_X},N_{P_X})_F=(T_C X,T_C X)_F=(X,X).$$

Since X is arbitrary in $\mathfrak{N}_C$, it follows that $\mathfrak{N}_C=0$ and by definition the set C is interpolable w.r.t. $(X_g)_{g\in G}$. ∎

DEFINITION 5.3. (a) $\mathfrak{Q}$ *will denote the set of* $q\times q$ *matrix-valued functions* $Q(g)$ *on* G *such that*

(*) $$Q(g)\in L_{1,dg}\cap\mathscr{D}(G),$$

(**) *the support of* $Q(g)$ *is contained in any proper compact subset of* G.

(b) $\mathfrak{P}$ *will denote the set of Fourier transforms of all functions from* $\mathfrak{Q}$.

We note that according to Definition 4.6 $\mathfrak{P}=\bigcup_C\mathfrak{P}_C$, where C is any proper compact subset of G. Hence the following generalization of Salehi's results (cf. [12], Th. 2, [13], Th. 3 and [14], Th. 3.6) concerning q-variate stationary processes over the group Z, Z^n and R follows directly from Theorem 5.2.

COROLLARY 5.4. *A q-variate stationary process* $(X_g)_{g\in G}$ *over an LCA group* G, *with spectral measure* F *is interpolable if and only if for any matrix-valued function* $P\in\mathfrak{P}$ *the Hellinger integral* $\int_\Gamma dN_P dN_P^*/dF$ *is zero or does not exist.*

Now, for discrete Abelian groups we obtain the following group analogue of Yaglom's result (cf. [17]) concerning simple discrete parameter processes, i.e., with $q=1$ and $G=Z$.

THEOREM 5.5. *Let* G *be a discrete Abelian group and* $(X_g)_{g\in G}$ *a q-variate stationary process over* G, *where the spectral measure* F *is a.c. w.r.t. the Haar measure* $d\gamma$ *on dual group* Γ. *Then* $(X_g)_{g\in G}$ *is interpolable if and only if for any trig-polynomials* $W(\gamma)$ *with matrix coefficients, the integral* $\int_\Gamma W(\gamma)(dF/d\gamma)^- W(\gamma)^* d\gamma$ *is zero or does not exist.*

Proof. Since G is discrete, then according to the Pontryagin theorem (cf. [10], p. 9) the dual group of G is compact and consequently the Haar measure $d\gamma$ is finite on Γ. Since by assumption F is a.c. w.r.t. $d\gamma$, the Radon–Nikodym derivative $(dF/d\gamma)$ exists. By Lemma 4.7 (b) $\mathfrak{P}$ is exactly the set of all trig-polynomials $W(\gamma)$ with matrix coefficients, on dual group Γ. Let for each $E\in\mathscr{B}$ $N_W(E)=\int_E W(\gamma)d\gamma$, then $(dN_W/d\gamma)=W(\gamma)$. Thus we get the formula

$$\int_\Gamma dN_W dN_W^*/dF=\int_\Gamma (dN_W/d\gamma)(dF/d\gamma)^-(dN_W/d\gamma)^* d\gamma$$
$$=\int_\Gamma W\big(\gamma(dF/d\gamma)\big)^- W(\gamma)^* d\gamma.$$

Now our assertion is a consequence of Corollary 5.4. ∎

We note that by Definition 3.3 (iii) minimal processes exist only on discrete groups. Let G be a discrete Abelian group. Let Y_g denote the orthogonal projection of X_g onto $\mathfrak{N}_{\{g\}}$ and let J denote the projection matrix on the subspace S^q of q-tuples of complex numbers onto the range of (Y_e, Y_e) in the privileged basis of S^q. The next theorem is an extension to the case of an LCA group of Salehi's results on processes over Z and Z^n (cf. [13], Th. 3 and [14], Th. 3.7).

THEOREM 5.6. *Let* G *be a discrete Abelian group and* $(X_g)_{g\in G}$ *a q-variate stationary process over* G, *with spectral measure* F. *Then* $(X_g)_{g\in G}$ *is minimal if and only if the Hellinger integral* $\int_\Gamma dN_J dN_J/dF\neq 0$, *where, for each* $E\in\mathscr{B}$, $N_J(E)=\int_E J\,d\gamma$.

Proof. Since the process $(X_g)_{g\in G}$ is stationary, $\mathfrak{N}_{\{e\}}\neq 0$ if and only if for each $g\in G$ $\mathfrak{N}_{\{g\}}\neq 0$. Hence it remains to prove that Y_e is not a null-point in $\mathfrak{N}_{\{e\}}$. Let, for each $E\in\mathscr{B}$, $N_{Y_e}(E)=\int_E (Y_e, Y_e)d\gamma=(Y_e, Y_e)\int_E d\gamma$. Let P_{Y_e} be as in (4.4); then according to the definition of Y_e we get $P_{Y_e}(\gamma)=\int_G \overline{\langle g,\gamma\rangle}(Y_e, X_g)dg=(Y_e, Y_e)$. It follows that $T_C Y_e=N_{Y_e}$. Consequently, by Theorem 4.9 (b) $(Y_e, Y_e)=(N_{Y_e}, N_{Y_e})_F$. From Lemma

