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The equivalence of two conditions for weight functions

by

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Abstract. A function $W(x)$ defined on $\boldsymbol{R}^n$ satisfies the condition A_p, $p > 1$, if it is non-negative and for every cube Q

$$\Big(\int_Q W(x)\,dx\Big)\Big(\int_Q [W(x)]^{-1/(p-1)}\,dx\Big)^{p-1} \leqslant C|Q|^p$$

where C is independent of Q and $0\cdot\infty$ is taken to be 0. A function $W(x)$ defined on $\boldsymbol{R}^n$ satisfies the condition A_∞ if it is non-negative and given $\varepsilon > 0$ there exists a $\delta > 0$ such that if Q is a cube, $E \subset Q$ and $|E| < \delta|Q|$, then $\int_E W(x)\,dx \leqslant \varepsilon \int_Q W(x)\,dx$. Such functions are the weight functions for various weighted norm inequalities for classical operators. It is shown that a locally integrable function satisfies the condition A_∞ if and only if it satisfies the condition A_p for some $p > 1$.

1. Introduction. In several recent papers, [4], [5] and [6], it was shown that the condition A_p characterizes all weight functions for which various weighted norm inequalities are true. On the other hand, in [1] and [2] similar weighted norm inequalities were proved using the condition A_∞. The equivalence proved here is of interest because it shows a relation between the results in the cited papers. It is also an essential part of the proof of the norm inequalities in [2].

The fact that a function $W(x)$ that satisfies A_p also satisfies A_∞ is an immediate result of formula (3.19), p. 214 of [5]; the fact that (3.19) is valid in the n dimensional case is proved in § 7 of [5]. This formula states that if $W(x)$ satisfies A_p, then for every cube Q

$$\int_Q [W(x)]^v\,dx \leqslant C|Q|^{1-v}\Big[\int_Q W(x)\,dx\Big]^v$$

where C and v are constants independent of Q and $v > 1$. Since Hölder's inequality implies that for any subset E of Q

$$\int_E W(x)\,dx \leqslant |E|^{1-1/v}\Big(\int_Q [W(x)]^v\,dx\Big)^{1/v},$$

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it is immediate that if $E \subset Q$, then

$$\int_E W(x)\,dx \leqslant C^{1/v}(|E|/|Q|)^{1-1/v}\int_Q W(x)\,ds. \tag{1.1}$$

If given $\varepsilon > 0$, δ is taken to be $[\varepsilon C^{-1/v}]^{v/(v-1)}$, (1.1) shows that W satisfies the definition of A_∞. The fact that condition (1.1) holds for some $v > 1$ and C is used as the definition of A_∞ in [2] and appears to be more restrictive than A_∞ as defined here since it gives a particular type of relation between the δ and ε. The main theorem of this paper shows, however, that the two definitions are equivalent. If W satisfies A_∞ as defined here, then for any cube Q such that $\int_Q W(x)\,dx < \infty$ the theorem in § 3 shows that W satisfies A_p on Q with fixed constants and $p > 1$; the argument above then gives (1.1) for some C and $v > 1$. If $\int_Q W(x)\,dx = \infty$, (1.1) is satisfied trivially for any C and $v > 1$.

The proof given here that A_∞ implies A_p for some $p > 1$ is a one dimensional one. It is not difficult, however, to adapt it to n dimensions. Some comments about how this is done are made at the end of this paper. This extension to the n dimensional version has also been done by Wo-Sang Young.

2. A basic lemma.

The following will be needed in the proof of the main theorem in § 3.

