Pour $1 < q < p \leqslant +\infty$, on peut choisir d'après les lemmes 1 et 9

$$\varphi_{q,p} = \varphi^*_{q',p'}$$

avec $\frac{1}{q}+\frac{1}{q'} = \frac{1}{p}+\frac{1}{p'} = 1$.

Le théorème 6 donne alors le

COROLLAIRE 11. *Pour $(p, q) \in [1, +\infty] \times]1, +\infty]$, $\mathscr{C}_{q,p}$ est un espace strictement intermédiaire.*

THÉORÈME 12. *Le dual de $\mathscr{C}_{q,p}$ est $\mathscr{C}_{q',p'}$ avec $\frac{1}{q}+\frac{1}{q'} = \frac{1}{p}+\frac{1}{p'} = 1$ si $(p, q) \in [1, +\infty[\times]1, +\infty[$ ou $p = q = +\infty$.*

Preuve. Le cas $1 < q \leqslant p < +\infty$ résulte des lemmes 1 et 9 et du théorème 4 après avoir constaté que la fonction de norme $\varphi_{q,p} = \varphi^*_{q',p}$ n'est pas équivalente à la φ-norme maximale. Pour ce faire, on s'appuie sur l'inégalité $\sum_{i=1}^{n} i^{\frac{p}{q}-1} \leqslant \frac{q}{p}[(n+1)^{\frac{p}{q}}-1]$ et sur l'équivalence de la norme $|\ |_{\varphi^*_{q',p'}}$ et de la quasi-norme $|\ |_{q,p}$.

Nous avons alors le

COROLLAIRE 13. *Pour $(p, q) \in]1, +\infty[\times]1, +\infty[$ $\mathscr{C}_{q,p}$ est un espace réflexif.*

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Some remarks on the spectra of unitary dilations

by

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Abstract. We generalize several well-known theorems concerning the spectral behavior of the minimal unitary dilation of a single contraction to the setting of contractive representations of certain semigroups. We prove, for example, that if such a representation is completely non-unitary, then the spectral measure for its minimal unitary dilation is quasi-invariant under a certain flow. This generalizes the fact that the spectral measure for the minimal unitary dilation of a single completely non-unitary contraction is mutually continuous with respect to Lebesgue measure on the circle.

§ 1. Introduction. Throughout this note Γ will denote a fixed dense subgroup of the real numbers $\boldsymbol{R}$. We shall give Γ the discrete topology and we shall denote its subsemigroup of nonnegative elements by Γ_+. Also, we shall fix a contractive representation $\{T_\gamma\}_{\gamma\in\Gamma_+}$ of Γ_+ on a (complex) Hilbert space $\mathscr{H}$ and we shall let $\{U_\gamma\}_{\gamma\in\Gamma}$ be its minimal unitary dilation acting on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$. This means first that $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is a family of linear operators on $\mathscr{H}$ such that $\|T_\gamma\| \leqslant 1$ for each γ in Γ_+, $T_{\gamma+\sigma} = T_\gamma T_\sigma$, and such that T_0 is the identity operator on $\mathscr{H}$, and secondly, that $\{U_\gamma\}_{\gamma\in\Gamma}$ is a unitary representation of Γ on $\mathscr{K}$ such that $T_\gamma = PU_\gamma|\mathscr{K}$ for all γ in Γ_+ and such that the smallest subspace of $\mathscr{K}$ containing $\mathscr{H}$ and reducing $\{U_\gamma\}_{\gamma\in\Gamma}$ is $\mathscr{K}$ itself. (Here P denotes the projection of $\mathscr{K}$ onto $\mathscr{H}$, and the vertical bar denotes restriction here and always.) In this note we investigate some of the spectral properties of $\{U_\gamma\}_{\gamma\in\Gamma}$ and prove analogues of well-known theorems concerning the spectral behavior of the minimal unitary dilation of a single contraction (see [7], Chap. II, n° 6). We note that Mlak [3] proved that the minimal unitary dilation of a contractive representation of Γ_+ always exists and, consequently, we are not working in a vacuum.

The group dual to Γ will be denoted by G, and the pairing between the two will be denoted thus: $\langle\gamma, x\rangle$, $\gamma\in\Gamma$, $x\in G$. We shall write $\langle\gamma, \cdot\rangle$ for γ if we wish to regard γ as a function on G. For each t in $\boldsymbol{R}$ we shall write e_t for the element in G defined by the equation $\langle\gamma, e_t\rangle = e^{it\gamma}$. The family $\{e_t\}_{t\in\boldsymbol{R}}$ is a one-parameter subgroup of G and the action of $\boldsymbol{R}$ on G

