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Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh–Paley series*

by

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Abstract. We prove weighted integral inequalities between the Lusin area function and nontangential maximal function of a harmonic function. We also obtain results for Walsh–Paley series as a corollary of the method.

Introduction. In this paper we prove weighted integral inequalities for the Lusin area function and the nontangential maximal function. Specifically, we are able to answer some questions raised in [11], and extend the inequalities proved there. Our results indicate that many of the norm inequalities for H^p-spaces in $\boldsymbol{R}_+^{n+1}$ remain true for a wide class of measures on the boundary. Our method consists of showing that certain distribution function inequalities, proved in [2] for the area function and the nontangential maximal function, are valid not only for Lebesgue measure, but also for this wide class of measures. These inequalities lead easily to the desired norm inequalities.

The technique used in studying the area integral may also be used to obtain weighted norm inequalities for Walsh–Paley series. Inequalities of this kind were first studied by Hirschman [6]; we are able to recover and extend his results.

Theorems concerning the area function and nontangential maximal function are in Section 1; Walsh–Paley series are treated in Section 2. Section 3 contains a remark on the radial maximal function.

1. We use the notation of [2]. A cone of opening a in $\boldsymbol{R}_+^{n+1} = \{(x, y):$ $x \in \boldsymbol{R}^n, y > 0\}$ is defined as

$$\Gamma(x) = \Gamma(x, a) = \{(s, y):\ |x-s| < ay\}.$$

The area function, corresponding to a harmonic function u, is given by

$$A(x) = A_a(u)(x) = \Big(\iint_{\Gamma(x)} y^{1-n} |\nabla u(s, y)|^2 ds\, dy\Big)^{\frac{1}{2}},$$

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and the nontangential maximal function by

$$N(x) = N_a(u)(x) = \sup_{(s,y)\in\Gamma(x)} |u(s,y)|.$$

Another auxiliary function is

$$D(x) = \sup_{(s,y)\in\Gamma(x)} y|\nabla u(s,y)|.$$

"Local" versions of these functions are defined as follows: If R is a measurable subset of $\boldsymbol{R}_+^{n+1}$, let

$$A_R^2(x) = \iint_{\Gamma(x)\cap R} y^{1-n}|\nabla u(s,y)|^2 ds\,dy,$$

$$N_R(x) = \sup_{(s,y)\in\Gamma(x)\cap R} |u(s,y)|,$$

and

$$D_R(x) = \sup_{(s,y)\in\Gamma(x)\cap R} y|\nabla u(s,y)|.$$

These definitions make sense if $\Gamma(x)\cap R$ is nonempty; otherwise, each function is defined to be zero. Notice that as R expands to the entire space $\boldsymbol{R}_+^{n+1}$, each function increases to its unrestricted version.

We often use $m(\cdot)$ to denote Lebesgue measure on $\boldsymbol{R}^n$; when it occurs under an integral, we simply write dx.

The following class of measures was discovered by Benjamin Muckenhoupt in connection with his discussion of norm inequalities for the Hardy–Littlewood maximal function [8]. Let $1 < p < \infty$, $q = p/(p-1)$ and

$$m_w(dx) = w(x)\,dx,$$

where $w(x)$ is a non-negative, locally integrable function that satisfies

$$(\mathrm{A}_p) \qquad \left(\frac{1}{m(I)}\int_I w(x)\,dx\right)\left(\frac{1}{m(I)}\int_I [w(x)]^{-q/p}\,dx\right)^{p/q} \leqslant C < \infty$$

for all cubes $I\subset \boldsymbol{R}^n$. The constant C is independent of the cube I. Charles Fefferman observed that any function $w(x)$ that satisfies condition A_p for some $p>1$ has the following property: If E is a measurable subset of a cube I, then

$$(\mathrm{A}_\infty) \qquad \frac{m(E)}{m(I)} \leqslant \varepsilon \Rightarrow \frac{m_w(E)}{m_w(I)} \leqslant C\varepsilon^{1/r}$$

for some C and $r>1$, independent of E and I. In fact, as Fefferman pointed out, property A_∞ follows easily from another fact about A_p-functions, due to Muckenhoupt ([8], see inequality (3.19) and Section 7). If $w(x)$ satisfies A_p, then it also satisfies a "backwards Hölder inequality" of the form

$$\left[\frac{1}{m(I)}\int_I [w(x)]^s\,dx\right]^{1/s} \leqslant C\frac{1}{m(I)}\int_I w(x)\,dx$$

for some C and $s>1$, independent of the cube I. To see that A_∞ follows from this, we use Hölder's inequality to write

$$\frac{1}{m(I)}\int_E w(x)\,dx \leqslant \left(\frac{m(E)}{m(I)}\right)^{1/r}\left(\frac{1}{m(I)}\int_I [w(x)]^s\,dx\right)^{1/s}$$
$$\leqslant C\left(\frac{m(E)}{m(I)}\right)^{1/r}\left(\frac{1}{m(I)}\int_I w(x)\,dx\right)$$

where $s^{-1}+r^{-1}=1$. We may arrange this to obtain

$$\frac{m_w(E)}{m_w(I)} \leqslant C\left(\frac{m(E)}{m(I)}\right)^{1/r},$$

which is equivalent to A_∞.

Remarks. (a) It follows easily from Hölder's inequality that if $w(\cdot)$ satisfies A_p, then it also satisfies $\mathrm{A}_{p+\varepsilon}$ for every $\varepsilon>0$. That is, in general, $\mathrm{A}_{p+\varepsilon}$ is weaker than A_p for $\varepsilon>0$.

