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RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY

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A note on multipliers on a Segal algebra

by

K. R. UNNI (Madras, India)

Abstract. It is the purpose of this paper to show that if $S(G)$ is a Segal algebra on the locally compact abelian group G and T is a multiplier on $S(G)$ then there exists a unique pseudomeasure σ such that $Tf = \sigma * f$ for each $f \in S(G)$.

Various properties of $S(G)$ are given in Reiter [5]. We denote by $\hat{G}$ the character group of G. Let dx and $d\gamma$ denote the Haar measures on G and $\hat{G}$ respectively where $d\gamma$ is so chosen that the Fourier inversion theorem holds. Let $\mathscr{K}(\hat{G})$ denote the space of continuous functions on $\hat{G}$ with compact support and let

$$B(G) = \{f \in L^1(G) \colon \hat{f} \in \mathscr{K}(\hat{G})\}.$$

Then $B(G)$ is dense in $S(G)$.

A multiplier on $S(G)$ is a bounded linear operator on $S(G)$ which commutes with translations. The problem of characterizing multipliers on various special cases of Segal algebras has been studied by Lai [3], Larsen [4], Keshava Murthy and Unni [1], [2], and Unni [7]. In another paper [6] we introduced the space of parameasures which contains the space of pseudomeasures as a subclass and showed that if T is a multiplier on $S(G)$ then there exists a unique parameasures β such that $Tf = \beta * f$ for each $f \in B(G)$.

Recently Keshava Murthy has brought to my attention a paper by Yap [8] who proves that every Segal algebra on a locally compact abelian group is a semisimple Banach algebra. Though parameasures are of independent interest, the semisimplicity of the Segal algebra makes it possible to prove the following

THEOREM. *Let G be a locally compact abelian group and $S(G)$ a Segal algebra. If T is a multiplier on $S(G)$ then there exists a unique pseudomeasure σ such that*

$$Tf = \sigma * f \quad \textit{for each } f \in S(G).$$

Proof. Let us recall that the algebra $A(G)$ is the space of Fourier transforms of functions integrable over $\hat{G}$ with the topology given by

$$\|\hat{f}\|_{A(G)} = \|f\|_{L^1(\hat{G})}$$

and the space of pseudomeasures is the dual space of $A(G)$ in the above norm. We have seen in [6] that $B(G)$ is dense in $A(G)$.

Let T be a multiplier on $S(G)$. Then $T(f*g) = Tf*g$ for all $f, g \in S(G)$. (This is proved in [6].) Since $S(G)$ is a semisimple Banach algebra with convolution as multiplication, the arguments of Larsen ([4], p. 241) shows that there exists a continuous bounded function Φ defined on $\hat{G}$ such that $\widehat{Tf} = \Phi\hat{f}$ for each $f \in S(G)$. Hence it follows that $B(G)$ is invariant under T and thus Tf is a continuous function for each $f \in B(G)$. If we set $L(f) = Tf(0)$ then

$$|L(f)| = |Tf(0)| \leqslant \|Tf\|_\infty \leqslant \|\widehat{Tf}\|_1 = \|\Phi\hat{f}\|_1 \leqslant \|\Phi\|_\infty \|\hat{f}\|_1 = \|\Phi\|_\infty \|f\|_{A(G)}$$

and L is a continuous linear functional on $B(G)$ which is dense in $A(G)$, and hence can be extended uniquely as a continuous linear functional on $A(G)$ without increasing the norm. Hence there exists a unique pseudomeasure σ such that $L(f) = Tf(0) = \langle f, \tilde{\sigma}\rangle$ for $f \in B(G)$. Then $Tf = \sigma * f$ for all $f \in B(G)$. The uniqueness of pseudomeasure σ can be proved either as in Larsen [4] or as follows. We have proved in [6] the existence of a unique parameasure β such that $Tf = \beta * f$ for each $f \in B(G)$. Hence $Tf = \sigma * f = \beta * f$ for each $f \in B(G)$. Since the space of parameasures contains the space of pseudomeasures, the uniqueness of σ follows from the uniqueness of β.

It remains to show that $Tf = \sigma * f$ holds for all $f \in S(G)$. Let $f \in S(G)$. Since $S(G) \subset L^1(G)$ for each $\sigma \in P(G)$, we have $\sigma * f \in P(G)$. Let $\{\mu_\alpha\}$ be an approximate identity such that $\mu_\alpha \in B(G)$, $\|\mu_\alpha\|_1 = 1$ and $\|\mu_\alpha * f - f\|_S \to 0$. Since $\mu_\alpha * f \in B(G)$ we have $T(\mu_\alpha * f) = \sigma * (\mu_\alpha * f)$. From the relation

$$\|\sigma * (\mu_\alpha * f) - \sigma * (\mu_\beta * f)\|_S = \|T(\mu_\alpha * f) - T(\mu_\beta * f)\|_S \leqslant \|T\| \, \|\mu_\alpha * f - \mu_\beta * f\|_S$$

we see that $\{\sigma * (\mu_\alpha * f)\}$ is a Cauchy net in $S(G)$. Then there exists an $F \in S(G)$ such that $\|F - \sigma * (\mu_\alpha * f)\|_S \to 0$. Now

$$\begin{aligned} \|\hat{F} - \hat{\sigma}\hat{f}\|_\infty &\leqslant \|\hat{F} - \hat{\sigma}(\hat{\mu}_\alpha \hat{f})\|_\infty + \|\hat{\sigma}(\hat{\mu}_\alpha \hat{f}) - \hat{\sigma}\hat{f}\|_\infty \\ &\leqslant \|F - \sigma * (\mu_\alpha * f)\|_1 + \|\hat{\sigma}\|_\infty \|\mu_\alpha * f - f\|_1 \end{aligned}$$

from which it follows that $\hat{F} = \hat{\sigma}\hat{f}$. Since f and F are in $S(G) \subset L^1(G)$, by the Fourier inversion theorems we obtain $F = \sigma * f$ and

$$\|Tf - \sigma * (\mu_\alpha * f)\|_S = \|Tf - T(\mu_\alpha * f)\|_S \leqslant \|T\| \, \|f - \mu_\alpha * f\|_S \to 0$$

shows that $Tf = \sigma * f$ for each $f \in S(G)$. This completes the proof.

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THE INSTITUTE OF MATHEMATICAL SCIENCES
MADRAS, INDIA

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