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it determines will be called the *natural action*. The spectral measure for $\{U_\gamma\}_{\gamma\in\Gamma}$, whose existence is guaranteed by the SNAG Theorem, will be denoted by E and we shall say that E is *quasi-invariant* under the natural action of $\boldsymbol{R}$ on G provided that for each Borel set M in G such that $E(M) = 0$ it happens that $E(M+e_t) = 0$ as well for all t in $\boldsymbol{R}$. The representation $\{T_\gamma\}_{\gamma\in\Gamma_+}$ will be called *completely non-unitary* (c.n.u.) in case there is no nontrivial subspace $\mathscr{M}$ of $\mathscr{H}$ which reduces $\{T_\gamma\}_{\gamma\in\Gamma_+}$ such that $T_\gamma|\mathscr{M}$ is unitary for all γ in Γ_+. According to a theorem of Mlak ([3], Theorem 2.2) every contractive representation of Γ_+ decomposes uniquely into the direct sum of a c.n.u. representation and a unitary representation. Our first theorem is an analogue of the result which states that the spectral measure of the minimal unitary dilation of a c.n.u. contraction is mutually absolutely continuous with respect to Lebesgue measure on the unit circle (see [7], Chap. II, Theorem 6.4).

THEOREM I. *The spectral measure of the minimal unitary dilation of a c.n.u. contractive representation of Γ_+ is quasi-invariant.*

In this paper all scalar measures are nonnegative, finite, regular, and Borel and so we will not append these adjectives when we refer to one. A scalar measure, just like a spectral measure, is called quasi-invariant in case the class of its null sets is preserved under the natural action of $\boldsymbol{R}$ on G. For each vector f in $\mathscr{H}$ we shall write ν_f for the measure defined by the formula $\nu_f(M) = \|E(M)f\|^2$ for all Borel sets M in G.

Our second theorem is an analogue of the second half of Théorème 6.4 on page 78 of [7].

THEOREM II. *If $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is c.n.u., then ν_h is quasi-invariant for each nonzero vector h in $\mathscr{H}$.*

The *measure class* of a scalar measure is the collection of all measures mutually absolutely continuous with respect to it and we shall say that a spectral measure on G belongs to the measure class of a scalar measure in case the two have the same null sets. A vector f in $\mathscr{H}$ is called a *separating vector* for $\{U_\gamma\}_{\gamma\in\Gamma}$ or for E in case E belongs to the measure class of ν_f. It is well known that $\{U_\gamma\}_{\gamma\in G}$ need not have a separating vector, but that a necessary and sufficient condition that it does is that E belongs to the measure class of *some* scalar measure on G. Of course it is always possible to decompose $\mathscr{H}$ into a direct sum of a family of orthogonal subspaces which reduce $\{T_\gamma\}_{\gamma\in\Gamma_+}$ such that the minimal unitary dilation of the restriction of $\{T_\gamma\}_{\gamma\in\Gamma_+}$ to each has a separating vector. If ν is a quasi-invariant measure on G, then for t in $\boldsymbol{R}$, ν_t will denote the measure defined by the formula $\nu_t(M) = \nu(M-e_t)$ for all Borel sets M in G, and the Radon–Nikodym derivative $\varrho(t,x) = \dfrac{d\nu_t}{d\nu}(x)$ will be referred to as the *ϱ-function* for ν.

Our final theorem is an analogue of Proposition 6.5 on page 78 of [7].

THEOREM III. *Suppose $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is c.n.u. and that $\{U_\gamma\}_{\gamma\in\Gamma}$ has a separating vector. Let ν be any scalar measure determining the measure class of E, let ϱ be its ϱ-function, and for each f in $\mathscr{H}$, let F_f denote the Radon–Nikodym derivative $d\nu_f/d\nu$. If h is a nonzero vector in $\mathscr{H}$, then ν_h is supported by the set $\mathfrak{S}_h$ consisting of all x in G such that the function of t*

$$\log\big(F_h(x-e_t)\,\varrho(t,x)\big)/1+t^2$$

belongs to $L^1(\boldsymbol{R})$.

The proof of Theorem I is given in the next section while the proofs of Theorems II and III appear in Section 3. In Section 4 we present some corollaries.

§ 2. The proof of Theorem I.

To prove Theorem I it clearly suffices to show that when $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is c.n.u., $\mathscr{H}$ can be written as the span of two (not necessarily orthogonal) subspaces which reduce $\{U_\gamma\}_{\gamma\in\Gamma}$ such that the restriction of E to each is quasi-invariant. We shall show that this is possible in the two lemmas to follow. But first we must recall certain facts about isometric representations of Γ_+; we refer the reader to ([5], § 2) for definitions of terms used but not defined here.