(b) B. Muckenhoupt [9] has recently proved that condition A_∞ implies A_p for some $p>1$. (This, together with the converse indicated above, shows that A_∞ is equivalent to requiring condition A_p for some $p>1$, and explains our notation "A_∞".) Muckenhoupt proved this fact for functions w on $\boldsymbol{R}^1$; Wo-Sang Young has adapted the proof for functions on $\boldsymbol{R}^n$.

(c) If $w(x) = |x|^a$, $-1<a<p-1$, then $w(x)$ satisfies condition A_p on $\boldsymbol{R}^1$. These weight functions have been considered by many authors.

(d) The inequality defining condition A_∞ implies a converse also holds. That is, if $w(\cdot)$ satisfies A_∞, then there exist constants p and c, $1<p<\infty$ and $c>0$, such that for any measurable subset E of a cube I,

$$(1.1) \qquad \frac{m_w(E)}{m_w(I)} \geqslant c\left(\frac{m(E)}{m(I)}\right)^p.$$

To show (1.1), observe by remark (b) that for some p, $1<p<\infty$, we have $w\in\mathrm{A}_p$. By Hölder's inequality,

$$m(E) = \int_E w^{1/p}w^{-1/p}\,dx \leqslant \Big(\int_E w\,dx\Big)^{1/p}\Big(\int_E w^{-q/p}\,dx\Big)^{1/q}.$$

Since $E\subset I$ and w satisfies condition A_p, we have

$$\int_E w^{-q/p}\,dx \leqslant \int_I w^{-q/p}\,dx \leqslant Cm(I)^{1+q/p}\Big(\int_I w\,dx\Big)^{-q/p}.$$

Combining these estimates, we obtain

$$m(E) \leqslant Cm(I)\Big(\int_E w\,dx\Big)^{1/p}\Big(\int_I w\,dx\Big)^{-1/p},$$

which is (1.1).

In particular, if tI denotes the cube with the same center as I whose edges are t times as long as those of I then

$$\frac{m_w(I)}{m_w(tI)} \geqslant c_t > 0 \tag{1.2}$$

for any $t > 1$. This implies that the A_∞ criterion, stated originally for cubes, holds if cubes are replaced by balls. Other variants are also possible.

(e) In the discussion of Walsh–Páley series, slightly weaker A_p and A_∞ conditions are appropriate. Instead of requiring that $w(\cdot)$ satisfy the A_p or A_∞ condition on all intervals, we demand that it hold only on intervals I of the form $\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)$. This condition is referred to as A_p (dyadic) or A_∞ (dyadic).

THEOREM 1. *Let Φ be a nondecreasing continuous function on the interval $0 \leqslant \lambda < \infty$, such that $\Phi(0) = 0$ and $\Phi(2\lambda) \leqslant C_1\Phi(\lambda)$. If $u(\cdot)$ is harmonic on $\boldsymbol{R}_+^{n+1}$ and $m_w(dx)$ satisfies condition A_∞, then*

$$c\int \Phi\big(A(u)\big)m_w(dx) \leqslant \int \Phi\big(N(u)\big)m_w(dx). \tag{i}$$

Conversely, if the left hand side of (i) *is finite, u may be normalized by subtracting a suitable constant so that it vanishes at infinity; assuming this normalization we have,*

$$\int \Phi\big(N(u)\big)m_w(dx) \leqslant C\int \Phi\big(A(u)\big)m_w(dx). \tag{ii}$$

The constants c and C depend only on the growth constant for Φ, the size of the opening of the cone Γ, the measure $m_w(dx)$, and the dimension n.

In particular, for $0 < p < \infty$,

$$\int [A(u)]^p m_w(dx) \simeq \int [N(u)]^p m_w(dx)$$

for u suitably normalized if necessary.

These inequalities complete and extend the partial results contained in [11] (see Theorem 3).

COROLLARY 1. *Let u be a harmonic function on $\boldsymbol{R}_+^{n+1}$. Then*

$$c\int [A(u)]^p m_w(dx) \leqslant \sup_{y>0}\int |u(x,y)|^p m_w(dx)$$

for $1 < p < \infty$ provided $m_w(dx)$ satisfies A_p. If the left-hand side is finite, u may be normalized to vanish at infinity so that

$$\sup_{y>0}\int |u(x,y)|^p m_w(dx) \leqslant C\int [A(u)]^p m_w(dx).$$

In this inequality, $m_w(dx)$ is required to satisfy only the condition A_∞.

This corollary extends Theorem 1 of [11].

Fefferman and Stein [3] have shown that the H^p spaces in $\boldsymbol{R}_+^{n+1}$, for $0 < p < \infty$, can be characterized as the spaces of harmonic functions u such that

$$\int [N(u)]^p\,dx < \infty.$$

The key to this definition is the fact that

$$\int [N(u)]^p\,dx \simeq \int [A(u)]^p\,dx.$$

Theorem 1 shows that this important equivalence holds for all measures in the class A_∞. We will discuss the analogous characterization for weighted H^p spaces briefly at the end of this section.

The same ideas that apply in Theorem 1 may be used to prove norm inequalities for Walsh–Paley series. (See Section 2 for definitions.)

THEOREM 2. *Let $f = (f_1, f_2, \ldots)$ be the sequence of 2^kth partial sums of a Walsh–Paley series. If Φ is a nondecreasing continuous function with $\Phi(0) = 0$, $\Phi(2\lambda) \leqslant C_1\Phi(\lambda)$, and $m_w(dx) = w(x)\,dx$ is a measure on the unit interval that satisfies A_∞ (dyadic), then*

$$c\int \Phi\big(S(f)\big)m_w(dx) \leqslant \int \Phi(f^*)\,m_w(dx) \leqslant C\int \Phi\big(S(f)\big)m_w(dx)$$

where

$$f^* = \sup_n |f_n|$$

and

$$S(f) = \Big(\sum_{n=1}^{\infty}(f_{n+1}-f_n)^2 + f_1^2\Big)^{\frac{1}{2}}.$$

In general, the proofs in this paper are minor modifications of those in [2], and we do not reproduce all the details. However, some lemmas from [2] are treated again to take account of the change of measures.

