STUDIA MATHEMATICA T. XLVIII. (1973)

Characterization of saturation classes in $L(\boldsymbol{R}^n)$

by

MICHITAKA KOJIMA (Kazanawa, Japan)
and GEN-ICHIRÔ SUNOUCHI (Sendai, Japan)

Abstract. The saturation class of a large set of radial approximation processes is given by the class $\tilde{K}$ of functions such as

$$\tilde{K} = \{f \in L(\boldsymbol{R}^n):\ |v|^\alpha \hat{f}(v) = \hat{g}(v) \quad \text{for some } g \in M(\boldsymbol{R}^n)\}$$

where $\hat{g}(v)$ is Fourier–Stieltjes transform of a measure g. The main purpose of this paper is to a characterization of this class by the term of $f(x)$, that is to say, when $f \in L(\boldsymbol{R}^n)$, $f \in \tilde{K}$ if and only if

$$\Big\| \int\limits_{|y| \geqslant \varrho^{-1}} \frac{\Delta^{2s}(f)(\cdot, y)}{|y|^{n+\alpha}} dy \Big\|_{L(\boldsymbol{R}^n)} = O(1) \quad \text{uniformly in } \varrho.$$

Our approach is also approximation theoretic and may be applicable to another problems.

§ 0. Introduction. Let $\boldsymbol{R}^n$ be the n-dimensional Euclidean space whose element be denoted by $x = (x_1, \ldots, x_n)$ with norm $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. For a function $f \in L(\boldsymbol{R}^n)$ we consider an approximation process of convolution type with a kernel $k \in M(\boldsymbol{R}^n)$, that is to say,

$$K(x, \varrho; f) = (2\pi)^{-n/2} \int\limits_{\boldsymbol{R}^n} f(x-y)\, dk(\varrho y) \tag{1}$$

where k is normalized as

$$(2\pi)^{-n/2} \int\limits_{\boldsymbol{R}^n} dk(y) = 1. \tag{2}$$

Then it is easy to see that

$$\|K(\cdot, \varrho; f) - f\|_{L(\boldsymbol{R}^n)} \to 0 \quad \text{as } \varrho \to \infty.$$

DEFINITION. Suppose that there exists a positive number α and a class $\tilde{K} \subset L(\boldsymbol{R}^n)$ such that

(i) $\|K(\cdot, \varrho; f) - f\|_{L(\boldsymbol{R}^n)} = o(\varrho^{-\alpha})$ as $\varrho \to \infty$ implies $f = 0$ a.e.,

(ii) $\|K(\cdot, \varrho; f) - f\|_{L(\boldsymbol{R}^n)} = O(\varrho^{-\alpha})$ as $\varrho \to \infty$ implies $f \in \tilde{K}$

and vice versa.

Then the singular integral $K(x, \varrho; f)$ is called to be *saturated with the order* $\varrho^{-\alpha}$ *in the space* $L(\mathbf{R}^n)$ and $\tilde{K}$ is called its *saturation class*.

The saturation class of the approximation process (1) is often given by the class $\tilde{K}$ of functions such as

$$\tilde{K} = \{f \in L(\mathbf{R}^n)\colon |v|^\alpha \hat{f}(v) = \hat{g}(v) \text{ for some } g \in M(\mathbf{R}^n)\}$$

where $\hat{g}(v)$ is Fourier–Stieltjes transform of a measure g.

For this see P. L. Butzer and R. J. Nessel [4]. In this book, the treatise is given in the one dimension, but the situation is the same in $\mathbf{R}^n$.

For the special case in $\mathbf{R}$, G. Sunouchi [8], [9] gives a characterization of the class $\tilde{K}$. In G. Sunouchi [10] and H. Berens, P. L. Butzer and U. Westphal [2], there are another simple proofs. In $\mathbf{R}^n$-case, the similar theorem is given by W. Trebels [11]. The purpose of this paper is to give a simple and approximation theoretic proof based upon the fundamental theorem of saturation and a criterion given in our paper [6].

§ 1. Fundamental theorem of saturation. Let the Fourier–Stieltjes transform $\hat{k}(v)$ of the given kernel $k \in M(\mathbf{R}^n)$ be radial and set $\hat{\varkappa}(|v|) = \hat{k}(v)$. We suppose that $\hat{\varkappa}(t)$ satisfies

$$\lim_{t\to 0} \frac{\hat{\varkappa}(t)-1}{t^\alpha} = c \neq 0 \quad \text{for some } \alpha > 0. \tag{3}$$

The fundamental theorem of saturation is as follows (see P. L. Butzer and R. J. Nessel [4]).