If $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is an isometric representation of Γ_+, i.e., if $\|T_\gamma f\| = \|f\|$ for all γ in Γ_+ and all f in $\mathscr{H}$, then $\{U_\gamma\}_{\gamma\in\Gamma}$ is its minimal unitary extension and to say that $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is c.n.u. is to say that $\{T_\gamma\}_{\gamma\in\Gamma}$ is *pure*. According to Theorem 0 of [5] every pure isometric representation of Γ_+ can be decomposed uniquely into the direct sum of a shift representation and an evanescent isometric representation. The discussion at the end of § 2 in [5] (see the proof of [5], Proposition 6.6 also) shows that the spectral measure for the minimal unitary extension of a shift representation of Γ_+ belongs to the measure class of Haar measure on G and so must be quasi-invariant. On the other hand, Theorem I of [5] (see in particular equation 3.3) shows that the spectral measure of the minimal unitary extension of an evanescent isometric representation of Γ_+ is quasi-invariant. Therefore, taken together, these two facts constitute a proof of

LEMMA 2.1. *The spectral measure of the minimal unitary extension of a pure isometric representation of Γ_+ is quasi-invariant.*

If S is a subset of $\mathscr{H}$, we shall write $\mathfrak{M}_\pm(S)$ for the space $\bigvee_{\gamma\in\Gamma_+} U_{\pm\gamma}S$ and we shall write $\mathfrak{N}_\pm(S)$ for the space $\bigwedge_{\gamma\in\Gamma_+} U_{\pm\gamma}\mathfrak{M}_\pm(S)$ where the symbols $\bigvee$ and $\bigwedge$ stand for span and intersection respectively. For either choice of sign, the space $\mathfrak{M}_\pm(S)$ is invariant under $\{U_{\pm\gamma}\}_{\gamma\in\Gamma_+}$ while both $\mathfrak{N}_+(S)$ and $\mathfrak{N}_-(S)$ reduce $\{U_\gamma\}_{\gamma\in\Gamma}$. Therefore, if $\mathfrak{K}^+_\pm(S) = \mathfrak{M}_\pm(S)\ominus\mathfrak{N}_\pm(S)$, then $\mathfrak{K}^+_\pm(S)$ is also invariant under $\{U_{\pm\gamma}\}_{\gamma\in\Gamma_+}$ and $\{U_{\pm\gamma}|\mathfrak{K}^+_\pm(S)\}_{\gamma\in\Gamma_+}$ is a pure isometric representation of Γ_+. Finally, if $\mathfrak{K}_\pm(S) = \bigvee_{\gamma\in\Gamma} U_\gamma\mathfrak{K}^+_\pm(S)$, then

$\mathfrak{R}_\pm(S)$ reduces $\{U_\gamma\}_{\gamma\in\Gamma}$ and $\{U_{\pm\gamma}|\mathfrak{R}_\pm(S)\}_{\gamma\in\Gamma}$ is the minimal unitary extension of $\{U_{\pm\gamma}|\mathfrak{R}_\pm^+(S)\}_{\gamma\in\Gamma_+}$. This observation together with Lemma 2.1 and the following lemma clearly completes the proof of Theorem I.

LEMMA 2.2. *The representation $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is c.n.u. if and only if*

$$\mathscr{K} = \mathfrak{R}_+(\mathscr{H})\vee\mathfrak{R}_-(\mathscr{H}).$$

Proof. Since $\{U_\gamma\}_{\gamma\in\Gamma}$ is the minimal unitary dilation of $\{T_\gamma\}_{\gamma\in\Gamma_+}$, $\mathscr{K} = \mathfrak{R}_+(\mathscr{H})\oplus\mathfrak{R}_+(\mathscr{H}) = \mathfrak{R}_-(\mathscr{H})\oplus\mathfrak{N}_-(\mathscr{H})$. Therefore a vector f in $\mathscr{K}$ is orthogonal to $\mathfrak{R}_+(\mathscr{H})\vee\mathfrak{R}_-(\mathscr{H})$ if and only if f belongs to $\mathfrak{N}_+(\mathscr{H})\wedge\mathfrak{N}_-(\mathscr{H})$. Hence the lemma follows from Theorem 3.1 of [3] which shows that $\mathfrak{N}_+(\mathscr{H})\wedge\mathfrak{N}_-(\mathscr{H})$ is the largest subspace $\mathscr{M}$ of $\mathscr{H}$ which reduces $\{T_\gamma\}_{\gamma\in\Gamma_+}$ such that $T_\gamma|\mathscr{M}$ is unitary for each γ in Γ_+.

§ 3. The proofs of Theorems II and III. Let $\mathscr{M}$ be a subspace of $\mathscr{K}$ and let $P_{\mathscr{M}}$ be the projection of $\mathscr{K}$ onto $\mathscr{M}$. Then we shall call $\mathscr{M}$ a *spectral subspace* for $\{U_\gamma\}_{\gamma\in\Gamma}$ in case $P_{\mathscr{M}}$ lies in the von Neumann algebra generated by $\{U_\gamma\}_{\gamma\in\Gamma}$. If $\{U_\gamma\}_{\gamma\in\Gamma}$ has a separating vector, and in particular if $\mathscr{K}$ is separable, then it is well known that a spectral subspace for $\{U_\gamma\}_{\gamma\in\Gamma}$ can always be written as the range of $E(M)$ for some Borel set M in G (see [1]). In the absence of separating vector, this is not always so.