LEMMA 1 (Lemma 2 of [2]). *Let $b > a > 0$. Then for all $\lambda > 0$*

$$m_w(N_b > \lambda) \leqslant c\,m_w(N_a > \lambda)$$

where $w(\cdot)$ satisfies A_∞. The choice of c depends only on n, the ratio a/b, and the function $w(\cdot)$.

Proof. We use Muckenhoupt's result that A_∞ implies A_p for some p, $1 < p < \infty$, (Remark b), and his theorem from [8]:

If $w(\cdot)$ satisfies A_p, $1 < p < \infty$, and f^* is the Hardy–Littlewood maximal function of f, then

$$\int [f^*(x)]^p w(x)\,dx \leqslant C \int |f(x)|^p w(x)\,dx$$

where C depends only on p, $w(\cdot)$, and n.

The proof of Lemma 1 now follows the same lines as Lemma 2 of [2]. We replace the set

$$\{N_a > \lambda\} = \bigcup \{B(x, ay): |u(x, y)| > \lambda\}$$

by a larger set, using the Hardy–Littlewood maximal function f^* of the characteristic function f of the set $\{N_a > \lambda\}$. In fact, by an elementary geometric argument, given in [2],

$$\{N_b > \lambda\} \subset \{f^* \geqslant a\}$$

where $a = a^n/(a+b)^n$. Therefore, by Muckenhoupt's theorem quoted above, we have

$$\begin{aligned} m_w(N_b > \lambda) &\leqslant m_w(f^* \geqslant a) \\ &\leqslant a^{-p} \int [f^*(x)]^p w(x)\,dx \\ &\leqslant C a^{-p} \int |f(x)|^p w(x)\,dx \\ &= C' m_w(N_a > \lambda). \end{aligned}$$

This proves Lemma 1.

The inequalities of Theorem 1 are consequences of certain distribution function inequalities. These distribution function inequalities are stated as Theorems 3 and 4, corresponding to Theorems 2 and 3 of [2].

THEOREM 3. *Let G be a bounded open subset of $\mathbf{R}^n$ and R the interior of the complement of $\bigcup_{x \notin G} \Gamma(x)$; let $m_w(dx)$ be a measure on $\mathbf{R}^n$ that satisfies A_∞. Given $a > 1$, $\beta > 1$, there exist constants γ and δ such that*

$$a m_w(A_R \geqslant \beta\lambda, N_R \leqslant \gamma\lambda, D_R \leqslant \delta\lambda) \leqslant m_w(A_R > \lambda)$$

for all $\lambda > 0$. The conclusion also holds for $R = \mathbf{R}_+^{n+1}$ by passage to the limit.

In the following theorem, corresponding to Theorem 3 of [2], we need a variant of $N_R(x)$, defined as follows. If $\Gamma(x) \cap R$ is empty, let $N_R^0(x) = 0$; otherwise, let

$$N_R^0(x) = \sup_{(s,y) \in \Gamma(x) \cap R} |u(s, y) - u(s, y_s)|$$

where (s, y_s) is the point on the upper boundary of R directly above (s, y):

$$y_s = \sup\{y': (s, y') \in R\}.$$

THEOREM 4. *Let G be a bounded open subset of $\mathbf{R}^n$ and R the interior of the complement of $\bigcup_{x \notin G} \Gamma(x)$. Given $a > 1$, $\beta > 1$, there exist constants γ and δ such that*

$$a m_w(N_R^0 > \beta\lambda, A_R \leqslant \gamma\lambda, D_R \leqslant \delta\lambda) \leqslant m_w(N_R^0 > \lambda)$$

for all $\lambda > 0$. The conclusion also holds for $R = \mathbf{R}_+^{n+1}$ by passage to the limit.

We begin by proving Theorem 1 (i), assuming the validity of Theorem 3. To avoid technical difficulties, we may assume that u is continuous in the closed half-space $\{(x, y): y \geqslant 0\}$. If necessary, replace $u(x, y)$ by $u_\delta(x, y) = u(x, y+\delta)$, $\delta > 0$; if inequality (1.3) holds for u_δ, we may obtain it for $u = \lim_{\delta \to 0} u_\delta$ by the monotone convergence theorem.

The first step is to show that

$$\int \Phi(A_R)\,dm_w \leqslant C \int \Phi(N_R)\,dm_w + C \int \Phi(D_R)\,dm_w \tag{1.3}$$

where R is any region of the kind described in Theorem 1, for u harmonic in $\mathbf{R}_+^{n+1}$ and continuous in the closed half-space. In this case, the integrals in (1.3) are all finite since A_R, N_R and D_R are bounded and have compact support. To prove (1.3), write[1]

$$\begin{aligned} \int \Phi(A_R)\,m_w(dx) &= \int_{\mathbf{R}^n} \int_0^{A_R(x)} \Phi(d\lambda)\,m_w(dx) \\ &= \int_0^\infty m_w(A_R > \lambda)\,\Phi(d\lambda). \end{aligned}$$

Notice that

$$\begin{aligned} \int \Phi(A_R)\,m_w(dx) &\leqslant C_1 \int \Phi(\tfrac{1}{2}A_R)\,m_w(dx) \\ &= C_1 \int_0^\infty m_w(A_R > 2\lambda)\,\Phi(d\lambda), \end{aligned}$$

by the growth condition imposed on Φ. Now we use Theorem 3 with $a = 2C_1$ and $\beta = 2$:

$$\begin{aligned} m_w(A_R > 2\lambda) &\leqslant m_w(A_R \geqslant 2\lambda, N_R \leqslant \gamma\lambda, D_R \leqslant \delta\lambda) + m_w(N_R > \gamma\lambda) + \\ &\qquad + m_w(D_R > \delta\lambda) \\ &\leqslant \frac{1}{2C_1} m_w(A_R > \lambda) + m_w(N_R > \gamma\lambda) + m_w(D_R > \delta\lambda). \end{aligned}$$

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[1] The following computations are simpler and somewhat more general than those in [2]; they were communicated to us by D. L. Burkholder.