THEOREM 1.1. *Let* $f \in L(\mathbf{R}^n)$ *and the kernel* $k \in M(\mathbf{R}^n)$ *of a singular integral* (1) *satisfy* (2) *and* (3).

Then the following statements are valid;

(i) $\|K(\cdot, \varrho; f) - f\|_{L(\mathbf{R}^n)} = o(\varrho^{-\alpha})$ *as* $\varrho \to \infty$ *implies* $f = 0$ *a.e.,*

(ii) $\|K(\cdot, \varrho; f) - f\|_{L(\mathbf{R}^n)} = O(\varrho^{-\alpha})$ *as* $\varrho \to \infty$ *implies* $|v|^\alpha \hat{f}(v) = \hat{g}(v)$ *for some* $g \in M(\mathbf{R}^n)$,

(iii) *Conversely, if* $|v|^\alpha \hat{f}(v) = \hat{g}(v)$ *for some* $g \in M(\mathbf{R}^n)$ *and if*

$$\frac{\hat{\varkappa}(|v|)-1}{|v|^\alpha} \text{ is a Fourier–Stieltjes transform of some measure in } M(\mathbf{R}^n), \tag{4}$$

then $\|K(\cdot, \varrho; f) - f\|_{L(\mathbf{R}^n)} = O(\varrho^{-\alpha})$ *as* $\varrho \to \infty$. *Hence, the singular integral* (1) *with the kernel* $k \in M(R^n)$ *satisfying the conditions* (2), (3) *and* (4) *is saturated with the order* $\varrho^{-\alpha}$ *and with the saturation class* $\tilde{K}$ *such as*

$$\tilde{K} = \{f \in L(\mathbf{R}^n)\colon |v|^\alpha \hat{f}(v) = \hat{g}(v) \text{ for some } g \in M(\mathbf{R}^n)\}. \tag{5}$$

We remarks here for the later use that there is a criterion for the validity of the condition (4). (See M. Kojima and G. Sunouchi [6].)

THEOREM 1.2. *If, for the radial Fourier–Stieltjes transform* $\hat{\varkappa}(|v|) = \hat{k}(v)$ *of the kernel* $k \in M(\mathbf{R}^n)$, *there exists a radial Fourier–Stieltjes transform* $\hat{\mu}(|v|) = \hat{m}(v)$ *of some* $m \in M(\mathbf{R}^n)$ *such that*

$$\hat{\varkappa}(t) - 1 = \alpha \int_0^t \hat{\mu}(\tau)\tau^{\alpha-1}\,d\tau. \tag{6}$$

then the condition (4) *is satisfied.*

§ 2. A Characterization of the saturation class $\tilde{K}$. In the preceeding section, we derived the saturation class $\tilde{K}$. We shall characterize this class by the term of $f(x)$.

THEOREM. *When* $f \in L(\mathbf{R}^n)$, $f \in \tilde{K}$ *of* (5) *if and only if*

$$\left\| \int_{|y| \geq \varrho^{-1}} \frac{\Delta^{2s}(f)(\cdot, y)}{|y|^{n+\alpha}}\,dy \right\|_{L(\mathbf{R}^n)} = O(1) \quad \text{uniformly in } \varrho,$$

where

$$\Delta^{2s}(f)(x, y) = \sum_{j=0}^{2s} (-1)^j \binom{2s}{j} f\{x + (s-j)y\}$$

and integer s *is choosen such that* $0 < \alpha < 2s$.

§ 3. Proof of the theorem in the case of $n = 1$. For the sake of clarification of our method, we begin with the case $n = 1$. See also [10]. We can write

$$\int_{|y| \geq \varrho^{-1}} \frac{\Delta^{2s}(f)(x, y)}{|y|^{1+\alpha}}\,dy = A_\alpha \varrho^\alpha \left\{(2\pi)^{-1/2} \varrho \int_R f(x-y)k(\varrho y)\,dy - f(x)\right\} \tag{7}$$

where

$$A_\alpha = (-1)^{s+1}\binom{2s}{s} 2\alpha^{-1},$$

$$k(x) = \begin{cases} 2A_\alpha^{-1}(2\pi)^{1/2} c_j |x|^{-1-\alpha} & \text{for } j \leq |x| < j+1 \ (j = 0, 1, \ldots, s-1), \\ 2A_\alpha^{-1}(2\pi)^{1/2} c_s |x|^{-1-\alpha} & \text{for } |x| \geq s, \end{cases}$$

$$c_j = \sum_{k=0}^{j} (-1)^{s-k} \binom{2s}{s-k} k^\alpha \qquad (j = 0, 1, \ldots, s).$$

Consequently, by Theorem (1.1) and Theorem (1.2) in order to prove the theorem, it is sufficient to verify that the measure $k(x)dx$ satisfies the conditions (2), (3) and (6).