LEMMA 3.1. *Let ν be a measure on G and let $\mathscr{K}_\nu$ be the set of all vectors f in $\mathscr{K}$ such that ν_f is absolutely continuous with respect to ν and let P_ν be the projection of $\mathscr{K}$ onto $\mathscr{K}_\nu$. Then $\mathscr{K}_\nu$ is a spectral subspace for $\{U_\gamma\}_{\gamma\in\Gamma}$. If, in addition, ν is quasi-invariant and if $\mathscr{M}$ is any subspace of $\mathscr{K}$ which is invariant under $\{U_\gamma\}_{\gamma\in\Gamma_+}$, then P_ν and $P_{\mathscr{M}}$ commute where $P_{\mathscr{M}}$ is the projection of $\mathscr{K}$ onto $\mathscr{M}$.*

Proof. Basic spectral theory [1] tells us that $\mathscr{K}_\nu$ is a spectral subspace for $\{U_\gamma\}_{\gamma\in\Gamma}$. Assume, now, that ν is quasi-invariant and that $\mathscr{M}$ is invariant under $\{U_\gamma\}_{\gamma\in\Gamma_+}$. Since $\mathfrak{N}_+(\mathscr{M})$ and $\mathfrak{R}_+(\mathscr{M})$ reduce $\{U_\gamma\}_{\gamma\in\Gamma}$, P_ν commutes with the projections onto $\mathfrak{N}_+(\mathscr{M})$ and $\mathfrak{R}_+(\mathscr{M})$. Therefore, to prove the lemma, we may assume without loss of generality that $\{U_\gamma|\mathscr{M}\}_{\gamma\in\Gamma_+}$ is a pure isometric representation of Γ_+ and that $\{U_\gamma\}_{\gamma\in\Gamma}$ is its minimal unitary extension. By Theorem 0 in [5] we may write $\mathscr{M}$ as $\mathscr{M}_s\oplus\mathscr{M}_e$ where $\mathscr{M}_s$ and $\mathscr{M}_e$ reduce $\{U_\gamma|\mathscr{M}\}_{\gamma\in\Gamma_+}$ so that $\{U_\gamma|\mathscr{M}_s\}_{\gamma\in\Gamma_+}$ (resp. $\{U_\gamma|\mathscr{M}_e\}_{\gamma\in\Gamma_+}$) is a shift representation of Γ_+ (resp. an evanescent isometric representation of Γ_+). Likewise, we may write $\mathscr{K} = \mathscr{K}_s\oplus\mathscr{K}_e$ where $\mathscr{K}_s$ and $\mathscr{K}_e$ reduce $\{U_\gamma\}_{\gamma\in\Gamma}$ so that $\{U_\gamma|\mathscr{K}_s\}_{\gamma\in\Gamma}$ (resp. $\{U_\gamma|\mathscr{K}_e\}_{\gamma\in\Gamma}$) is the minimal unitary extension of $\{U_\gamma|\mathscr{M}_s\}_{\gamma\in\Gamma_+}$ (resp. $\{U_\gamma|\mathscr{M}_e\}_{\gamma\in\Gamma_+}$). Since P_ν commutes with the projections onto $\mathscr{K}_s$ and $\mathscr{K}_e$, it suffices to consider the two cases $\mathscr{M} = \mathscr{M}_s$ and $\mathscr{M} = \mathscr{M}_e$ separately. If $\mathscr{M} = \mathscr{M}_s$, so $\mathscr{K} = \mathscr{K}_s$, then as we pointed out in Section 2, E belongs to the measure class of Haar measure σ on G. Since σ is ergodic under the natural action of $\boldsymbol{R}$ on G (i.e., the only invariant Borel sets are null or have null complements) and since ν is quasi-invariant, it follows that either ν belongs to the measure class of σ, in which case P_ν is the identity on $\mathscr{K}$, or σ and ν are singular, in which case $P_\nu = 0$. Therefore, in either case P_ν commutes with $P_{\mathscr{M}}$. If, on the other hand, $\mathscr{M} = \mathscr{M}_e$, then by Theorem I of [5], $P_{\mathscr{M}}$ is a spectral projection of a strongly continuous unitary representation $\{S_t\}_{t\in\boldsymbol{R}}$ of $\boldsymbol{R}$ on $\mathscr{K}$ such that $S_t^*E(M)S_t = E(M-e_t)$ for all Borel sets M in G and all t in $\boldsymbol{R}$. Therefore, if f is in $\mathscr{K}_\nu$ and if M is a null set for ν, then this equation and the fact that ν is quasi-invariant imply the following equation which shows that P_ν commutes with $\{S_t\}_{t\in\boldsymbol{R}}$.

$$\nu_{S_tf}(M) = \|E(M)S_tf\|^2 = \|S_t^*E(M)S_tf\|^2 = \|E(M-e_t)f\|^2 = \nu_f(M-e_t) = 0.$$

Consequently, P_ν commutes with $P_{\mathscr{M}}$ and the proof is complete.

