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On a problem of moments of S. Rolewicz

by

IVAN SINGER (Bucharest)

Abstract. We solve a problem of moments raised by S. Rolewicz.

1. S. Rolewicz has proved the following result, with applications to minimum time problems of the theory of control:

THEOREM [5]. *Let E, F be two Banach spaces, u a continuous linear mapping of E into F, and y an element of F such that the equation $u(x) = y$ has a solution. If $u(S_E)$ is closed in F, where $S_E = \{x \in E \mid \|x\| \leqslant 1\}$ (the unit ball of E), then*

$$\inf_{\substack{x \in E \\ u(x)=y}} \|x\| = \sup_{g \in F^*} \inf_{\substack{x \in E \\ g(u(x))=g(y)}} \|x\|. \tag{1}$$

It is known (see [5], remark 1 and the references of [5]) that in the particular case when $\dim F < \infty$, formula (1) holds without any additional assumption; in particular, in this case the assumption that $u(S_E)$ is closed in F, is superfluous. At the Conference on Functional Analysis in October 1970 at Oberwolfach, S. Rolewicz has raised the problem whether (1) always holds without any additional assumption. In the present Note we shall solve this problem by giving a necessary and sufficient condition for the validity of (1) and an example in which this condition is not satisfied. Also, using our criterion, we shall show that the assumption that $u(E)$ is closed in F is sufficient, but not necessary, in order that we have (1).

2. The following theorem gives a necessary and sufficient condition for the validity of (1):

THEOREM 1. *Let E, F be two Banach spaces, u a continuous linear mapping of E into F, and y an element of F such that the equation $u(x) = y$ has a solution, say x_0. We have (1) if and only if*

$$\inf_{\substack{x \in E \\ u(x)=y}} \|x\| = \sup_{\substack{f \in u^*(F^*) \\ \|f\|=1}} |f(x_0)|. \tag{2}$$

Proof. If for each $g \in F^*$ we denote

$$H_g = \{x \in E \mid (u^*(g))(x) = g(y)\} = \{x \in E \mid g(u(x)) = g(y)\}, \tag{3}$$

then formula (1) can be written in the form

(4) $$\inf_{\substack{x\in E\\ u(x)=y}} \|x\| = \sup_{g\in F^*} \operatorname{dist}(0, H_g).$$

Now, if $u^*(g) = 0$, then $g(y) = g(u(x_0)) = (u^*(g))(x_0) = 0$, whence $H_g = E$, and thus $\operatorname{dist}(0, H_g) = 0$. On the other hand, if $u^*(g) \neq 0$, then by (3) and e.g. [1], p. 46, remark a) (on the distance from 0 to a hyperplane), we have

(5) $$\operatorname{dist}(0, H_g) = \frac{|g(y)|}{\|u^*(g)\|} = \frac{|g(u(x_0))|}{\|u^*(g)\|} = \frac{|(u^*(g))(x_0)|}{\|u^*(g)\|}.$$

Consequently, (4) becomes

$$\inf_{\substack{x\in E\\ u(x)=y}} \|x\| = \sup_{g\in F^*} \frac{|(u^*(g))(x_0)|}{\|u^*(g)\|} = \sup_{\substack{f\in u^*(F^*)\\ \|f\|=1}} |f(x_0)|,$$

which completes the proof of theorem 1.

Remark 1. By the above, we can write (1) in the form

(6) $$\operatorname{dist}(0, A) = \sup_{\substack{g\in F^*\\ u^*(g)\neq 0}} \operatorname{dist}(0, H_g),$$

where A is the closed linear manifold $\{x \in E \mid u(x) = y\}$. Since clearly $A \subset H_g (g \in F^*)$, formula (6) is a sharpening of a result of M. Eidelheit [4], according to which $\operatorname{dist}(0, A) = \sup_{\substack{H \text{ hyperplane}\\ H\supset A}} \operatorname{dist}(0, H)$.

Remark 2. Generalizing the terminology of [3], let us say that a linear subspace V of a conjugate Banach space B^* is *of characteristic* 1 *at a point* $b_0 \in B$, if

(7) $$\sup_{\substack{\Phi\in V\\ \|\Phi\|=1}} |\Phi(b_0)| = \|b_0\|.$$

Then, *under the assumptions of Theorem* 1, *we have* (1) *if and only if the linear subspace* $V = u^*(F^*) \subset (\operatorname{Ker} u)^\perp \equiv (E/\operatorname{Ker} u)^*$ *is of characteristic* 1 *at the point* $x_0 + \operatorname{Ker} u \in E/\operatorname{Ker} u$; here, as usually,

$$(\operatorname{Ker} u)^\perp = \{f \in E^* \mid f(x) = 0 \text{ for all } x \in \operatorname{Ker} u\}.$$

The idea of Remark 2 will be helpful in constructing Examples 1 and 2 below. Before doing this, let us give the following corollary of Theorem 1:

Corollary 1. *Under the assumptions of Theorem* 1, *if* $u(E)$ *is closed in* F, *then we have* (1).

Proof. If $u(E)$ is closed in F, then (see e.g. [2], p. 149, Theorem 8) we have $u^*(F^*) = (\operatorname{Ker} u)^\perp$. On the other hand, by a well known corollary of the Hahn–Banach theorem, we have

$$\inf_{\substack{x\in E\\ u(x)=y}} \|x\| = \|x_0 + \operatorname{Ker} u\|_{E/\operatorname{Ker} u} = \sup_{\substack{f\in(\operatorname{Ker} u)^\perp\\ \|f\|=1}} |f(x_0)|.$$

Consequently, we have (2), whence also (1), which completes the proof.