By simple calculations we have $k\in L(\boldsymbol{R})$, and

$$\int_R k(x)\,dx = 2\alpha(-1)^{-(s+1)}\binom{2s}{s}^{-1}(2\pi)^{1/2}\Big\{\sum_{j=1}^{s-1} c_j\int_j^{j+1}\frac{1}{x^{1+\alpha}}\,dx + c_s\int_s^\infty \frac{1}{x^{1+\alpha}}\,dx\Big\}$$

$$= (2\pi)^{1/2}\frac{2}{(-1)^{s+1}\binom{2s}{s}}\Big[\sum_{j=1}^{s-1} c_i\{j^{-\alpha}-(j+1)^{-\alpha}\}+c_s s^{-\alpha}\Big]$$

$$=(2\pi)^{1/2}\frac{2}{(-1)^{s+1}\binom{2s}{s}}\sum_{j=1}^{s}(-1)^{s-j}\binom{2s}{s-j}$$

$$=(2\pi)^{1/2}\frac{2}{(-1)^{s+1}\binom{2s}{s}}\sum_{j=0}^{s-1}(-1)^j\binom{2s}{j}=(2\pi)^{1/2}$$

and also

$$\hat{\varkappa}(t)-1 = \frac{2\alpha}{(-1)^s\binom{2s}{s}}\Big\{\sum_{j=1}^{s-1}c_j\int_j^{j+1}\frac{1-\cos xt}{x^{1+\alpha}}\,dx + c_s\int_s^\infty\frac{1-\cos xt}{x^{1+\alpha}}\,dx\Big\}$$

$$=\frac{4\alpha}{(-1)^s\binom{2s}{s}}\Big\{\sum_{j=1}^{s-1}c_j\int_j^{j+1}\frac{\left(\sin\frac{xt}{2}\right)^2}{x^{1+\alpha}}\,dx + c_s\int_s^\infty \frac{\left(\sin\frac{xt}{2}\right)^2}{x^{1+\alpha}}\,dx\Big\}$$

$$=\frac{4\alpha}{(-1)^s\binom{2s}{s}}\int_1^\infty \frac{s\left(\frac{xt}{2}\right)}{x^{1+\alpha}}\,dx = \frac{4\alpha}{(-1)^s\binom{2s}{s}}\,t^\alpha\int_t^\infty\frac{s\left(\frac{x}{2}\right)}{x^{1+\alpha}}\,dx$$

where

$$s(x)=\sum_{j=1}^s(-1)^{s-j}\binom{2s}{s-j}\sin^2 jx = (-1)^{s+1}2^{2(s-1)}(\sin x)^{2s}$$

because

$$(\sin x)^{2s} = \frac{1}{(2i)^{2s}}\{e^{ix}-e^{-ix}\}^{2s} = \frac{1}{(-1)^s2^{2s}}\sum_{l=0}^{2s}(-1)^l\binom{2s}{l}e^{ix(2s-2l)}$$

$$=\frac{1}{(-1)^s2^{2s}}\Big[\sum_{l=0}^{s-1}(-1)^l\binom{2s}{l}\{e^{ix2(s-l)}+e^{-ix2(s-l)}\}+(-1)^s\binom{2s}{s}\Big]$$

$$=\frac{1}{(-1)^s2^{2s}}\sum_{l=0}^{s-1}(-1)^l\binom{2s}{l}\{e^{ix(s-l)}-e^{-ix(s-l)}\}^2$$

$$=\frac{1}{(-1)^{s+1}2^{2(s-1)}}\sum_{l=0}^{s-1}(-1)^l\binom{2s}{l}(\sin x(s-l))^2$$

$$=\frac{1}{(-1)^{s+1}2^{2(s-1)}}\sum_{j=1}^s(-1)^{s-j}\binom{2s}{s-j}\sin^2 jx.$$

So we have

$$\lim_{t\to0}\frac{\hat{\varkappa}(t)-1}{t^\alpha} = -\frac{4^s\alpha}{\binom{2s}{s}}\int_0^\infty\frac{\left(\sin\frac{x}{2}\right)^{2s}}{x^{1+\alpha}}\,dx$$

where the integral exists for $0<\alpha<2s$. Hence the conditions (2) and (3) are satisfied.