COROLLARY 3.2. *Let P_ν be as in Lemma 3.1 with ν quasi-invariant. Then P_ν commutes with the projection onto $\mathscr{H}$.*

Proof. This follows from Lemma 3.1 and the fact that $\mathscr{H}$ may be written as the orthogonal difference of two subspaces which are invariant under $\{U_\gamma\}_{\gamma\in\Gamma_+}$ ([6], Lemma 0).

The proofs of Theorems II and III are based primarily upon our next lemma which is an analogue of the following well-known and often used fact: Suppose for the moment that U is a unitary operator on the Hilbert space $\mathscr{K}$ and that $\mathscr{M}$ is an invariant subspace for U such that $U|\mathscr{M}$ is a pure isometry. Then for each nonzero vector k in $\mathscr{M}$, ν_k is mutually absolutely continuous with respect to Lebesgue measure m on the unit circle and $\log(d\nu_k/dm)$ lies in $L^1(m)$. In order to prove the lemma, we need some additional notation and terminology. We shall denote the space of functions f on $\boldsymbol{R}$ such that $f(t)/(1-it)$ belongs to the Paley–Wiener class by $H^2\left(\dfrac{dt}{\pi(1+t^2)}\right)$. If ν is a quasi-invariant measure on G and if $\theta(t,x)$ is a unimodular function on $\boldsymbol{R}\times G$, then we shall call $\theta(t,x)$ a *cocycle* in case (i) when regarded as a function from $\boldsymbol{R}$ into $L^2(\nu)$ it is continuous and (ii) $\theta(t_1+t_2,x) = \theta(t_1,x)\theta(t_2,x-e_{t_1})$ a.e. (ν) for each pair of numbers t_1 and t_2 in $\boldsymbol{R}$. Finally, if k is a vector in $\mathscr{K}$, then we shall write $\mathfrak{M}_\pm(k)$, $\mathfrak{R}_\pm(k)$ … for $\mathfrak{M}_\pm(\{k\})$, $\mathfrak{R}_\pm(\{k\})$, etc.

LEMMA 3.3. *Let $\mathscr{M}$ be a subspace of $\mathscr{K}$ such that $U_\gamma\mathscr{M}\subseteq\mathscr{M}$ for each γ in Γ_+ and such that $\{U_\gamma|\mathscr{M}\}_{\gamma\in\Gamma_+}$ is a pure isometric representation of Γ_+. Then for each nonzero k in $\mathscr{M}$, ν_k is quasi-invariant and as a function of t, $(\log\varrho_k(t,x))/(1+t^2)$ belongs to $L^1(\boldsymbol{R})$ a.e. (ν_k) where $\varrho_k(t,x)$ is the ϱ-function for ν_k.*

Proof. The hypothesis implies that $\mathfrak{M}_+(k) = \mathfrak{R}_+^+(k)$ and so we may restrict our attention to the pure isometric representation $\{U_\gamma|\mathfrak{R}_+^+(k)\}_{\gamma\in\Gamma_+}$ of Γ_+ and to its minimal unitary extension $\{U_\gamma|\mathfrak{R}_+(k)\}_{\gamma\in\Gamma}$. Since k is a cyclic vector for $\{U_\gamma|\ \mathfrak{K}_+(k)\}_{\gamma\in\Gamma}$, $E|\ \mathfrak{K}_+(k)$ and ν_k are mutually absolutely continuous and so, by Lemma 2.1, ν_k is quasi-invariant. Also, since k is a cyclic vector for $\{U_\gamma|\ \mathfrak{K}_+(k)\}_{\gamma\in\Gamma}$, the map which assigns to each finite linear combination $\sum c_\gamma U_\gamma k$ in $\mathfrak{K}_+(k)$ the sum $\sum c_\gamma\langle\gamma,\cdot\rangle$ in $L^2(\nu_k)$ extends to a Hilbert space isomorphism W from $\mathfrak{K}_+(k)$ onto $L^2(\nu_k)$ such that $WU_\gamma W^{-1}$ is the operator of multiplication by the function $\langle\gamma,\cdot\rangle$ on $L^2(\nu_k)$. When convenient, we shall identify $\mathfrak{K}_+(k)$ with $L^2(\nu_k)$, $\{U_\gamma\}_{\gamma\in\Gamma}$ with $\{WU_\gamma W^{-1}\}_{\gamma\in\Gamma}$, and $\mathfrak{M}_+(k)$ with $W\mathfrak{M}_+(k)$.

