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When the topology of an infinite-dimensional Banach space coincides with a Hilbert space topology

by

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Abstract. The topology of an infinite-dimensional Banach space V coincides with a Hilbert space topology if and only if the partially ordered set of all closed subspaces of V admits a full set of probability measures.

Let V be an infinite-dimensional Banach space over D, where D is one of three division rings $\boldsymbol{R}$ (real numbers), C (complex numbers) and Q (quaternions). Wa say that the topology of V coincides with a Hilbert space topology if there exists a D-valued inner product $(\cdot,\cdot)$ on $V\times V$ such that V becomes, under $(\cdot,\cdot)$, a Hilbert space over D, and the topology of V induced by the norm associated with $(\cdot,\cdot)$ coincides with its original Banach space topology. It is clear that if the topology of a Banach space V coincides with a Hilbert space topology, then the partially ordered set of closed subspaces of V (which is in fact a lattice) admits an orthocomplementation namely, the orthocomplementation induced by the inner product. The converse implication is the content of a deep theorem proved by S. Kakutani and G. W. Mackey.

THEOREM 1 (Kakutani–Mackey [1]). *Let V be an infinite-dimensional Banach space over D, and let $L(V)$ be the set of all closed subspaces of V. The natural partial order in $L(V)$ (induced by set inclusion) will be denoted by $\leqslant$. Assume that $A\to A^{\perp}$ is an orthocomplementation on $(L(V),\leqslant)$. Then there exists a D-valued inner product $(\cdot,\cdot)$ on $V\times V$ such that*

(i) *V becomes, under $(\cdot,\cdot)$, a Hilbert space over D;*

(ii) *the topology of V, induced by the norm associated with $(\cdot,\cdot)$, coincides with its original topology;*

(iii) *the map $A\to A^{\perp}$ coincides with the orthocomplementation induced by $(\cdot,\cdot)$.*

A proof of this theorem can also be found in [4] (Theorem 7.1).

We see that, owing to Kakutani and Mackey's theorem, the question of deciding when the topology of an infinite-dimensional Banach space V coincides with a Hilbert space topology is reduced to that of deciding when the partially ordered set $L(V)$ of all closed subspaces of V admits an orthocomplementation. In this paper we will give an equivalent condition for this expressed in terms of some measure-theoretical properties of $L(V)$.

Let us recall that by an *orthocomplementation* on a partially ordered set $(L, \leqslant)$ we mean a map $a \to a^{\perp}$ of L into L with the following properties:

(i) $a^{\perp\perp} = a$;

(ii) $a \leqslant b$ implies $b^{\perp} \leqslant a^{\perp}$;

(iii) if $a_1, a_2, \ldots$ is a sequence in L such that $a_i \leqslant a_j^{\perp}$ for $i \neq j$, then the least upper bound $a_1 \cup a_2 \cup \ldots$ exists in $(L, \leqslant)$;

(iv) $a \cup a^{\perp} = b \cup b^{\perp}$ for all $a, b \in L$ ($a \cup a^{\perp}$ is denoted by 1);

(v) $a \leqslant b$ implies $b = a \cup (a \cup b^{\perp})^{\perp}$.

If $\perp$ is an orthocomplementation on $(L, \leqslant)$, then $(L, \leqslant, {}^{\perp})$ is called an *orthocomplemented partially ordered set* (see Mackey [2]).

Let $(L, \leqslant, {}^{\perp})$ be an orthocomplemented partially ordered set. A map $m\colon L \to [0, 1]$ is said to be a *probability measure* on $(L, \leqslant, {}^{\perp})$ if

(i) $m(1) = 1$;

(ii) $m(a_1 \cup a_2 \cup \ldots) = \sum_{j=1}^{\infty} m(a_j)$ whenever $a_i \leqslant a_j^{\perp}$ for $i \neq j$.