Next we pass to verify the condition (6). Taking the Fourier transforms of the both side (7) with $\varrho = 1$, we get

$$\hat{\varkappa}(t)-1 = A_\alpha^{-1}\int_1^\infty\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}\frac{e^{i(s-j)yt}+e^{-i(s-j)yt}}{y^{1+\alpha}}\,dy.$$

So if we put

$$\hat{\mu}(t) = \alpha^{-1}A_\alpha^{-1}i\int_t^{\to\infty}\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}(s-j)\frac{e^{i(s-j)y}-e^{-i(s-j)y}}{y^\alpha}\,dy$$

where the integral exists as $y\to\infty$ in the improper sense, and if we can show that $\hat{\mu}(|v|)$ is Fourier–Stieltjes transform of some measure in $M(\boldsymbol{R}^n)$, then the condition (6) would be satisfied and the proof will be completed.

By a formula in a book of I. M. Gelfand and G. E. Shilov [5], p. 359 and p. 361, the Fourier transform $H(x)$ of $\hat{\mu}(|v|)$ in the sense of distribution is

$$H(x) = \begin{cases} B_\alpha\dfrac{1}{x}\displaystyle\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}(s-j)\{|x+(s-j)|^{\alpha-1}-|x-(s-j)|^{\alpha-1}\} \\ \qquad\qquad (\text{if } \alpha\neq 2m+1\ (m=0,1,\ldots)), \\ B_\alpha^{(1)}\dfrac{1}{x}\displaystyle\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}(s-j)\{|x+(s-j)|^{\alpha-1}-|x-(s-j)|^{\alpha-1}\}+ \\ \quad +B_\alpha^{(2)}\dfrac{1}{x}\displaystyle\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}(s-j)\{|x+(s-j)|^{\alpha-1}\log|x+(s-j)|- \\ \quad -|x-(s-j)|^{\alpha-1}\log|x-(s-j)|\}, \\ \qquad\qquad (\text{if } \alpha = 2m+1\ (m=0,1,\ldots)). \end{cases}$$

So we have only to show that $H\in L(\boldsymbol{R})$ for $0<\alpha<2s$.

Case 1. $\alpha \neq 2m+1$ $(m = 0, 1, \ldots)$. Since for $|x| \leqslant s$, $H(x) = O(1)$ and since for $|x| \geqslant s$,

$$H(x) = B_\alpha \frac{|x|^{\alpha-1}}{x} \sum_{j=0}^{2s} (-1)^j \binom{2s}{j} (s-j) \left\{ \left(1+\frac{s-j}{x}\right)^{\alpha-1} - \left(1-\frac{s-j}{x}\right)^{\alpha-1} \right\}$$

$$= B_\alpha \frac{|x|^{\alpha-1}}{x} \sum_{k=0}^{s-2} 2\binom{\alpha-1}{2k+1} x^{-(2k+1)} \sum_{j=0}^{2s} (-1)^j \binom{2s}{j} (s-j)^{2k+2} + $$

$$+ O(|x|^{-(2s-\alpha+1)})$$

$$= O(|x|^{-(2s-\alpha+1)}),$$

we have $H \in L(\mathbf{R})$. Here we used the identify

$$\sum_{j=0}^{2s} (-1)^j \binom{2s}{j} (s-j)^{2k+2} = 0 \qquad \text{for } 0 \leqslant k \leqslant s-2.$$

Case 2. $\alpha = 2m+1$ $(m = 0, 1, \ldots)$.

$$H(x) = H_1(x) + H_2(x) \qquad \text{say},$$

where $H_1(x)$ or $H_2(x)$ denotes respectively the first or second sum of $H(x)$. We have as before $H_1 \in L(\mathbf{R})$.