Let σ denote Haar measure on G as before, let P_σ be the projection of $\mathscr{K}$ onto the space of all f in $\mathscr{K}$ such that $\nu_f\ll\sigma$, write $k'=P_\sigma k$ and write $k''=k-k'$. Then by Lemma 3.1, k' and k'' belong to $\mathscr{M}$ and so by what was just proved, $\nu_{k'}$ and $\nu_{k''}$ are quasi-invariant. Since σ is ergodic, $\nu_{k'}$ and σ belong to the same measure class and therefore, since $\nu_{k''}$ and σ are supported on disjoint Borel sets, it suffices to consider the following two cases separately in order to complete the proof.

Case 1. ν_k and σ are mutually absolutely continuous.

Write w for the Radon–Nikodym derivative $d\nu_k/d\sigma$ and note that by the chain rule and the fact that σ is invariant we have $\varrho_k(t,x)=w(x-e_t)/w(x)$. Now, in $L^2(\nu_k)=L^2(wd_\sigma)$, $\mathfrak{M}_+(k)$ is the span of the set of functions $\{\langle\gamma,\cdot\rangle\}_{\gamma\in\Gamma_+}$ and since $\mathfrak{M}_+(k)$ is not all of $L^2(\nu_k)$, a famous theorem of Helson and Lowdenslager ([2], Theorem 5) implies that as a function of t, $\log w(x-e_t)/(1+t^2)$ belongs to $L^1(\boldsymbol{R})$ a.e. (σ). Therefore, in this case, $\log\varrho_k(t,x))/(1+t^2)$ belongs to $L^1(\boldsymbol{R})$ a.e. ν_k.

Case 2. ν_k and σ are mutually singular.

In this case, Proposition 6.7 of [5] implies that $\{U_\gamma|\ \mathfrak{M}_+(k)\}_{\gamma\in\Gamma_+}$ s an evanescent isometric representation of Γ_+ and so by the discussion at the end of § 5 in [5] we may assert that there is a cocycle $\theta(t,x)$ such that $\mathfrak{M}_+(k)$ consists of all functions f in $L^2(\nu_k)$ with the property that as functions of t $\bar\theta(t,x)\bar f(x-e_t)\varrho_k^{\frac12}(t,x)$ belongs to $H^2\left(\dfrac{dt}{\pi(1+t^2)}\right)$ a.e. (ν_k). By basic Hardy space theory then, we find that for each f in $\mathfrak{M}_+(k)$, $\log\big(|f(x-e_t)|^2\varrho_k(t,x)\big)/(1+t^2)$ belongs to $L^1(\boldsymbol{R})$ as a function of t a.e. (ν_k). But k belongs to $\mathfrak{M}_+(k)$ and in $L^2(\nu_k)$, k is the constant function 1. Thus, as was promised, $\log\big(\varrho_k(t,x)\big)/(1+t^2)$ belongs to $L^1(\boldsymbol{R})$ as a function of t a.e. ν_k, and the proof is complete.

We are now in a position to present the

Proof of Theorem II. We shall prove more than we need in order to obtain Theorem II; however, the excess will be used in the proof of Theorem III. Given a nonzero vector h in $\mathscr{H}$ our goal is to produce four vectors h_1, h_2, k_1 and k_2 with the following properties:

(3.1.a) $h=h_1+h_2$.

(3.1.b) ν_{k_1} and ν_{k_2} are mutually singular.

(3.1.c) $\mathfrak{M}_+(k_1)=\mathfrak{K}_+^\dagger(k_1)$, $\mathfrak{M}_-(k_2)=\mathfrak{K}_-^\dagger(k_2)$.

(3.1.d) For $i=1,2$, $h_i=P_{k_i}h$ while $\nu_{k_i}(M)\leqslant\nu_{h_i}(M)$ for all Borel sets M in G (where, for the remainder of this section, we shall write P_{k_i} (resp. P_{h_i}) for the projection of $\mathscr{K}$ onto the space of vectors f such that $\nu_f\ll\nu_{k_i}$ (resp. $\nu_f\ll\nu_{h_i}$), $i=1,2$.)

Of course we allow the possibility that one of the pairs (h_i,k_i), $i=1,2$, may consists of zero vectors. Observe that (3.1.c) and Lemma 3.3 imply that each ν_{k_i} is quasi-invariant while (3.1.d) and Lemma 3.1 show that $\nu_h=\nu_{h_1}+\nu_{h_2}$. Since (3.1.d) implies that each ν_{h_i} is equivalent to ν_{k_i}, we may conclude that each ν_{h_i} is quasi-invariant and so ν_h is quasi-invariant. Thus if we can produce the four desired vectors the proof of Theorem II will be complete.