2.6. we obtain

$$(Y_e, Y_e)^- = (Y_e, Y_e)^-(Y_e, Y_e)(Y_e, Y_e)^-;$$

thus

$$(Y_e, Y_e)^- = (Y_e, Y_e)^-(N_{Y_e}, N_{Y_e})_F(Y_e, Y_e)^-$$
$$= (Y_e, Y_e)^-(Y_e, Y_e)(N_I, N_I)_F(Y_e, Y_e)(Y_e, Y_e)^-$$

where for each $E \in \mathscr{B}$ $N_I(E) = \int_E I\,d\gamma$ and I denotes the unity matrix. Since $(Y_e, Y_e)^-(Y_e, Y_e) = J$, we get

$$(Y_e, Y_e)^- = J(N_I, N_I)_F J = (N_J, N_J)_F = \int_\Gamma dN_J dN_J/dF.$$

From Lemma 2.6 it follows also that $(Y_e, Y_e) \neq 0$ if and only if $(Y_e, Y_e)^- \neq 0$. Hence it follows that Y_e is not a null-point in $\mathfrak{N}_{\{e\}}$ if and only if $\int_\Gamma dN_J dN_J/dF \neq 0$. ■

Finally we obtain a natural extension to the case of an LCA group of a result due to Kolmogorov (cf. [5]) concerning univariate processes, which was generalized by Rozanov (cf. [9], p. 138) to the multivariate case.

THEOREM 5.7. *Let G be a discrete Abelian group and $(X_g)_{g\in G}$ a q-variate stationary process over G, where the spectral measure F is a.c. w.r.t. the Haar measure on Γ. Then $(X_g)_{g\in G}$ is minimal if and only if the integral $\int_\Gamma \operatorname{tr}(dF/d\gamma)^- d\gamma$ exists.*

Proof. From Definition 5.1 (c) for each $g \in G$ the set $\{g\}$ is not interpolable w.r.t. $(X_g)_{g\in G}$ if and only if $\mathfrak{N}_{\{g\}} \neq 0$. If g is in G then $\mathfrak{P}_{\{g\}}$ is exactly the set of all trig-monomial $A\langle g, \gamma\rangle$ with matrix-coefficients. Let, for each $E \in \mathscr{B}$, $N_A(E) = \int_E A\overline{\langle g, \gamma\rangle}\,d\gamma$, hence $(dN_A/d\gamma) = A\overline{\langle g, \gamma\rangle}$. Since F is a.c. w.r.t. $d\gamma$, similarly as in the proof of Theorem 5.5 $(dF/d\gamma)$ exists. Consequently

$$\int_\Gamma dN_A dN_A^*/dF = \int_\Gamma (dN_A/d\gamma)(dF/d\gamma)^-(dN_A/d\gamma)\,d\gamma$$
$$= \int_\Gamma \overline{\langle g, \gamma\rangle}\langle g, \gamma\rangle A(dF/d\gamma)^- A^*\,d\gamma$$
$$= A\int_\Gamma (dF/d\gamma)^- d\gamma\, A^*.$$

Since g is arbitrary in G, it follows by Theorem 5.2 that $(X_g)_{g\in G}$ is minimal if and only if the integral $\int_\Gamma (dF/d\gamma)^- d\gamma$ exists. If this integral exists then obviously the integral $\int_\Gamma \operatorname{tr}(dF/d\gamma)^- d\gamma$ exists. The necessity of the condition is thus proved. It remains to prove that if $\operatorname{tr}(dF/d\gamma)^-$ is in $L_{1,d\gamma}$ then also $(dF/d\gamma)^-$ is in $L_{1,d\gamma}$. Put $\Phi = (dF/d\gamma)^-$. Since $(dF/d\gamma)$ is a nonnegative Hermitian-valued function, from the properties of a generalized inverse matrix (cf. [6]) we conclude that Φ is also a nonnegative Hermitian-valued function on Γ. Consider Φ as a linear operator on the space S^q of q-tuples of complex numbers. Let $\|\Phi\|$ denote the norm of Φ and $\|\Phi\|_{H-S}$ – the Hilbert–Schmidt norm of Φ (i.e., $\|\Phi\|_{H-S}^2 = \operatorname{tr}(\Phi\Phi^*)$). Since in the finite dimensional case all norms are equivalent and for a nonnegative matrix $\Phi = \Phi^{1/2}\Phi^{1/2*}$, we have

$$\|\Phi\| = \|\Phi^{1/2}\Phi^{1/2*}\| = \|\Phi^{1/2}\|^2 \leqslant c\|\Phi^{1/2}\|_{H-S}^2 = c\operatorname{tr}(\Phi^{1/2}\Phi^{1/2*}) = c\operatorname{tr}\Phi.$$

But in the privileged basis of S^q

$$|\varphi_{ij}| = |(e_i, \Phi e_j)| \leqslant \|\Phi\|.$$

Hence if $\operatorname{tr}\Phi$ is in $L_{1,d\gamma}$ then $\varphi_{ij} \in L_{1,d\gamma}$ for $1 \leqslant i, j \leqslant q$. ■

6. REGULARITY AND SINGULARITY

The interpolation problem for univariate stationary processes over any discrete Abelian group was studied by Bruckner in [1]. In this section we shall demonstrate how the results of the previous section extend his results to the case of q-variate stationary processes over any LCA group.