LEMMA. *If $W(x)$, defined on $(-\infty, \infty)$, satisfies the condition A_∞, d is the value of δ that corresponds to $\varepsilon = \frac{1}{2}$ in the definition of A_∞, Q is an interval, $\int_Q W(x)\,dx < \infty$, k is a positive integer and E is the subset of Q where $W(x) < (8^{-k}/|Q|)\int_Q W(x)\,dx$, then $|E| \leqslant (1-\frac{1}{4}d)^k|Q|$.*

The following simple property of d will be needed. If G is a subset of an interval J and $|G| \geqslant (1-d)|J|$, then since $|J-G| \leqslant d|J|$, $\int_{J-G} W(x)\,dx \leqslant \frac{1}{2}\int_J W(x)\,dx$ so that $\int_G W(x)\,dx \geqslant \frac{1}{2}\int_J W(x)\,dx$. Equivalently, if $G \subset J$ and $\int_G W(x)\,dx < \frac{1}{2}\int_J W(x)\,dx$, then $|G| < (1-d)|J|$.

Now fix Q and k. By the definition of E

$$\int_E W(x)\,dx < 8^{-k}\int_Q W(x)\,dx. \tag{2.1}$$

Since $8^{-k} < \frac{1}{2}$, by the property of d above $|E| < (1-d)|Q|$.

Therefore, for each point of density, x, of E it is possible to choose a closed interval, R_x, centered about x such that

$$|R_x \cap Q \cap E| = (1-d)|R_x \cap Q| \tag{2.2}$$

and R_x is a subset of the interval $3Q$ with the same center as Q and three times as long.

Let S be the set of the R_x's. A sequence $\{Q_j\}$, possibly finite, of members of S will be chosen as follows. Let Q_1 be an interval in S such that $|Q_1| \geqslant \frac{1}{2}\sup_{R\in S}|R|$. Given $Q_1, \ldots, Q_k$, let S_k be the set of all members of S whose centers are not in $\bigcup_{j=1}^{k} Q_j$. Choose Q_{k+1} from S_k such that $|Q_{k+1}| \geqslant \frac{1}{2}\sup_{R\in S_k}|R|$.

Given y in $\bigcup Q_j$, it will be shown that y can not be in the the right half of more than two Q_j's; the right half is taken to include the center. If y were in the right half of three or more Q_j's let Q_{j_1}, Q_{j_2} and Q_{j_3} be the first three such intervals in their order of occurrence. The center of Q_{j_2} lies to the left of Q_{j_1} since it cannot lie in Q_{j_1}. Similarly the center of Q_{j_3} lies to the left of Q_{j_2}. Then the right half of Q_{j_2} strictly contains the left half of Q_{j_1} so $|Q_{j_2}| > |Q_{j_1}|$. Similarly, the right half of Q_{j_3} contains the left halves of Q_{j_2} and Q_{j_1} so that $|Q_{j_3}| \geqslant |Q_{j_2}| + |Q_{j_1}|$. Combining these two inequalities shows that $|Q_{j_3}| > 2|Q_{j_1}|$. This contradicts the way the sequence $\{Q_j\}$ was chosen.

It is immediate that an y in $\bigcup Q_j$ cannot be in the left half of more than two Q_j's; therefore, no point can be in more than four Q_j's. This fact and the fact that all the Q_j's are subsets of $3Q$ show that the sequence $\{Q_j\}$ terminates or the length of the Q_j's approaches 0. Therefore, since each R_x is eliminated from S_k for sufficiently large k, each point of density of E lies in the union of the Q_j's.

Now let $E_1 = \bigcup_j (Q_j \cap Q)$. It is trivial that $E_1 \subset Q$. Furthermore, $\int_{E_1} W(x)\,dx$ is bounded by

$$\sum_j \int_{Q_j\cap Q} W(t)\,dt. \tag{2.3}$$

By (2.2) and the property of d derived above, (2.3) is bounded by

$$2\sum_j \int_{Q_j\cap Q\cap E} W(x)\,dx. \tag{2.4}$$

Since no point is in more that $4Q_j$'s, (2.4) is bounded by $8\int_E W(x)\,dx$. Combining these inequalities and (2.1) then shows that