By Theorem 3.1 of [3] one or the other of the spaces $\mathfrak{K}_+(h)$ and $\mathfrak{K}_-(h)$ must be different from the zero space and so we may assume without loss of generality that $\mathfrak{K}_+(h)\neq\{0\}$. Let Q_1 be the projection of $\mathscr{K}$ onto $\mathfrak{K}_+(h)$ and let $k_1=Q_1h$. Then $k_1\neq 0$ by assumption and a moment's reflection reveals that

(3.2.a) $$\mathfrak{M}_+(k_1)=\mathfrak{K}_+^\ddagger(k_1)=\mathfrak{K}_+^\ddagger(h)$$

and

(3.2.b) $$\mathfrak{K}_+(k_1)=\mathfrak{K}_+(h).$$

Equation (3.2.a) and Lemma 3.3 imply that ν_{k_1} is quasi-invariant and equation (3.2.b) implies that $Q_1\leqslant P_{k_1}$. Therefore, if $h_1=P_{k_1}h$ then for each Borel set M in G we see that $\nu_{k_1}(M)=\|E(M)Q_1h\|^2=\|E(M)Q_1P_{k_1}h\|^2=\|QE(M)h_1\|^2\leqslant\|E(M)h_1\|^2=\nu_{h_1}(M)$.

Let $h'=h-h_1$ and note that since ν_{k_1} is quasi-invariant, h_1 and h' lie in $\mathscr{H}$ by Corollary 3.2. Also since ν_{k_1} is quasi-invariant, Lemma 3.1 implies that $\mathfrak{M}_+(h)=\mathfrak{M}_+(h_1)\oplus\mathfrak{M}_+(h')$ while $\mathfrak{N}_+(h)=\mathfrak{N}_+(h_1)\oplus\mathfrak{N}_+(h')$ and, consequently, $\mathfrak{K}_+^\ddagger(h)=\mathfrak{K}_+^\ddagger(h_1)\oplus\mathfrak{K}_+^\ddagger(h')$. But since $Q_1\leqslant P_{k_1}$, $k_1=Q_1h_1$ and so $\mathfrak{K}_+^\ddagger(h_1)=\mathfrak{K}_+^\ddagger(k_1)=\mathfrak{K}_+^\ddagger(h)$; or equivalently, $\mathfrak{K}_+^\ddagger(h')=\{0\}$ and $\mathfrak{M}_+(h')=\mathfrak{N}_+(h')$. Since h' belongs to $\mathscr{H}$ this, along with Theorem 3.1 of [3] implies that $\mathfrak{K}_-(h')\neq\{0\}$. Let Q_2 be the projection of $\mathscr{K}$ onto $\mathfrak{K}_-(h')$ and let $k_2=Q_2h'$. Then $Q_2\leqslant I-P_{k_1}$ so that ν_{k_2} and ν_{k_1} are mutually singular and we have the following equations:

(3.3.a) $$\mathfrak{M}_-(k_2)=\mathfrak{K}_-^\pm(k_2)=\mathfrak{K}_-^\pm(h'),$$

and

(3.3.b) $$\mathfrak{K}_-(k_2)=\mathfrak{K}_-(h').$$

Equation (3.3.a) and Lemma 3.3 imply that ν_{k_2} is quasi-invariant and equation (3.3.b) implies that $Q_2\leqslant P_{k_2}$. Therefore, as before, if we set

$h_2 = P_{k_2}h'$, then since ν_{k_1} and ν_{k_2} are mutually singular $h_2 = P_{k_2}h$ and the following inequality is valid for all Borel sets M in G,

$$\nu_{k_2}(M) = \|E(M)Q_2h'\|^2 = \|Q_2E(M)P_{k_2}h'\|^2 \leqslant \|E(M)h_2\|^2 = \nu_{h_2}(M).$$

Thus to complete the proof, we need to show that $h = h_1+h_2$. To this end, let $h'' = h-(h_1+h_2) = h'-h_2$. Then by Corollary 3.2, h'' belongs to $\mathscr{H}$ and by what was shown above, h'' also belongs to $\mathfrak{N}_+(h)$. Hence, since $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is c.n.u. by hypothesis, $\mathfrak{N}_+(h)\wedge\mathfrak{N}_-(h) = \{0\}$ by Theorem 3.1 of [3], and so all we need to show is that h'' belongs to $\mathfrak{N}_-(h)$. Since ν_{k_1} and ν_{k_2} are mutually singular, P_{k_1} and P_{k_2} are orthogonal, and since the two measures are quasi-invariant, Lemma 3.1 allows us to write the following two equations

$$\text{(3.4)}\qquad \mathfrak{M}_-(h) = \mathfrak{M}_-(h_1)\oplus\mathfrak{M}_-(h') = \mathfrak{M}_-(h_1)\oplus\mathfrak{M}_-(h_2)\oplus\mathfrak{M}_-(h''),$$

$$\text{(3.5)}\qquad \mathfrak{N}_-(h) = \mathfrak{N}_-(h_1)\oplus\mathfrak{N}_-(h') = \mathfrak{N}_-(h_1)\oplus\mathfrak{N}_-(h_2)\oplus\mathfrak{N}_-(h'').$$