Let I be any family of nonempty subsets of an LCA group G. A q-variate stationary process $(X_g)_{g\in G}$ is called (cf. [1], p. 253) *I-regular* if $\bigcap_{A\in I} \mathfrak{M}_A = 0$ and *I-singular* if $\bigcap_{A\in I} \mathfrak{M}_A = \mathfrak{M}$, where $\mathfrak{M}_A$ denotes the Hilbert subspace of $\mathfrak{M}$ spanned by X_g; $g \in A$.

Following Bruckner (cf. [1], Th. 3.1) we obtain *Wold's decomposition theorem* in the multivariate case.

THEOREM 6.1. *If the family I is closed under translations for all $g \in G$ (i.e., $A \in I$ and $g \in G$ implies $Ag = \{hg;\ h \in A\} \in I$), then every q-variate stationary process $(X_g)_{g\in G}$ over any LCA group G is a sum of two q-variate stationary processes $(X_g^1)_{g\in G}$ and $(X_g^2)_{g\in G}$ over G (i.e., for $g \in G$ $X_g = X_g^1 + X_g^2$) such that the following statements are true:*

(i) *for the spaces $\mathfrak{M}^1$ and $\mathfrak{M}^2$ spanned by $(X_g^1)_{g\in G}$ and $(X_g^2)_{g\in G}$ respectively, we have*

$$\mathfrak{M} = \mathfrak{M}^1 \oplus \mathfrak{M}^2,$$

(ii) *$(X_g^1)_{g\in G}$ is I-regular,*

(iii) *$(X_g^2)_{g\in G}$ is I-singular.*

The proof of this result is essentially the same as in the classical case (cf. [9], p. 75) and we omit it.

Let I_c denote the family of complements of all compact subsets in G. If G is a discrete group then I_c coincides with the family I_∞ of complements of all finite subsets. Denote by I_0 the family of complements of all singletons of G.

LEMMA 6.2. *Let $(X_g)_{g\in G}$ be a q-variate stationary process over any LCA group G; then*

(a) *$(X_g)_{g\in G}$ is I_c-singular if and only if is interpolable,*

(b) *$(X_g)_{g\in G}$ is not I_0-singular if and only if is minimal.*

Proof. (a) If $(X_g)_{g\in G}$ is I_c-singular then for each compact subset C of a group G we have $\mathfrak{M} = \mathfrak{M}_{G-C}$. Hence $\mathfrak{N}_C = 0$ and consequently by Definition 5.1 (b) $(X_g)_{g\in G}$ is interpolable. Conversely, if for each compact subset C $\mathfrak{N}_C = 0$ then $\mathfrak{M} = \mathfrak{M}_{G-C}$ and $\bigcap_{A\in I_c} \mathfrak{M}_A = \mathfrak{M}$. Hence $(X_g)_{g\in G}$ is I_c-singular.

Of course, part (b) may be proved in the same way. ∎

Corollary 5.4, Theorem 5.6 and Lemma 6.2 immediately yield the following

COROLLARY 6.3. *Let $(X_g)_{g\in G}$ be a q-variate stationary process over any LCA group G, with the spectral measure F; then*

(a) *$(X_g)_{g\in G}$ is I_c-singular if and only if for any matrix-valued function $P\in\mathfrak{P}$ the Hellinger integral $\int_\Gamma dN_P dN_P^*/dF$ is zero or does not exist.*

(b) *$(X_g)_{g\in G}$ over a discrete group is not I_0-singular if and only if the Hellinger integral $\int_\Gamma dN_J dN_J/dF \neq 0$.*

Similarly Theorems 5.5 and 5.7 with Lemma 6.2 yield the following multivariate analogue of the results obtained by Bruckner (cf. [1], Th. 4.1 and Th. 5.2)

COROLLARY 6.4. *Let G be a discrete Abelian group and $(X_g)_{g\in G}$ a q-variate stationary process, where the spectral measure F is a.c. w.r.t. the Haar measure $d\gamma$ on dual group Γ; then*

(a) *$(X_g)_{g\in G}$ is I_∞-singular if and only if for any trig-polynomial $W(\gamma)$ with matrix coefficients the integral $\int_\Gamma W(\gamma)(dF/d\gamma)^- W(\gamma)^* d\gamma$ is zero or does not exist.*

(b) *$(X_g)_{g\in G}$ is not I_0-singular if and only if the integral $\int_\Gamma \operatorname{tr}(dF/d\gamma)^- d\gamma$ exists.*

Added in proof. In the case of discrete ICA groups similar problems, concerning interpolation of multivariate stationary processes, were considered also by H. Salehi and J. K. Scheidt, Journal of Multivariate Analysis 2 (1972), pp. 307–331.

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