Concerning $H_2(x)$, for $|x| \leqslant s$, $H_2(x) = O(1)$ and for $|x| \geqslant s$, $H_2(x)$ $\parallel$ $H_2^{(1)}(x) + H_2^{(2)}(x)$, where

$$H_2^{(1)}(x) = B_\alpha^{(2)} \frac{1}{x} \sum_{j=0}^{2s} (-1)^j \binom{2s}{j} (s-j) \{|x+(s-j)|^{\alpha-1} - |x-(s-j)^{\alpha-1}\} \log|x|.$$

$$H_2^{(2)}(x) = B_\alpha^{(2)} \frac{1}{x} \sum_{j=0}^{2s} (-1)^j \binom{2s}{j} (s-j) \left\{ |x+(s-j)|^{\alpha-1} \log\left(1+\frac{s-j}{x}\right) \right.$$

$$\left. - |x-(s-j)|^{\alpha-1} \log\left(1-\frac{s-j}{x}\right) \right\}.$$

$$H_2^{(1)}(x) = O(\{\log|x|\} |x|^{-(2s-\alpha+1)}), \qquad \text{as } x \to \infty,$$

$$H_2^{(2)}(x) = B_\alpha^{(2)} \frac{|x|^{\alpha-1}}{x} \sum_{j=0}^{2s} (-1)^j \binom{2s}{j} (s-j) \left\{ \left(1+\frac{s-j}{x}\right)^{\alpha-1} \log\left(1+\frac{s-j}{x}\right) - \right.$$

$$\left. - \left(1-\frac{s-j}{x}\right)^{\alpha-1} \log\left(1-\frac{s-j}{x}\right) \right\}$$

$$= B_\alpha^{(2)} \frac{|x|^{\alpha-1}}{x} \sum_{j=0}^{2s} (-1)^j \binom{2s}{j} (s-j) \sum_{v=0}^{\infty} a_v \left(\frac{s-j}{x}\right)^{2v+1}$$

$$= B_\alpha^{(2)} \frac{|x|^{\alpha-1}}{x} \sum_{v=0}^{s-2} a_v \left(\frac{1}{x}\right)^{2v+1} \sum_{j=0}^{2s} (-1)^j \binom{2s}{j} (s-j)^{2v+2} + O(|x|^{-(2s-\alpha+1)})$$

$$= O(|x|^{-(2s-\alpha+1)}); \qquad \text{as } x \to \infty.$$

Therefore we have $H_2 \in L(\mathbf{R})$. So we complete the proof of the theorem when $n = 1$.

§ 4. Proof of the theorem in the case of $n \geqslant 2$. Similary as in the case of $n = 1$, we can write

$$(8) \qquad \int_{|y| \geqslant \varrho^{-1}} \frac{\Delta^{2s}(f)(x, y)}{|y|^{n+\alpha}} dy = A_\alpha \varrho^\alpha \left\{ (2\pi)^{-n/2} \varrho^n \int_{\mathbf{R}^n} f(x-y) k(\varrho y)\, dy - f(x) \right\}$$

where

$$A_\alpha = (-1)^{s+1} \binom{2s}{s} \alpha^{-1} \omega_n, \qquad \omega_n = 2\pi^{n/2} \{\Gamma(n/2)\}^{-1}$$

$$k(x) = \begin{cases} 2A_\alpha^{-1} (2\pi)^{n/2} c_j |x|^{-n-\alpha} & \text{for } j \leqslant |x| < j+1,\ j = 0, 1, \ldots, s-1, \\ 2A_\alpha^{-1} (2\pi)^{n/2} c_s |x|^{-n-\alpha} & \text{for } |x| \geqslant s, \end{cases}$$

$$c_j = \sum_{k=0}^{j} (-1)^{s-k} \binom{2s}{s-k} k^\alpha \qquad (j = 0, 1, \ldots, s).$$

Consequently by Theorem (1.1) and Theorem (1.2) in order to prove the theorem, it is sufficient to verify that the measure $k(x)\,dx$ satisfies the conditions (2), (3) and (6).