Remark 3. Obviously, the condition of Corollary 1 is satisfied whenever $\dim F < \infty$. Thus, we obtain again the result mentioned in § 1 that for $\dim F < \infty$ (1) always holds.

3. Now we shall give an example in which condition (2), and hence (1), is not satisfied.

Example 1. Let E be a Banach space with a basis $\{x_n\}$ such that the closed linear subspace $[f_n]$ of E^* spanned by the coefficient functionals $\{f_n\} \subset E^* (f_i(x_j) = \delta_{ij}$ for $i, j = 1, 2, \ldots)$ is of characteristic < 1, i.e. [3], there exists an element $x_0 \in E$ such that

(8) $$\sup_{\substack{f\in[f_n]\\ \|f\|=1}} |f(x_0)| < \|x_0\|;$$

such a basis $\{x_n\}$ exists e.g. in $E = l^1$ and $E = c_0$ (see [6]). Furthermore, let $F = E$ and let

(9) $$u(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x) x_i \quad (x \in E).$$

Then u is one-to-one, $u(E)$ is dense in F and

(10) $$u^*(g) = \sum_{i=1}^{\infty} \frac{1}{2^i} g(x_i) f_i \quad (g \in F^* = E^*).$$

Hence $u^*(F^*)$ is a norm-dense linear subspace of $[f_n]$ and therefore, by (8),

(11) $$\sup_{\substack{f\in u^*(F^*)\\ \|f\|=1}} |f(x_0)| = \sup_{\substack{f\in[f_n]\\ \|f\|=1}} |f(x_0)| < \|x_0\|.$$

Finally, let $y = u(x_0)$. Then, since u is one-to-one, we have $\inf_{\substack{x\in E\\ u(x)=y}} \|x\| = \|x_0\|$. Thus (2), and hence (1), is not satisfied.

Now we shall slightly modify example 1 to show that the condition of Corollary 1 is not necessary in order to have (1):

EXAMPLE 2. Let E be a Banach space with a basis $\{x_n\}$ such that the closed linear subspace $[f_n]$ of E^* spanned by the coefficient functionals $\{f_n\}$ is of characteristic 1, i.e. [3],

$$\|x\| = \sup_{\substack{f\in[f_n]\\ \|f\|=1}} |f(x)| \quad (x \in E); \tag{12}$$

such a basis is e.g. the unit vector basis in $E = c_0$ or $E = l^p$, $1 \leqslant p < \infty$. Furthermore, let $F = E$ and define u by (9). Then, as above, $u^*(F^*)$ is a norm-dense subspace of $[f_n]$ and hence, by (12),

$$\|x\| = \sup_{\substack{f\in u^*(F^*)\\ \|f\|=1}} |f(x)| \quad (x \in E). \tag{13}$$

On the other hand, since u is one-to-one, whenever the equation $u(x) = y$ has a solution x_0, we have $\inf\limits_{\substack{x\in E\\ u(x)=y}} \|x\| = \|x_0\|$. Consequently (2), and hence (1), is satisfied, although $u(E)$ is not closed.

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A trace inequality for generalized potentials*

by

DAVID R. ADAMS (Houston, Tex.)

Abstract. In the spirit of the Sobolev–Il'in inequality, the trace or restriction of generalized potentials of L_p functions to arbitrary measurable sets in Euclidean space are studied. When the potential kernel is homogenous, then necessary and sufficient conditions are given for the trace inequality to hold.

Introduction. In [1] the author showed that the necessary and sufficient condition for the continuous imbedding of $L_p(\mathbf{R}^n, l_n)$ into $L^q(\mathbf{R}^n, \nu)$, $1 < p < q < \infty$ via the Riesz potential operator $T: f \to h_\alpha * f(x) = \int |x-y|^{\alpha-n} f(y)\,dy$, $0 < \alpha < n$, is that the maximal function of ν of dimension d, $0 < d \leqslant n$, be bounded ($d/q = n/p - \alpha$), i.e.,

$$M_d(\nu)(x) \equiv \sup_{r>0} r^{-d} \nu\big(B(x, r)\big)$$

is a bounded function of x. Here $B(x,r) = \{y \in \mathbf{R}^n: |x-y| < r\}$ and $L_p(\mathbf{R}^n, \nu)$ denotes the usual Lebesgue pth power summable functions on $\mathbf{R}^n$ with respect to the Borel measure ν. l_n = Lebesgue n-dimensional measure on $\mathbf{R}^n$. When d is a positive integer, this result has, as a corollary, the well known Sobolev-Il'in theorem concerning the restrictions of Riesz potentials of L_p functions to smooth manifolds in $\mathbf{R}^n$ of dimension d. That is, the *trace* of the potential belongs to L_q on the manifold with respect to some d-dimensional measure with a bounded d-dimensional maximal function, e.g., surface measure. See [5].

The purpose of this note is not only to extend this result to a more general class of potentials, but to give a much more direct and simplified approach than presented in [1], even in the case of Riesz potentials. Furthermore, the method of proof allows for a much more accurate estimate of the norm of the operator T than known before. In particular, we get $\|T\|_{pq} \approx \sup_x M_d(\nu)(x)^{1/q}$, see corollary to Theorem B.

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