Therefore, we may integrate both sides of this inequality to obtain

$$\int \Phi(A_R)\, m_w(dx) \leqslant C_1 \int_0^\infty m_w(A_R > 2\lambda)\, \Phi(d\lambda)$$

$$\leqslant \tfrac{1}{2} \int \Phi(A_R)\, m_w(dx) + C_1 \int \Phi(\gamma^{-1} N_R)\, m_w(dx) + C_1 \int \Phi(\delta^{-1} D_R)\, m_w(dx).$$

Since both sides of this inequality are finite, we may subtract the first term on the right-hand side from both sides, and use the growth condition on Φ to obtain

$$\int \Phi(A_R)\, m_w(dx) \leqslant C \int \Phi(N_R)\, m_w(dx) + C \int \Phi(D_R)\, m_w(dx).$$

Now we let R expand indefinitely and apply the monotone convergence theorem to obtain

$$\int \Phi(A)\, m_w(dx) \leqslant C \int \Phi(N)\, m_w(dx) + C \int \Phi(D)\, m_w(dx).$$

The second step of the proof is to show that

$$\int \Phi(D)\, m_w(dx) \leqslant C \int \Phi(N)\, m_w(dx). \tag{1.4}$$

We use the inequality

$$D(x) = D_a(x) \leqslant C N_b(x)$$

if $b > a > 0$. (See Stein [12], p. 207.) Therefore,

$$m_w(D > \lambda) \leqslant m_w(C N_b > \lambda)$$
$$\leqslant C\, m_w(C N_a > \lambda)$$

by Lemma 1. This inequality, together with the growth condition imposed on Φ, implies inequality (1.4).

Therefore,

$$\int \Phi(A)\, m_w(dx) \leqslant C \int \Phi(N)\, m_w(dx) + C \int \Phi(D)\, m_w(dx)$$
$$\leqslant C \int \Phi(N)\, m_w(dx),$$

so that Theorem 1 is proved.

Proof of Theorem 3. The key to the proof of this theorem is simply to isolate the relevant part of the proof of inequality (29) of [2].

Let $R_\varepsilon = \{(x, y) \in R,\, y > \varepsilon\}$; then A_{R_ε} is a continuous function on $\boldsymbol{R}_+^{n+1}$ that vanishes outside of G. Therefore, $G_0 = \{A_{R_\varepsilon} > \lambda\}$ is an open set whose closure is contained in G. Let

$$E = \{A_{R_\varepsilon} > \beta\lambda,\, N_{R_\varepsilon} \leqslant \gamma\lambda,\, D_{R_\varepsilon} \leqslant \delta\lambda\}$$

where γ and δ are suitably chosen, the choice to depend only on a, β, and n. To prove Theorem 3, it is enough to prove

$$a\, m_w(E) < m_w(G_0). \tag{1.5}$$

(Since the choice of constants γ and δ is independent of ε, we may obtain

$$a\, m(A_R > \beta\lambda,\, N_R \leqslant \gamma\lambda,\, D_R \leqslant \delta\lambda) < m(A_R > \lambda)$$

by letting $R_\varepsilon \to R$ as ε tends to zero.)

In order to prove (1.5), we decompose G_0 into disjoint cubes I_k using the Whitney decomposition theorem (see Stein [12]). Each cube I_k is covered by an open ball B_k with the same center, contained in G_0, and such that

$$\text{diameter } B_k \leqslant c(\text{diameter } I_k).$$

Furthermore, each ball is required to have at least one of its boundary points contained in the complement of G_0. For each cube I_k in the Whitney decomposition, there is such a ball B_k: Consider all balls centered at the center of I_k, containing I_k, and contained in G_0. This family is non-empty since the Whitney cube I_k has the property that its diameter is less than its distance to the boundary of G_0. Let B be the union of all members of this family. The maximality of B guarantees that at least one of its boundary points belongs to the complement of G_0. The crucial fact needed to prove (1.5) is the following:

For such balls B, there exist constants γ and δ, depending only on a_0, β, and n such that

$$a_0\, m(E \cap B) < m(B). \tag{1.6}$$

As before, the constants a_0 and β are restricted only by the assumption that they are greater than one. We refer to inequality (1.6) as the "special" inequality. It differs from the general inequality (1.5) in that it refers only to Lebesgue measure and to balls B, contained in G_0, with at least one boundary point in the complement of G_0.

Before proving the special inequality, let us indicate how the general inequality (1.5) follows from it. If B is any "Whitney ball", as described above, then

$$a_0\, m(E \cap B) < m(B)$$

implies

$$a\, m_w(E \cap B) < m_w(B)$$

where $a = C a_0^{1/r}$, by the A_∞ condition applied to balls instead of cubes (see Remark (d)). Then

$$a\, m_w(E) = a \sum_k m_w(E \cap I_k) \leqslant a \sum_k m_w(E \cap B_k)$$
$$< \sum_k m_w(B_k) \leqslant c \sum_k m_w(I_k) = c\, m_w(G_0),$$

since condition A_∞ implies

$$m_w(B_k) \leqslant c\, m_w(I_k)$$

by Remark (d). Thus,

$$a\, m_w(E) < c\, m_w(G_0)$$

for all $a > 1$. Since c is independent of a, this is equivalent to the general inequality (1.5).