Hence it suffices to show that $\mathfrak{M}_-(h'') = \mathfrak{N}_-(h'')$. But $Q_2\leqslant P_{k_2}$ and so $\mathfrak{K}_-^{\pm}(h_2) = \mathfrak{K}_-^{+}(k_2) = \mathfrak{K}_-^{+}(h')$. Therefore by equations (3.4) and (3.5) we find that $\mathfrak{M}_-(h') = \mathfrak{K}_-^{+}(h')\oplus\mathfrak{N}_-(h') = \mathfrak{K}_-^{+}(h_2)\oplus\mathfrak{N}_-(h_2)\oplus\mathfrak{N}_-(h'') = \mathfrak{M}_-(h_2)\oplus\mathfrak{N}_-(h'')$. Whence $\mathfrak{N}_-(h'') = \mathfrak{M}_-(h'')$ and the proof is complete.

Proof of Theorem III. We continue to use the notation and auxiliary results developed in the proof of Theorem II. Let $\mu = \nu_{k_1}+\nu_{k_2}$ and observe that the properties (3.1) imply that μ is quasi-invariant, equivalent to ν_h, and $\mu(M)\leqslant\nu_h(M)$ for all Borel sets M. Furthermore, if F is the Radon–Nikodym derivative $d\mu/d\nu$, then $F = F_{k_1}+F_{k_2}$ and by the chain rule for Radon–Nikodym derivatives we find that the ϱ-function for μ can be written as $F(x-e_t)\varrho(t,x)/F(x)$. Hence by Lemma 3.3 and the properties (3.1), it follows that as a function of t, $\log\big(F(x-e_t)\varrho(t,x)\big)/(1+t^2)$ belongs to $L^1(\boldsymbol{R})$ a.e. ν on $\mathfrak{S}_{k_1}\cup\mathfrak{S}_{k_2}$ and $\mathfrak{S}_{k_1}\cup\mathfrak{S}_{k_2}$ supports μ. But μ is equivalent to ν_h, so $\mathfrak{S}_{k_1}\cup\mathfrak{S}_{k_2}$ supports ν_h, and since $F_h = F$, $\nu_h\big(\mathfrak{S}_h\triangle(\mathfrak{S}_{k_1}\cup\mathfrak{S}_{k_2})\big) = 0$. It follows that $\mathfrak{S}_h$ supports ν_h and that the proof is complete.

§ 4. Corollaries. As our first corollary we give another proof of the following result due to Mlak ([4], Theorem 3).

Corollary 4.1. *If $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is a contractive representation of Γ_+ which is not a unitary representation, then the closed support of E is all of G.*

Proof. By Theorem 2.2 of [3], it suffices to assume that $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is c.n.u. Then E is quasi-invariant by Theorem I and so the closed support of E is invariant under the natural action of $\boldsymbol{R}$ on G. But it is well known that the only non-empty closed invariant set of G is G itself and so the proof is complete.

The spectral measure E is called erogdic provided that if M is any Borel set in G which is invariant under the natural action of $\boldsymbol{R}$ on G, then $E(M)$ is the identity or the zero operator. We note that the proof of Theorem III in [5] shows that if E is quasi-invariant and ergodic, then E has a separating vector.

The proofs of Lemma 3.1 and Corollary 3.2 show that if M is an invariant Borel set, then $E(M)\mathscr{H}$ reduces $\{T_\gamma\}_{\gamma\in\Gamma_+}$ and therefore we have

Corollary 4.2. *If $\{T_\gamma\}_{\gamma\in\Gamma_+}$ is irreducible then E is ergodic.*

The analysis in [5] shows that the converse of this corollary is far from being true.

We conclude with the following corollary which contains Theorem 1 of [4] as a special case.

Corollary 4.3. *Let the notation be as in Theorem* III *and in addition to the hypotheses there, assume that ν is ergodic. Then $\nu(G\setminus\mathfrak{S}_h) = 0$ for each nonzero vector h in $\mathscr{H}$.*

Proof. Indeed, Theorem III implies that $\nu(\mathfrak{S}_h) > 0$ and since $\mathfrak{S}_h$ is easily seen to be invariant, the result follows from the ergodicity of ν.

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