A family M of probability measures on $(L, \leqslant, {}^{\perp})$ is said to be *full* if $m(a) \leqslant m(b)$ for all $m \in M$ implies $a \leqslant b$.

If $(L, \leqslant)$ is a partially ordered set, say with the least element 0 and the greatest element 1, and without *a priori* any additional structure, then it is not clear how to give a sensible definition of a probability measure on $(L, \leqslant)$. We might say that a measure is a map $m\colon L \to [0, 1]$ such that $m(0) = 0$, $m(1) = 1$ and $a \leqslant b$ implies $m(a) \leqslant m(b)$, but then we would have nothing more than a monotonic map on $(L, \leqslant)$. Such a definition would not justify calling m a measure, since with the notion of measure we always associate some kind of σ-additivity, and no such property is postulated here. It is interesting that we can give a global definition of a set of mappings from a partially ordered set L into $[0, 1]$ in such a way that an additional structure will be induced on $(L, \leqslant)$, and under this structure every member of our set of mappings will turn out to be a probability measure in the usually accepted sense. In other words, we will globally define a set of probability measures on a partially ordered set without defining an individual measure itself. Accordingly, we accept the following definition.

DEFINITION. Let $(L, \leqslant)$ be a partially ordered set, and let M be a set of mappings from L into $[0, 1]$. We say that M is a *full set of probability measures on* $(L, \leqslant)$ if the following conditions hold:

(i) $a \leqslant b$ if and only if $m(a) \leqslant m(b)$ for all $m \in M$;

(ii) for any sequence $a_1, a_2, \ldots$ in L (finite or countable), where $i \neq j$ implies $m(a_i) + m(a_j) \leqslant 1$ for all $m \in M$, there is $b \in L$ such that $m(b) + m(a_1) + m(a_2) + \ldots = 1$ for all $m \in M$.

Of course, not every partially ordered set admits a full set of probability measures. A necessary (but by no means sufficient) condition is that L admit an orthocomplementation. Namely, we have the following lemma.

LEMMA. *Assume that a partially ordered set $(L, \leqslant)$ admits a full set of probability measures M. For $a, b \in L$, let $a^{\perp} = b$ if and only if $m(a) + m(b) = 1$ for all $m \in M$. Then $a \to a^{\perp}$ is a well-defined map of L into L which is an orthocomplementation on $(L, \leqslant)$. Moreover, every member of M is a probability measure on $(L, \leqslant, {}^{\perp})$, and the family of measures M is full on $(L, \leqslant, {}^{\perp})$.*

Proof. For each $a \in L$, let $\bar{a}$ be a map of M into $[0, 1]$ defined by $\bar{a}(m) = m(a)$ for all $m \in M$. Let $\bar{M} = \{\bar{a}\colon a \in L\}$. We have $\bar{M} \subset [0, 1]^M$ and $\bar{M}$ is naturally ordered by pointwise order of real functions ($\bar{a} \leqslant \bar{b}$ if and only if $\bar{a}(x) \leqslant \bar{b}(x)$ for all $x \in M$). In view of (i) above, $(\bar{M}, \leqslant)$ is isomorphic to $(L, \leqslant)$: $a \leqslant b$ if and only if $\bar{a} \leqslant \bar{b}$. Since for each $a \in L$ the one-element sequence $\{a\}$ satisfies the condition of (ii), there is $b \in L$ such that $m(a) + m(b) = 1$ for all $m \in M$; that is, for each $\bar{a} \in \bar{M}$ there is $\bar{b} \in \bar{M}$ such that $\bar{a} + \bar{b} = 1$. Consequently, the map $a \to a^{\perp}$ is well defined, since $b = a^{\perp}$ if and only if $\bar{b} = 1 - \bar{a}$, and for each $\bar{a} \in \bar{M}$, $1 - \bar{a} \in \bar{M}$. Now for any $a \in M$, the sequence $a, a^{\perp}$ satisfies the condition of (ii), and consequently there is $b \in L$ such that $m(a) + m(a^{\perp}) + m(b) = 1$ for all $m \in L$, i.e. $\bar{a} + \bar{a}^{\perp} + \bar{b} = 1$. This implies $\bar{b} \equiv 0$, i.e. the zero function belongs to $\bar{M}$. Moreover, it follows from (ii) that for any sequence $\bar{a}_1, \bar{a}_2, \ldots$ in $\bar{M}$ satisfying $\bar{a}_i + \bar{a}_j \leqslant 1$ for $i \neq j$ we have $\bar{a}_1 + \bar{a}_2 + \ldots = 1 - \bar{b} \in \bar{M}$. Hence $\bar{M}$ is a set of functions from M into $[0, 1]$ satisfying the following conditions:

1° The zero function belongs to $\bar{M}$.

2° For every $\bar{a} \in \bar{M}$, $1 - \bar{a} \in \bar{M}$.

3° For every sequence $\bar{a}_1, \bar{a}_2, \ldots$ in $\bar{M}$ satisfying $\bar{a}_i + \bar{a}_j \leqslant 1$ for $i \neq j$, we have $\bar{a}_1 + \bar{a}_2 + \ldots \in \bar{M}$.

We may now appeal to Theorem 1 of [3] stating that if $\bar{M}$ is a set of functions satisfying conditions 1°–3°, then $\bar{M}$ is an orthocomplemented partially ordered set with respect to the natural (pointwise) order of real functions with orthocomplementation $\bar{a} = 1 - \bar{a}$. Consequently, $(L, \leqslant, {}^{\perp})$ is also an orthocomplemented partially ordered set. Since $a \leqslant b^{\perp}$ is equivalent to $m(a) + m(b) \leqslant 1$ for all $m \in M$, it is evident that each $m \in M$ is a probability measure on $(L, \leqslant, {}^{\perp})$, and the family M is full. This ends the proof of the lemma.

We can now state the main theorem of this paper.

THEOREM 2. *Let V be an infinite-dimensional Banach space, and let $L(V)$ be the set of all closed subspaces of V partially ordered by set inclusion. Then $(L(V), \leqslant)$ admits an orthocomplementation if and only if $(L(V), \leqslant)$ admits a full set of probability measures.*

Proof. Assume that $(L(V), \leqslant)$ admits a full set of probability measures. Then from the lemma it follows that $(L(V), \leqslant)$ admits an orthocomplementation. Conversely, assume that $(L(V), \leqslant)$ admits an orthocomplementation $A \to A^{\perp}$ where $A \in L(V)$. From Kakutani and Mackey's theorem (Theorem 1) it follows that there exists an inner product $(\cdot, \cdot)$ on $V \times V$ such that V is a Hilbert space with respect to $(\cdot, \cdot)$ (we denote this space by H), and the orthocomplemented partially ordered set $(L(V), \leqslant, {}^{\perp})$ concides with the orthocomplemented partially ordered set of closed subspaces of H. For each $A \in L(V)$, let P_A be the orthogonal projection onto A. For every vector u in the unit sphere S^1 of H, let m_u be a function from $L(V)$ into $[0, 1]$ defined by $m_u(A) = (P_A u, u)$ for all $A \in L(V)$. We claim that $M = \{m_u : u \in S^1\}$ is a full set of probability measures on $(L(V), \leqslant)$ in the sense of the definition given above. In fact, we clearly have $A_1 \subset A_2$ if and only if $(P_{A_1} u, u) \leqslant (P_{A_2} u, u)$ for all $u \in S^1$, i.e. if and only if $m(A_1) \leqslant m(A_2)$ for all $m \in M$. Let $A_1, A_2, \ldots$ be a sequence of members of $L(V)$ satisfying $m(A_i) + m(A_j) \leqslant 1$ for $i \neq j$ and all $m \in M$. This means that $A_1, A_2, \ldots$ is an orthogonal sequence of closed subspaces of H. Let $A = A_1 \oplus A_2 \oplus \ldots$ and let $B = A^{\perp}$, the orthogonal complement of A. We have $B \oplus A_1 \oplus A_2 \oplus \ldots = H$, which implies $m(B) + m(A_1) + m(A_2) + \ldots = 1$ for all $m \in M$. Hence both condition (i) and (ii) of the definition hold and M is a full set of probability measures on the partially ordered set $(L(V), \leqslant)$ of closed subspaces of V. This concludes the proof of the theorem.