The proof of the special inequality (1.6) is contained in the proof of Theorem 3, [2]. In the notation of [2] (see especially, the discussion following inequality (29)), the argument consists of showing that

$$m(B) < a_0 m(E \cap B)$$

is *not* possible with a_0 greater than a fixed constant $2c_{(25)} > 1$ and constants γ and δ arbitrarily small. This, of course, is equivalent to our assertion. For the details of this argument we refer the reader to [2].

Theorem 1 (ii) follows from Theorem 4 in almost the same way as Theorem 1 (i) follows from Theorem 3. If

$$\int \Phi(A)\, m_w(dx) < \infty,$$

then u may be normalized to vanish at infinity by the same argument used in the proof of Theorem 1 of [2]. (The essential point to check is that $m_w(dx)$ is always an unbounded measure on $\boldsymbol{R}^n$. This follows immediately from remark (d).) Therefore

$$\lim_{R \to \boldsymbol{R}_+^{n+1}} N_R^0(x) = N(x),$$

so that Fatou's lemma implies

$$\int \Phi(N)\, m_w(dx) \leqslant \liminf_{R \to \boldsymbol{R}_+^{n+1}} \int \Phi(N_R^0)\, m_w(dx)$$
$$\leqslant C \int \Phi(A)\, m_w(dx);$$

the last inequality is proved from Theorem 4 in the same way Theorem 3 is used to prove part (i).

The proof of Theorem 4 follows the same lines as the proof of Theorem 3. The crucial fact needed is the following. Given a ball $B \subset \{N_{R_\varepsilon}^0 > \lambda\}$ with at least one boundary point in $\{N_{R_\varepsilon}^0 \leqslant \lambda\}$, constants α and β larger than 1, there exist constants γ and δ such that

$$a\, m(E \cap B) \leqslant m(B), \tag{1.7}$$

where $E = \{N_{R_\varepsilon}^0 > \beta\lambda, f^* \leqslant \tfrac{1}{2}, D_{R_\varepsilon} \leqslant \delta\lambda\}$ and f^* is the Hardy–Littlewood maximal function of the characteristic function of $\{A_{R_\varepsilon} > \gamma\lambda\}$. Again,

(1.7) is a routine extension of inequalities proved in [2]—see in particular inequalities (34) and (35) and the ensuing argument. The remainder of the proof of Theorem 4, using A_∞ to mediate between Lebesgue measure and m_w, is virtually the same as in Theorem 3.

Proof of Corollary 1. To prove the first part of Corollary 1, we claim that if $1 < p < \infty$ and

$$\sup_{y>0} \int_{\boldsymbol{R}^n} |u(x, y)|^p m_w(dx) = C_1 < \infty,$$

for a function w which satisfies A_p, then u is the Poisson integral of a function f with

$$\int_{\boldsymbol{R}^n} |f(x)|^p m_w(dx) \leqslant C_1.$$

From this we can derive the first inequality of the corollary as follows. By ([12], p. 92),

$$N(u)(x) \leqslant c f^*(x),$$

where f^* is the Hardy–Littlewood maximal function of f. Therefore, by Theorem 1 (i) and [8],

$$\int [A(u)]^p m_w(dx) \leqslant c \int [N(u)]^p m_w(dx) \leqslant c \int [f^*]^p m_w(dx)$$
$$\leqslant c \int |f|^p m_w(dx) \leqslant c C_1.$$

To establish the claim, observe that the $L^p(m_w(dx))$ norms of the functions $u_y(x) = u(x, y)$ are uniformly bounded by $C_1^{1/p}$. Hence there exists a sequence y_k converging to zero and a function f with

$$\int |f|^p m_w(dx) \leqslant C_1$$

such that $u(x, y_k)$ converges weakly to $f(x)$—that is, so that

$$\lim_{y_k \to 0} \int u(x, y_k) g(x) m_w(dx) = \int f(x) g(x) m_w(dx) \tag{1.8}$$

for $g \in L^q(m_w(dx))$, $q = p/(p-1)$. Now choose $g(x) = P(x-z, \delta) w(x)^{-1}$, where P is the Poisson kernel and (z, δ) is a fixed point of $\boldsymbol{R}_+^{n+1}$. Indeed, since w satisfies A_p it is easy to check that $w^{1-q} = w^{-1/(p-1)}$ satisfies A_q, and therefore, as in (2.3) of [7], $g \in L^q(m_w(dx))$. With this choice of g, the right side of (1.8) is the Poisson integral $f(z, \delta)$ of f, while the left side is $u(z, y_k + \delta)$. Since $\lim_{y_k \to 0} u(z, y_k + \delta) = u(z, \delta)$, the claim follows.

The second part of the corollary follows immediately from Theorem 1 (ii).

We shall now give a short discussion of weighted H^p spaces. For the most part, the arguments are just weighted versions of those of [3], Section 8, together with those of Corollary 1 above, so we shall be very brief. For a function $u(x, y)$ harmonic in $\boldsymbol{R}_+^{n+1}$ and a weight w satisfying A_∞, we write $u \in H_w^p$, $0 < p < \infty$, if

$$\|u\|_{H_w^p} = \Big(\int_{\boldsymbol{R}^n} [N(u)(x)]^p m_w(dx)\Big)^{\frac{1}{p}} < \infty. \tag{1.9}$$

Here we choose $N(u) = N_1(u)$ for definiteness, but by Lemma 1

$$c^{-1}\|u\|_{H_w^p} \leqslant \Big(\int_{\boldsymbol{R}^n} [N_a(u)(x)]^p m_w(dx)\Big)^{\frac{1}{p}} \leqslant c\|u\|_{H_w^p}$$

for any $a > 0$, where c depends only on a, n and w.