$$\int_{E_1} W(x)\,dx < 8^{-k+1}\int_Q W(x)\,dx. \tag{2.5}$$

Since E_1 contains all the points of density of E,

$$|E_1| = \Big|\bigcup_j (Q_j\cap Q)\Big| = |E| + \Big|\bigcup_j (Q_j \cap Q \cap CE)\Big| \tag{2.6}$$

where CE denotes the complement of E. Since no point is in more than $4Q_j$'s, (2.6) implies that

$$|E_1| \geqslant |E| + \tfrac{1}{4}\sum_j |Q_j \cap Q \cap CE|.$$

Because of (2.2), then

$$|E_1| \geqslant |E| + \tfrac{1}{4}\sum_j d\,|Q_j \cap Q|$$

so that $|E_1| \geqslant |E| + \frac{1}{4}d|E_1|$. Therefore, $|E_1| \geqslant (1-\frac{1}{4}d)^{-1}|E|$.

Now if $k \geqslant 2$, it is possible to start with (2.5) and repeat the argument following (2.3) with E replaced by E_1. This will produce a set $E_2 \subset Q$ such that

$$\int_{E_2} W(t)\,dt < 8^{-k+2}\int_Q W(x)\,dx$$

and $|E_2| \geqslant (1-\frac{1}{4}d)^{-2}|E|$. Repeating this process k times will give a set E_k such that $E_k \subset Q$ and $|E_k| \geqslant (1-\frac{1}{4}d)^{-k}|E|$. Since $|E_k| \leqslant |Q|$, $|E| \leqslant (1-\frac{1}{4}d)^k|Q|$.

3. Proof that A_∞ implies A_p. The following theorem and corollary will be proved. A sketch of the proof of the n dimensional version is given at the end of this section.

THEOREM. *If $W(x)$, defined on $(-\infty, \infty)$, satisfies the condition A_∞, Q is an interval and $\int_Q W(x)\,dx < \infty$, then*

$$\Big(\int_Q W(x)\,dx\Big)\Big(\int_Q [W(x)]^{-1/(p-1)}\,dx\Big)^{p-1} \leqslant C|Q|^p \tag{3.1}$$

where p and C are independent of Q, $p > 1$ and $0\cdot\infty$ is taken to be 0.

COROLLARY. *If $W(x)$, defined on $(-\infty, \infty)$, satisfies the condition A_∞ and is locally integrable, then it satisfies A_p for some $p > 1$.*

The corollary is, of course, a trivial consequence of the theorem. The hypothesis that $W(x)$ is locally integrable is essential. Consider, for example, a nowhere dense set, E, that has positive measure and define $W(x)$ to be 0 on E and ∞ off E. Then since the integral of W on any interval is ∞, W satisfies the condition A_∞. It does not satisfy A_p for any $p > 1$, however, since there are intervals on which $1/W(x)$ is ∞ on a subset of positive measure.

To prove the theorem fix an interval Q such that $0 < \int_Q W(x)\,dx < \infty$, and let d be the value of δ that corresponds to $\varepsilon = \frac{1}{2}$ in the definition of A_∞. It will be shown that (3.1) holds where p and C depend only on d and $p > 1$. The case $\int_Q W(x)\,dx = 0$ can be ignored; the convention $0\cdot\infty = 0$ makes (3.1) true for any p and C in this case.

Given a positive number a, let E_a be the subset of Q where $1/W(x) > a$ and let $B = |Q|/\int_Q W(x)\,dx$. If $a > B$, let k be the greatest integer less than or equal to $\log_8(a/B)$. Then $1/a \leqslant 8^{-k}/B$ so that E_a is contained in the subset of Q where $W(x) < (8^{-k}/|Q|)\int_Q W(t)\,dt$. Therefore, by the lemma $|E_a| \leqslant (1-\frac{1}{4}d)^k|Q|$. Using the facts that $a^{\log_b c} = c^{\log_b a}$ and $k \geqslant -1 + \log_8(a/B)$ shows that if $a > B$, then

$$|E_a| \leqslant (a/B)^r|Q|/(1-\tfrac{1}{4}d) \tag{3.2}$$

where $r = \log_8(1-\frac{1}{4}d)$. Next define p to be $1-2/r$; this is greater than 1 since r is negative. Now

$$\int_Q [W(x)]^{-1/(p-1)}\,dx = \frac{1}{p-1}\int_0^\infty a^{(2-p)/(p-1)}|E_a|\,da. \tag{3.3}$$