We have $k \in L(\mathbf{R}^n)$, and

$$\int_{\mathbf{R}^n} k(x)\, dx = \frac{2\alpha}{(-1)^{s+1}\binom{2s}{s}} (2\pi)^{n/2} \left\{ \sum_{j=1}^{s-1} c_j \int_j^{j+1} \frac{1}{r^{1+\alpha}} dr + c_s \int_s^\infty \frac{1}{r^{1+\alpha}} dr \right\} = (2\pi)^{n/2}.$$

Also we have

$$(9) \qquad \hat{\varkappa}(t) - 1 = \frac{2\alpha}{(-1)^s \binom{2s}{s}} \left\{ \sum_{j=1}^{s-1} c_j \int_j^{j+1} \frac{1-(2\pi)^{n/2} \omega_n^{-1} V_{(n-2)/2}(tr)}{r^{1+\alpha}} dr + \right.$$

$$\left. + c_s \int_s^\infty \frac{1-(2\pi)^{n/2} \omega_n^{-1} V_{(n-2)/2}(tr)}{r^{1+\alpha}} dr \right\}$$

$$= 2\alpha(-1)^{-s} \binom{2s}{s}^{-1} \int_1^\infty \frac{s(tr)}{r^{1+\alpha}} dr = 2\alpha(-1)^{-s} \binom{2s}{s}^{-1} t^\alpha \int_t^\infty \frac{s(r)}{r^{1+\alpha}} dr,$$

where

$$s(r) = \sum_{j=1}^{s} (-1)^{s-j} \binom{2s}{s} \{1-(2\pi)^{n/2} \omega_n^{-1} V_{(n-2)/2}(rj)\}$$

$$= \sum_{j=1}^{s} (-1)^{s-j} \binom{2s}{j} - \Gamma(n/2) \{\Gamma(n-1/2)\Gamma(1/2)\}^{1} \times$$

$$\times \int_0^\pi (\sin\theta)^{n-2} \left\{ \sum_{j=1}^{s} (-1)^{s-j} \binom{2s}{s-j} \cos(rj\cos\theta) \right\} d\theta,$$

and noting that

$$\sum_{j=1}^{s}(-1)^{s-j}\binom{2s}{s-j}\cos(jr\cos\theta) = \frac{1}{2}\sum_{l=0}^{s-1}(-1)^l\binom{2s}{l}\{e^{i(s-l)r\cos\theta}+e^{-i(s-l)r\cos\theta}\}$$
$$= \frac{1}{2}\left[(-1)^s 2^{2s}\left\{\sin\left(\frac{r}{2}\cos\theta\right)\right\}^{2s} - (-1)^s\binom{2s}{s}\right],$$

we have

$$s(r) = \sum_{j=1}^{s}(-1)^{s-j}\binom{2s}{j} - \Gamma(n/2)\{\Gamma((n-1)/2)\Gamma(1/2)\}^{-1}\times$$
$$\times\left[\frac{1}{2}(-1)^s 2^{2s}\int_0^\pi(\sin\theta)^{n-2}\left\{\sin\left(\frac{r}{2}\cos\theta\right)\right\}^{2s}d\theta - 2^{-1}(-1)^s\binom{2s}{s}\int_0^\pi(\sin\theta)^{n-2}d\theta\right]$$
$$= \Gamma(n/2)\{\Gamma((n-1)/2)\Gamma(1/2)\}^{-1}2(-1)^{s+1}2^{2(s-1)}\int_0^\pi(\sin\theta)^{n-2}\left\{\sin\left(\frac{r}{2}\cos\theta\right)\right\}^{2s}d\theta$$

because Wallis formula is

$$\int_0^\pi(\sin\theta)^{n-2}d\theta = \Gamma((n-1)/2)\Gamma(1/2)\{\Gamma(n/2)\}^{-1}.$$

Therefore

$$\lim_{t\to 0}\frac{\hat{\varkappa}(t)-1}{t^\alpha} = 2\alpha(-1)^{-s}\binom{2s}{s}^{-1}\int_0^\infty\frac{s(r)}{r^{1+\alpha}}dr$$

where the integral exists for $0<\alpha<2s$, and so the conditions (2) and (3) are satisfied.

In order to verify the condition (6), we first take the Fourier transforms of the both side of (8) with $\varrho=1$, and cancel $\hat{f}$, we get

$$\hat{\varkappa}(t)-1 = A_\alpha^{-1}(2\pi)^{n/2}\sum_{j=0}^{s}(-1)^j\binom{2s}{j}\int_1^\infty\frac{V_{(n-2)/2}(|s-j|tr)}{r^{1+\alpha}}dr.$$

Putting

$$\hat{\mu}(t) = -\alpha^{-1}A_\alpha^{-1}(2\pi)^{n/2}\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}|s-j|^2\int_t^{\to\infty}r^{-(\alpha-1)}V_{n/2}(|s-j|r)\,dr$$

where the integral exists as $r\to\infty$ in the improper sense.