We shall discuss the relation of this definition to a weighted version of the Stein–Weiss definition of H^p, as described in [3], Section 8. The basic ingredient of this definition is a vector

$$F(x, y) = \big(u_0(x, y), u_1(x, y), \ldots, u_m(x, y)\big)$$

whose length m depends on p and another constant r, $0 < r < p < \infty$. The components $u_i(x, y)$ are harmonic in $\boldsymbol{R}_+^{n+1}$ and the function

$$|F(x, y)|^r = \Big(\sum_{i=0}^{m} u_i^2(x, y)\Big)^{r/2}$$

is subharmonic in $\boldsymbol{R}_+^{n+1}$. (The details concerning m and its relation to p and r are given in [3], Section 8.) We say that the vector F is in Stein–Weiss H_w^p if

$$\|F\|_{p,w} = \sup_{y>0}\Big(\int_{\boldsymbol{R}^n} |F(x, y)|^p m_w(dx)\Big)^{1/p} < \infty.$$

Given p and r, $0 < r < p < \infty$, w in A_∞, and a harmonic function u with $\|u\|_{H_w^p} < \infty$, then there is a vector F with $u_0 = u$ such that

$$\|F\|_{p,w} \leqslant c\|u\|_{H_w^p}.$$

In fact, the other components u_i, $i = 1, \ldots, m$ satisfy

$$\|u_i\|_{H_w^p} \leqslant c\|u\|_{H_w^p}.$$

Conversely, given $0 < r < p < \infty$, a vector F and a weight function w in $A_{p/r}$, we have

$$\Big(\int_{\boldsymbol{R}^n} [N(|F|)]^p m_w(dx)\Big)^{1/p} \leqslant c\|F\|_{p,w}.$$

Moreover, if $w \in A_{p/r}$, the principle of harmonic majorization holds—that is, there is a non-negative function $h(x)$, with

$$\int_{\boldsymbol{R}^n} h(x)^{p/r} m_w(dx) \leqslant c\|F\|_{p,w}^p,$$

whose Poisson integral $h(x, y)$ satisfies $h(x, y) \geqslant |F(x, y)|^r$.

In summary, if w is in $A_{p/r}$, then the Stein–Weiss definition of H_w^p is equivalent to the nontangential definition.

To establish these facts, let us suppose first that we are given p and r, w in A_∞, and $u \in H_w^p$. The proof that there exists an F with all the properties listed then follows from the same arguments as those in the second part of the proof of Theorem 9 of [3], using the equivalence in Theorem 1 above with $\Phi(t) = t^p$, once we verify that a $u \in H_w^p$ satisfies

$$\Big|\frac{\partial^k}{\partial y^k}\frac{\partial^\alpha}{\partial x^\alpha} u(x, y)\Big| \leqslant Ay^{-k-|\alpha|-\delta}$$

for $y \geqslant 1$ for some $\delta > 0$, where the constant A depends on x, u, k and α. To see this, it is enough by standard homogeneity arguments (see [12], p. 275) to show that

$$\sup_{\{z:\, |z|<y\}} |u(x+z, y)| \leqslant Ay^{-\delta}$$

for $y \geqslant 1$ for some $\delta > 0$, $A = A(x, u)$. To prove this, let $x = 0$ and $\|u\|_{H_w^p} = 1$ for simplicity. By a lemma due to Hardy and Littlewood (see Lemma 2, Section 9 of [3]) we know that given ε, $0 < \varepsilon < 1$, there is a constant c so that

$$|u(z, y)|^{\varepsilon p} \leqslant cy^{-n-1} \iint_{B_{\frac{y}{2}}(z,y)} |u(\xi, \eta)|^{\varepsilon p}\, d\xi\, d\eta.$$

Here $B_{\frac{y}{2}}(z, y)$ denotes the ball with center (z, y) and radius $\frac{y}{2}$. Hence

$$|u(z, y)|^{\varepsilon p} \leqslant cy^{-n-1}\int_{y/2}^{3y/2} d\eta \int_{|\xi-z|<y/2} |u(\xi, \eta)|^{\varepsilon p} w(\xi)^{\varepsilon} w(\xi)^{-\varepsilon}\, d\xi$$

$$\leqslant cy^{-n-1}\int_{y/2}^{3y/2} d\eta \Big(\int_{|\xi-z|<y/2} |u(\xi, \eta)|^p m_w(d\xi)\Big)^{\varepsilon}\Big(\int_{|\xi-z|<y/2} w(\xi)^{-\frac{\varepsilon}{1-\varepsilon}}\, d\xi\Big)^{1-\varepsilon}$$

by Hölder's inequality. Since $\|u\|_{H_w^p} = 1$, we have

$$\sup_{\{z:\, |z|<y\}} |u(z, y)|^{\varepsilon p} \leqslant cy^{-n}\Big(\int_{|\xi|<2y} w(\xi)^{-\frac{\varepsilon}{1-\varepsilon}}\, d\xi\Big)^{1-\varepsilon}.$$

Since w satisfies A_∞, there is an s, $1 < s < \infty$, such that w satisfies A_s. Choose $\varepsilon = s^{-1}$, so that $\dfrac{\varepsilon}{1-\varepsilon} = \dfrac{1}{s-1}$. Then $w^{-1/(s-1)}$ satisfies $A_{\frac{s}{s-1}}$ and, by Lemma 5 of [8], satisfies A_t for some $t < \frac{s}{s-1}$. Hence as in (2.3) of [7],

$$B = \int_{R^n} \frac{w(\xi)^{-\frac{1}{s-1}}}{(1+|\xi|)^{nt}}\, d\xi < \infty.$$

In particular,

$$B \geqslant \int_{|\xi|<2y} \frac{w(\xi)^{-\frac{1}{s-1}}}{(1+|\xi|)^{nt}}\, d\xi \geqslant \frac{1}{(1+2y)^{nt}} \int_{|\xi|<2y} w(\xi)^{-\frac{1}{s-1}}\, d\xi.$$