Since $E_a \subset Q$ for all a and (3.2) is true for $a > B$, the right side of (3.3) is bounded by

$$\frac{1}{p-1}\int_0^B a^{(2-p)/(p-1)}|Q|\,da + \frac{1}{p-1}\int_B^\infty a^{(2-p)/(p-1)}\Big(\frac{a}{B}\Big)^r\frac{|Q|}{1-\frac{1}{4}d}\,da.$$

Both of these integrals can be computed; the result is that

$$\int_Q [W(x)]^{-1/(p-1)}\,dx \leqslant CB^{1/(p-1)}|Q|$$

where C depends only on d. This is equivalent to (3.1) and completes the proof of the theorem.

The principal changes for the n dimensional proof are as follows. The number d must be chosen so that it works for rectangular parallelepipeds with sides varying in length by no more than a factor of two. It is easy to prove that 2^{-n} times the δ that corresponds to $\varepsilon = 2^{-n-1}$ for cubes works by writing the parallelepiped as the union of 2^n cubes. The constants in the lemma are different and the Q_j's are chosen by use of Corollary 1.7, p. 304 of [3]. The rest of the proof of both the lemma and theorem are then similar except for changes in the constants.

References

[1] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, to appear.

[2] R. F. Gundy and R. L. Wheeden, *Weighted integral inequalities for the non-tangential maximal function, Lusin area integral, and Walsh–Paley series*, Studia Math. 49 (1974), pp. 107–124.

[3] M. de Guzmán, *A covering lemma with applications to differentiability of measures and singular integral operators*, Studia Math. 34 (1970), pp. 299–317.
[4] R. Hunt, B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), pp. 227–251.
[5] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), pp. 207–226.
[6] C. Segovia and R. Wheeden, *On weighted norm inequalities for the Lusin area integral*, Trans. Amer. Math. Soc. 176 (1973), pp. 103–123.

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Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh–Paley series*

by

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Abstract. We prove weighted integral inequalities between the Lusin area function and nontangential maximal function of a harmonic function. We also obtain results for Walsh–Paley series as a corollary of the method.

Introduction. In this paper we prove weighted integral inequalities for the Lusin area function and the nontangential maximal function. Specifically, we are able to answer some questions raised in [11], and extend the inequalities proved there. Our results indicate that many of the norm inequalities for H^p-spaces in $\boldsymbol{R}_+^{n+1}$ remain true for a wide class of measures on the boundary. Our method consists of showing that certain distribution function inequalities, proved in [2] for the area function and the nontangential maximal function, are valid not only for Lebesgue measure, but also for this wide class of measures. These inequalities lead easily to the desired norm inequalities.

The technique used in studying the area integral may also be used to obtain weighted norm inequalities for Walsh–Paley series. Inequalities of this kind were first studied by Hirschman [6]; we are able to recover and extend his results.

Theorems concerning the area function and nontangential maximal function are in Section 1; Walsh–Paley series are treated in Section 2. Section 3 contains a remark on the radial maximal function.

1. We use the notation of [2]. A cone of opening a in $\boldsymbol{R}_+^{n+1} = \{(x, y): x \in \boldsymbol{R}^n, y > 0\}$ is defined as

$$\Gamma(x) = \Gamma(x, a) = \{(s, y): |x-s| < ay\}.$$

The area function, corresponding to a harmonic function u, is given by

$$A(x) = A_a(u)(x) = \Big(\iint_{\Gamma(x)} y^{1-n} |\nabla u(s, y)|^2 ds\, dy\Big)^{\frac{1}{2}},$$

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