We shall first show that for $0<\alpha<n+2$ and $0<\alpha<2s$, $\hat{\mu}(|v|)$ is Fourier–Stieltjes transform of some measure in $M(\boldsymbol{R}^n)$. The Fourier transform $H(x)$ of $\hat{\mu}(|v|)$ is for $0<\alpha<n+2$,

$$H(x) = -\alpha^{-1}A_\alpha^{-1}(2\pi)^{n/2}\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}|s-j|^2|s-j|^{-n/2}|x|^{-n/2}\times$$
$$\times\int_0^{\to\infty}r^{-(\alpha-1)}J_{n/2}(|s-j|r)J_{n/2}(|x|r)\,dr$$
$$= B_\alpha\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}|s-j|^\alpha\int_{-\pi/2}^{\pi/2}(\cos\theta)^{\frac{1}{2}(n+\alpha)-1}\times$$
$$\times\begin{Bmatrix}(|x|^2e^{i\theta}+|s-j|^2e^{-i\theta})^{\frac{1}{2}(\alpha-n-2)} & \text{if } |x|\leqslant|s-j|\\ (|x|^2e^{-i\theta}+|s-j|^2e^{i\theta})^{\frac{1}{2}(\alpha-n-2)} & \text{if } |x|\geqslant|s-j|\end{Bmatrix}d\theta$$
$$= B_\alpha I(|x|),$$

by a formula in a book of H. Bateman [1], p. 51. We put for simplicity

$$\gamma = \tfrac{1}{2}(n+2-\alpha) > 0.$$

Since for $|x|\leqslant s$, $I(|x|) = O(1)$, and for $|x|\geqslant s$,

$$\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}|s-j|^2(|x|^2e^{-i\theta}+|s-j|^2e^{i\theta})^{-\gamma}$$
$$= \sum_{j=0}^{2s}(-1)^j\binom{2s}{j}|s-j|^2|x|^{-2\gamma}e^{i\theta\gamma}(1+|s-j|^2|x|^{-2}e^{2i\theta})^{-\gamma}$$
$$= \sum_{v=0}^{s-2}c_v|x|^{-2\gamma-2v}e^{i(2v+\gamma)\theta}\sum_{j=0}^{2s}(-1)^j\binom{2s}{j}(s-j)^{2v+2}+O(|x|^{-[2\gamma+2(s-1)]})$$
$$= O(|x|^{-(2\gamma+2(s-1))}) = O(|x|^{-(2s-\alpha+n)}),$$

we have $I\in L(\boldsymbol{R}^n)$ for $0<\alpha<2s$ and $0<\alpha<n+2$. Therefore the condition (6) is satisfied for such α.

In the case of $n+2\leqslant\alpha<2s$, from (9) putting

$$\hat{\mu}(t) = 2\alpha(-1)^{-s}\binom{2s}{s}^{-1}\int_t^\infty\frac{s'(r)}{r^\alpha}dr$$

where the integral exists in the Lebesgue sense, we shall prove that $\hat{\mu}(|v|)$ is Fourier–Stieltjes transform of some measure in $M(\boldsymbol{R}^n)$. For the sake of the proof, we use the following result due to J. Boman [3].

LEMMA. (a) *Assume that the function $h\in C^N(\boldsymbol{R}^n)$ where $N=[n/2]+1$, and that there exist constants $c>0$ and $\delta>0$ such that*

$$|D^m(h)(v)|\leqslant C|v|^{-\delta-m}\quad \textit{for all } v\in\boldsymbol{R}^n \textit{ and } 0\leqslant m\leqslant N.$$

Then the function $h(v)$ is Fourier transform of some function in $L(\boldsymbol{R}^n)$.

(b) *Assume that the function* $h \in C^N(\mathbf{R}^n - \{0\})$ *with compact support and that there exist constants* $c > 0$ *and* $\delta > 0$ *such that*

$$|D^m(h)(v)| \leqslant C|v|^{\delta-m} \quad \text{for all } v \in \mathbf{R}^n - \{0\} \text{ and } 0 \leqslant m \leqslant N.$$

Then the function $h(v)$ *is Fourier transform of some function in* $L(\mathbf{R}^n)$.

We put

$$h(v) = g(|v|) = \int_{|v|}^{\infty} \frac{s'(r)}{r^a} dr$$

and consider the function $\varrho(v)$ such that $\varrho \in C^\infty(\mathbf{R}^n)$, $\varrho(v) = 1$ for $|v| \leqslant 1$ and $\varrho(v) = 0$ for $|v| \geqslant 2$.