Therefore, if $y \geqslant 1$,

$$\sup_{\{z:\, |z|<y\}} |u(z, y)|^{ep} \leqslant cy^{-n}[B(1+2y)^{nt}]^{\frac{s-1}{s}} \leqslant Ay^{-\delta},$$

$$\delta = n - nt\left(\frac{s-1}{s}\right) > 0.$$

Conversely, suppose we are given p, r, F and w in $A_{p/r}$. Then $s(x, y) = |F(x, y)|^r$ is subharmonic, continuous and satisfies

$$\sup_{y>0} \int_{R^n} [s(x, y)]^{p/r} m_w(dx) = \|F\|^p_{p,w}.$$

Since $w \in A_{p/r}$, it follows essentially from known arguments (see [12] and the proof of Corollary 1 above) that there is a non-negative $h(x)$,

$$\int_{R^n} h(x)^{p/r} m_w(dx) \leqslant \|F\|^p_{p,w}$$

whose Poisson integral $h(x, y)$ satisfies $s(x, y) \leqslant h(x, y)$. Thus,

$$N(|F|)(x) \leqslant [N(h)(x)]^{1/r} \leqslant c[h^*(x)]^{1/r}.$$

where h^* is the Hardy–Littlewood maximal function of h. Therefore, by Theorem 9 of [8],

$$\int_{R^n} [N(|F|)(x)]^p m_w(dx) \leqslant c \int_{R^n} h(x)^{p/r} m_w(dx) \leqslant c\|F\|^p_{p,w}.$$

2. Walsh–Paley series. Let $r_0(x) = \operatorname{sgn}\sin 2\pi x$, and $r_n(x) = r_0(2^n x)$. The Walsh–Paley functions are defined as follows:

$$\psi_0(x) \equiv 1;$$

$$\psi_N(x) = r_{n_1}(x) \ldots r_{n_k}(x),$$

if $N = 2^{n_1} + \ldots + 2^{n_k}$, $n_1 > n_2 > \ldots > n_k$.

The collection $\{\psi_N(x), N = 0, 1, \ldots\}$ forms a complete orthonormal system for L^2 over the unit interval $0 \leqslant x \leqslant 1$.

We summarize the relevant facts concerning Walsh–Paley series (see [1] for more details). Let a_k, $k = 1, 2, \ldots$, be a sequence of real numbers; we define

$$f_{n+1}(x) = \sum_{k=0}^{2^n-1} a_k \psi_k(x);$$

$$d_0(x) = a_0;$$

$$d_{n+1}(x) = f_{n+1}(x) - f_n(x).$$

The sequence $f = (f_1, f_2, \ldots)$ is a martingale, and we may associate several growth-measuring functions with the sequence f. The two most prominent functions are

$$f^*(x) = \sup_n |f_n(x)|$$

and

$$S(f)(x) = \Big(\sum_{k=0}^{\infty} d_k^2(x)\Big)^{\frac{1}{2}}.$$

If we define $\|f\|_p = \sup_n \|f_n\|_p$ then a classical inequality of Paley [10] asserts

$$c_p\|S(f)\|_p \leqslant \|f\|_p \leqslant C_p\|S(f)\|_p$$

for $1 < p < \infty$. A parallel result, due essentially to Hardy and Littlewood [4] states that

$$c_p\|f^*\|_p \leqslant \|f\|_p \leqslant C_p\|f^*\|_p$$

for $1 < p < \infty$. If these two inequalities are combined, it is shown in [1] that the range $1 < p < \infty$ can be extended to the entire interval $0 < p < \infty$. Thus

$$c_p\|S(f)\|_p \leqslant \|f^*\|_p \leqslant C_p\|S(f)\|_p$$

holds for $0 < p < \infty$, and, more generally,

$$\int \Phi(S(f))\, dx \simeq \int \Phi(f^*)\, dx \tag{2.1}$$

for functions Φ satisfying the growth condition mentioned above.

We now turn to the question of extending these inequalities when Lebesgue measure on the unit interval is replaced by another measure $m_w(dx)$. The appropriate definitions of A_∞ (dyadic) and A_p (dyadic) are obvious: We require that the A_∞ (or A_p) condition hold, not with respect to all intervals, but only on all dyadic intervals (intervals I of the form $\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ for some k, n). With these classes of weight functions, Muckenhoupt's maximal function inequality holds:

$$\int [f^*(x)]^p w(x)\,dx \leqslant C_p \int |f(x)|^p w(x)\,dx \tag{2.2}$$

if and only if $w(\cdot)$ satisfies A_p (dyadic). The proof of this fact is virtually the same as the one given by Muckenhoupt.

If we combine Muckenhoupt's result (2.2) with Theorem 2 of this paper, we obtain the inequality

$$c_p \int [S(f)(x)]^p w(x)\,dx \leqslant \int |f(x)|^p w(x)\,dx \leqslant C_p \int [S(f)(x)]^p w(x)\,dx$$

if $1 < p < \infty$ and $w(\cdot)$ satisfies condition A_p (dyadic). This includes and extends the result of Hirschman [6], who treated the special case $w(x) = |x|^a$, $-1 < a < p-1$.

Theorem 2 itself may be proved in exactly the same way as Theorem 1, by appealing to the following distribution function inequalities for f^* and $S(f)$.

THEOREM 5. *Let $f = (f_1, f_2, \ldots)$ be the sequence of 2^kth partial sums of a Walsh–Paley series, and $m_w(dx) = w(x)\,dx$ where $w(\cdot)$ satisfies condition A_∞ (dyadic). Given any $a > 1$, $\beta > 1$, there exists a constant γ such that*

$$a m_w(f^* > \beta\lambda,\ S(f) \leqslant \gamma\lambda) \leqslant m_w(f^* > \lambda)$$

and

$$a m_w\big(S(f) > \beta\lambda,\ f^* \leqslant \gamma\lambda\big) \leqslant m_w\big(S(f) > \lambda\big)$$

for all $\lambda > 0$.