From the theorem of Kakutani and Mackey and from Theorem 2 we obtain the following corollary.

COROLLARY. *The topology of an infinite-dimensional Banach space V coincides with a Hilbert space topology if and only if the partially ordered set of closed subspaces of V admits a full set of probability measures.*

References

[1] S. Kakutani and G. W. Mackey, *Rings and lattice characterization of complex Hilbert space*, Bull. Amer. Math. Soc. 52 (1946), pp. 727–733.

[2] G. W. Mackey, *The Mathematical Foundation of Quantum Mechanics*, New York 1963.

[3] M. J. Mączyński, *On a functional representation of the lattice of projections on a Hilbert space*, Studia Math. 48 (1973), pp. 253–259.

[4] V. S. Varadarajan, *Geometry of Quantum Theory*, Vol. I., Princeton, New Jersey 1968.

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Relativ vollstetige Störungen von gewöhnlichen Differentialoperatoren höherer Ordnung

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Zusammenfassung. Für gewöhnliche Differentialoperatoren geradzahliger Ordnung, die halbbeschränkt nach unten sind und deren wesentliches Spektrum bekannt ist, werden mit Hilfe eines Satzes von Birman aus der Störungstheorie quadratischer Formen Störungen der Koeffizienten der Operatoren beschrieben, die das wesentliche Spektrum nicht verändern.

Im folgenden werden selbstadjungierte Differentialoperatoren betrachtet, die von dem Differentialausdruck

$$l[\cdot] = \sum_{\nu=0}^{n} (-1)^{n-\nu} \frac{d^{n-\nu}}{dx^{n-\nu}} a_\nu(x) \frac{d^{n-\nu}}{dx^{n-\nu}}, \qquad x \geqslant 0,$$

erzeugt werden. Dabei sei jeder Koeffizient $a_\nu(x)$, $x \geqslant 0$, $\nu = 1, 2, \ldots, n$, eine reelle Funktion, die bis zur Ordnung $n-\nu-1$ stetig differenzierbar ist und deren Ableitung der Ordnung $n-\nu-1$ eine auf $[0, \infty)$ absolut stetige Funktion ist, deren Ableitung auf jedem endlichen Teilintervall von $[0, \infty)$ im Lebesgueschen Sinne quadratisch integrierbar ist. Dieser Sachverhalt kann kürzer durch

$$\text{(I)} \qquad a_\nu(x) \in W_{2,\text{loc}}^{n-\nu}[0, \infty), \qquad \nu = 1, 2, \ldots, n,$$

beschrieben werden, wenn man den Begriff des Sobolevschen Raumes verwendet [8]. $a_0(x)$ sei gleich einer Konstanten $a_0 > 0$. Gehört die (komplexwertige) Funktion $y(x)$ zu $C_0^\infty(0, \infty)$(1), so liegt $l[y]$ nach den über die Koeffizienten $a_\nu(x)$ getroffenen Voraussetzungen im Hilbertraum $L_2(0, \infty)$. Durch die Festlegung

$$Ay = l[y], \qquad y \in D(A) = C_0^\infty(0, \infty),$$

wird dann ein symmetrischer Operator A mit dem Definitionsbereich $D(A)$ definiert. Im folgenden soll ein Satz über die Lokalisierung des

(1) $C_0^\infty(0, \infty)$ ist die Menge der auf der positiven x-Achse beliebig oft differenzierbaren Funktionen mit kompaktem Träger.