Then we can write

$$h(v) = \{h(v) - \varrho(v)h(v)\} + \varrho(v)h(v) = h_1(v) + h_2(v) \quad \text{say,}$$

where $h_1 \in C^\infty(\mathbf{R}^n)$ and $h_2 \in C^\infty(\mathbf{R}^n - \{0\})$ with compact support.

We now apply the above lemma (a) to the function $h_1(v)$ and (b) to the function $h_2(v)$.

Since

$$g^{(k)}(r) = \{s'(r)r^{-a}\}^{(k-1)} = O(r^{-a}) \quad \text{as } r \to \infty,$$

therefore, for $|v| \geqslant 1$,

$$|D^m(h_1)(v)| \leqslant \sum_{k=1}^{m} c_k |g^{(k)}(|v|)|\, |v|^{-(m-k)} = O\Big(\sum_{k=1}^{m} |v|^k |v|^{-(a+m)} = O(|v|^{-a})\Big)$$
$$= O(|v|^{-m-\delta}) \quad \text{for } \delta = a-1-n/2 > 0 \text{ and for all } 0 \leqslant m \leqslant N,$$

and so $h_1(v)$ is Fourier transform of some integrable function.

Also since

$$g^{(k)}(r) = \{s'(r)r^{-a}\}^{k-1} = O(r^{2s-a-k}) \quad \text{as } r \to 0$$

therefore, for $|v| \leqslant 2$.

$$|D^m(h_2)(v)| \leqslant \sum_{k=1}^{m} c_k \big|g^{(k)}(|v|)\big| |v|^{-(m-k)}$$
$$= O(|v|^{\delta-m}) \quad \text{for } \delta = 2s - a > 0 \text{ and for all } 0 \leqslant m \leqslant N,$$

and so $H_2(v)$ is Fourier transform of some integrable function.

Hence $\hat{\mu}(|v|)$ is Fourier transform of some integrable function for $n+2 \leqslant a \leqslant 2s$. Consequently the theorem for $n \geqslant 2$ is proved.

§ 5. Remarks. Since Fourier–Stieltjes transform implies (L^p, L^p) multiplier for $1 \leqslant p \leqslant \infty$, special case of our theorem implies E. M. Stein's result [7] which is given without proof. Also more general kernel but L^p-case for $1 < p < \infty$ is given by R. L. Wheeden [12] by completely different method.

References

[1] H. Bateman, *Higher Transcendental functions*, Vol. 2. New York 1953.
[2] H. Berens, P. L. Butzer and U. Westphal, *Representation of fractional powers of infinitesimal generators of semigroups*, Bull. Amer. Math. Soc., 74 (1968), pp. 191–196.
[3] J. Boman, *Saturation problems and distribution theory, Topics in approximation theory by H. S. Shapiro*, Lecture Notes in Mathematics, 1971, Appendix I.
[4] P. L. Butzer and R. J. Nessel, *Fourier analysis and approximation*, Vol. 1. 1971.
[5] I. M. Gelfand and G. E. Shilov, *Generalized functions*, Vol. 1. 1964.
[6] M. Kojima and G. Sunouchi, *On the approximation and saturation by general singular integrals*, Tohoku Math. J., 20 (1968), pp. 146–169.
[7] E. M. Stein, *The characterization of functions arising as potentials*, I and II, Bull. Amer. Math. Soc., 67 (1961), pp. 102–104 and 68 (1962), pp. 577–582.
[8] G. Sunouchi, *Characterization of certain classes of functions*, Tohoku Math. J., 14 (1962), pp. 127–134.
[9] — *Saturation in the local approximation*, Tohoku Math. J., 17 (1965), pp. 16–28.
[10] — *Direct theorems in the theory of approximation*, Acta Math. Acad. Sci Hungaricae, 20 (1969), pp. 409–420.
[11] W. Trebels, *Characterisierungen von Saturationsklassen in* $L^1(E_n)$, Dissertation T. H. Aachen, 1969.
[12] R. L. Wheeden, *On hypersingular integrals and Lebesgue spaces of differentiable functions, I and II*, Trans. Amer. Math. Soc., 134 (1968), pp. 421–435 and 139 (1969), pp. 37–53.

DEPARTMENT OF MATHEMATICS
KANAZAWA UNIVERSITY
KANAZAWA, JAPAN

MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI, JAPAN

Received June 20, 1972 (549)