The proof of this inequality follows the pattern of proof for Theorems 3 and 4. First, local inequalities are proved for Lebesgue measure by sharpening the estimates leading to Theorem 5.1 in [1]. The transition to other measures $m_w(dx)$ is made by appealing to condition A_∞ (dyadic) and the result for Lebesgue measure.

3. The radial maximal function. If $u(x, y)$ is harmonic in $\boldsymbol{R}_+^{n+1}$, its radial maximal function $N_0(u)(x)$ is defined by

$$N_0(u)(x) = \sup_{y>0} |u(x, y)|.$$

Clearly

$$N_0(u)(x) \leqslant N_a(u)(x), \qquad a > 0.$$

It is shown in [3] that for $0 < p < \infty$ and $a > 0$,

$$\int [N_0(u)]^p\,dx \simeq \int [N_a(u)]^p\,dx.$$

We remark here that this equivalence is true for any measure satisfying condition A_∞.

THEOREM 6. *If u is harmonic on $\boldsymbol{R}_+^{n+1}$ and $m_w(dx)$ satisfies condition A_∞ then*

$$\int [N_a(u)]^p m_w(dx) \leqslant C \int [N_0(u)]^p m_w(dx)$$

for $0 < p < \infty$, with C independent of u.

The proof of Theorem 6 is like that of Corollary 2 in Section 8, Chapter IV of [3]. We need the following lemma, which is proved in Section 9, Chapter IV of [3].

LEMMA (Hardy–Littlewood [5]). *Let B be a ball in $\boldsymbol{R}_+^{n+1}$ with center (x, y). If u is harmonic in $\boldsymbol{R}_+^{n+1}$, then for any $r > 0$*

$$|u(x, y)| \leqslant C_r \left(\frac{1}{m(B)} \iint_B |u(z, t)|^r\,dz\,dt\right)^{1/r}.$$

To prove Theorem 6, we may suppose by Remark (b) of Section 1 that $m_w(dx)$ satisfies condition A_s for some $s > 1$. Fix $x_0 \in \boldsymbol{R}^n$ and let $(x, y) \in \Gamma(x_0; a)$. Assuming for simplicity that $a = 1$, we see that the ball $B(x, y)$ with radius y and center (x, y) lies in $\boldsymbol{R}_+^{n+1}$ and has a projection onto $\boldsymbol{R}^n$ contained in $\{z\colon |z - x_0| < 2y\}$. By the lemma, we have

$$\begin{aligned} |u(x, y)|^r &\leqslant c_r y^{-n-1} \iint_{B(x,y)} |u(z, t)|^r\,dz\,dt \\ &\leqslant c_r y^{-n} \int_{|z-x_0|<2y} N_0(u)^r(z)\,dz \\ &\leqslant c_r [N_0(u)^r]^*(x_0), \end{aligned}$$

where $*$ denotes the Hardy–Littlewood maximal operator. Since (x, y) was an arbitrary point of $\Gamma(x_0)$, we obtain

$$N_1(u)(x_0) \leqslant c_r [N_0(u)^r]^*(x_0)^{1/r}.$$

Integrating with respect to x_0, we have

$$\int_{\boldsymbol{R}^n} [N_1(u)(x_0)]^p m_w(dx_0) \leqslant c_r \int_{\boldsymbol{R}^n} [N_0(u)^r]^*(x_0)^{p/r} m_w(dx_0).$$

Given p, $0 < p < \infty$, we now choose $r > 0$ so that $p/r = s$. Since $m_w(dx)$ satisfies condition A_s, Theorem 7 now follows immediately from [8].

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A note on multipliers on a Segal algebra

by

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Abstract. It is the purpose of this paper to show that if $S(G)$ is a Segal algebra on the locally compact abelian group G and T is a multiplier on $S(G)$ then there exists a unique pseudomeasure σ such that $Tf = \sigma * f$ for each $f \in S(G)$.

Various properties of $S(G)$ are given in Reiter [5]. We denote by $\hat{G}$ the character group of G. Let dx and $d\gamma$ denote the Haar measures on G and $\hat{G}$ respectively where $d\gamma$ is so chosen that the Fourier inversion theorem holds. Let $\mathscr{K}(\hat{G})$ denote the space of continuous functions on $\hat{G}$ with compact support and let

$$B(G) = \{f \in L^1(G) \colon \hat{f} \in \mathscr{K}(\hat{G})\}.$$

Then $B(G)$ is dense in $S(G)$.

A multiplier on $S(G)$ is a bounded linear operator on $S(G)$ which commutes with translations. The problem of characterizing multipliers on various special cases of Segal algebras has been studied by Lai [3], Larsen [4], Keshava Murthy and Unni [1], [2], and Unni [7]. In another paper [6] we introduced the space of parameasures which contains the space of pseudomeasures as a subclass and showed that if T is a multiplier on $S(G)$ then there exists a unique parameasures β such that $Tf = \beta * f$ for each $f \in B(G)$.

Recently Keshava Murthy has brought to my attention a paper by Yap [8] who proves that every Segal algebra on a locally compact abelian group is a semisimple Banach algebra. Though parameasures are of independent interest, the semisimplicity of the Segal algebra makes it possible to prove the following

THEOREM. *Let G be a locally compact abelian group and $S(G)$ a Segal algebra. If T is a multiplier on $S(G)$ then there exists a unique pseudomeasure σ such that*

$$Tf = \sigma * f \quad \textit{for each } f \in S(G).$$